On the generation of vorticity at a free-surface

By T. Lundgren[1] AND P. Koumoutsakos

1. Motivations and objectives

In free surface flows there are many situations where vorticity enters a flow in the form of a shear layer. This occurs at regions of high surface curvature and superficially resembles separation of a boundary layer at a solid boundary corner, but in the free surface flow there is very little boundary layer vorticity upstream of the corner, and the vorticity which enters the flow is entirely created at the corner. Rood (1994) has associated the flux of vorticity into the flow with the deceleration of a layer of fluid near the surface. These effects are quite clearly seen in spilling breaker flows studied by Duncan & Philomin (1994), Lin & Rockwell (1995) and Dabiri & Gharib (1997).

In this paper we propose a description of free surface viscous flows in a vortex dynamics formulation. In the vortex dynamics approach to fluid dynamics, the emphasis is on the vorticity vector which is treated as the primary variable; the velocity is expressed as a functional of the vorticity through the Biot-Savart integral. In free surface viscous flows the surface appears as a source or sink of vorticity, and a suitable procedure is required to handle this as a vorticity boundary condition.

As a conceptually attractive by-product of this study we find that vorticity is conserved if one considers the vortex sheet at the free surface to contain "surface vorticity". Vorticity which fluxes out of the fluid and appears to be lost is really gained by the vortex sheet. As an example of the significance of this, consider the the approach of a vortex ring at a shallow angle to a free surface. It has been observed (Bernal & Kwon, 1988; Gharib, 1994) that the vortex disconnects from itself as it approaches the surface and reconnects to the surface in a U-shaped structure with surface dimples at the vortex ends. There is a clear loss of vorticity from the fluid and an acceleration of the surface in the direction of motion of the ring as discussed by Rood (1994). Since vorticity is conserved the missing vorticity has been flattened out into a vortex sheet which connects the vortex ends. In a real water-air interface, the connection is in a thin vortex layer in the air. The completion of vortex lines along the surface allows one to maintain the physical picture of closed vortex tubes.

When vortex dynamics methods are used for viscous flows with solid boundaries, a vorticity boundary condition may be determined by following Lighthill's (1963) discussion of the problem. Lighthill noted that the velocity field induced by the vorticity in the fluid will not in general satisfy the no-slip boundary condition. This spurious slip velocity may be viewed as a vortex sheet on the surface of the body.

1 Permanent address: Dept. of Aerospace Engr. & Mechanics, Univ. of Minnesota, Minneapolis, MN 55455

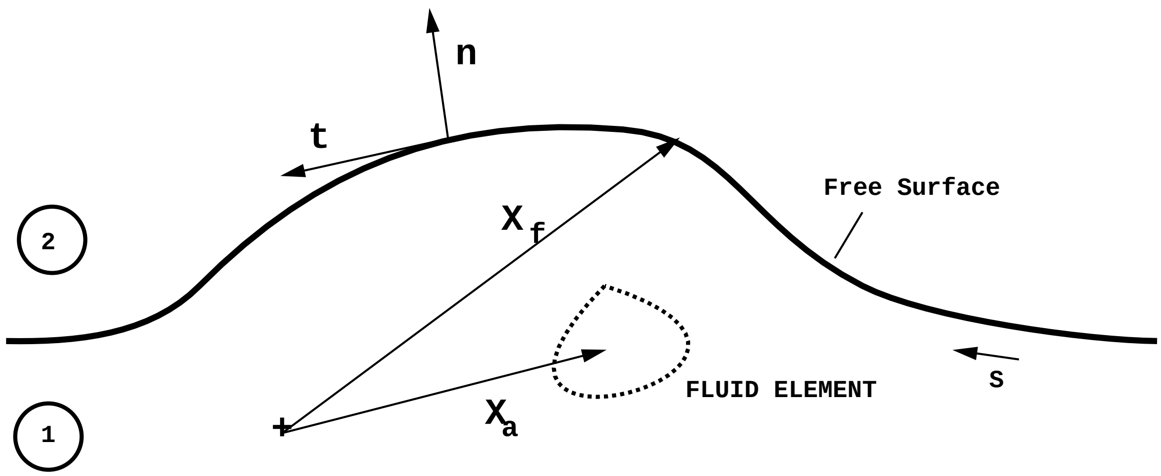

FIGURE 1. Definition sketch.

In order to enforce the no-slip boundary condition, the vortex sheet is distributed diffusively into the flow, transferring the vortex sheet to an equivalent thin viscous vortex layer by means of a vorticity flux. The vorticity flux is the strength of the spurious vortex sheet divided by the time increment.

For free surface flows a vortex sheet is employed in order to adjust the irrotational part of the flow. Unlike the case of a solid wall this vortex sheet is part of the vorticity field of the flow and is used in order to determine the velocity field. The strength of the vortex sheet is determined by enforcing the boundary conditions resulting from a force balance at the free surface.

The physical character of Lighthill's method has led to its direct formulation and implementation by Kinney and his co-workers (1974, 1977) in the context of finite difference schemes, and by Koumoutsakos, Leonard and Pepin (1994) in order to enforce the no-slip boundary condition in the context of vortex methods. Their method has produced benchmark quality simulations of some unsteady flows (Koumoutsakos and Leonard, 1995). The present strategy can be easily adapted to such a numerical scheme and can lead to improved numerical methods for the simulation of viscous free surface flows.

2. Accomplishments

In order to introduce the vorticity generation mechanism, we consider, without loss of generality, two-dimensional flow of a Newtonian fluid with a free surface (Fig. 1). We consider the stresses in *fluid 2* as negligible, and when not otherwise stated the flow quantities refer to *fluid 1.*

2.1 Mathematical formulation

Two-dimensional incompressible viscous flow may be described by the vorticity transport equation

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \nu \nabla^2 \omega \tag{1}$$

with the Lagrangian derivative defined as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \ ,$$

where $\mathbf{u}(\mathbf{x},t)$ is the velocity, $\boldsymbol{\omega} = \omega\,\hat{k} = \nabla\times\mathbf{u}$ the vorticity, and ν denotes the kinematic viscosity. The flow field evolves by following the trajectories of the vorticity carrying fluid elements $\mathbf{x}_a$ and the free-surface points $\mathbf{x}_f$ based on the following equation:

$$\frac{\mathrm{d}\mathbf{x}_p}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}_p)$$

where $\mathbf{x}_p$ denotes $\mathbf{x}_a$ or $\mathbf{x}_f$.

2.1.1 Boundary conditions

The *boundary conditions* at the free-surface are determined by a force balance calculation. For a Newtonian fluid the stress tensor is expressed as

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} \ . \tag{2}$$

where $\mathbf{D}$ is the symmetric part of the velocity gradient tensor. The local normal and tangential components of the surface traction force are expressed as $\hat{n}\cdot\mathbf{T}\cdot\hat{n}$ and $\hat{n}\cdot\mathbf{T}\cdot\hat{t}$ respectively. Balancing these two force components results in the following two boundary conditions at a free-surface.

1. Zero Shear Stress. Assuming negligible surface tension gradients, balancing the tangential forces at the free-surface results in

$$\hat{t}\cdot\mathbf{D}\cdot\hat{n} = 0 \ .$$

This may be expressed

$$\hat{n}\cdot\nabla\mathbf{u}\cdot\hat{t} + \hat{t}\cdot\nabla\mathbf{u}\cdot\hat{n} = 0 \ . \tag{3}$$

For the purposes of our velocity-vorticity formulation we wish to relate this boundary condition to the vorticity field and to the velocity components at the free-surface.

For a two-dimensional flow, by the definition of vorticity in a local coordinate system, we have

$$\omega = \hat{n}\cdot\nabla\mathbf{u}\cdot\hat{t} - \hat{t}\cdot\nabla\mathbf{u}\cdot\hat{n} \ . \tag{4}$$

Using (3) we may rewrite (4) as

$$\omega = -2\hat{t}\cdot\nabla\mathbf{u}\cdot\hat{n} \ . \tag{5}$$

By some further manipulation the free-surface vorticity may be expressed in terms of the local normal and tangential components of the velocity field;

$$\omega = -2\frac{\partial\mathbf{u}}{\partial s}\cdot\hat{n} \tag{6}$$

$$= -2\frac{\partial \mathbf{u}\cdot\hat{n}}{\partial s} + 2\mathbf{u}\cdot\frac{\partial \hat{n}}{\partial s} \tag{7}$$

$$= -2\frac{\partial \mathbf{u}\cdot\hat{n}}{\partial s} + 2\mathbf{u}\cdot\hat{t}\ \kappa \tag{8}$$

where κ is the curvature of the surface, defined by $\kappa = \hat{t}\cdot\partial\hat{n}/\partial s$. For steady flow, where the free-surface is stationary, $\mathbf{u}_1\cdot\hat{n}$ is zero and the first term on the right in (8) drops out. The steady version of (8) was given by Lugt (1987) and by Longuet-Higgins (1992), the unsteady form by Wu (1995). A three-dimensional version of (5) was derived by Lundgren (1989).

The sense of (8) is that vorticity develops at the surface whenever there is relative flow along a curved interface. This condition prevents a viscous free-surface flow from being irrotational. Enforcing the vorticity field given by the above equation at the free-surface is equivalent to enforcing the condition of zero shear stress.

2. Pressure Boundary Condition. This is the condition that the jump in normal traction across the free-surface interface is balanced by the surface tension. It is expressed as

$$\|\hat{n}\cdot\mathbf{T}\cdot\hat{n}\| = -T\kappa$$

where T is the surface tension and the vertical braces denote the jump in the quantity. Using (Eq. 2), this becomes

$$-p_1 + \mu\hat{n}\cdot\nabla\mathbf{u}\cdot\hat{n} + p_2 = -T\kappa\ .$$

Using the continuity equation, expressed in local coordinates, we get

$$\hat{n}\cdot\nabla\mathbf{u}\cdot\hat{n} = -\hat{t}\cdot\nabla\mathbf{u}\cdot\hat{t}$$

$$= -\frac{\partial \mathbf{u}\cdot\hat{t}}{\partial s} + \mathbf{u}\cdot\frac{\partial \hat{t}}{\partial s}$$

$$= -\frac{\partial \mathbf{u}\cdot\hat{t}}{\partial s} - \mathbf{u}\cdot\hat{n}\ \kappa\ .$$

Therefore

$$p_1 = p_2 + T\kappa - \rho\nu\Big(\frac{\partial \mathbf{u}\cdot\hat{t}}{\partial s} + \mathbf{u}\cdot\hat{n}\ \kappa\Big) \tag{9}$$

where p_2 is the constant pressure on the zero density side of the interface.

Since pressure does not occur in the vorticity equation, the pressure condition must be put in a form which accesses the primary variables. From the momentum equation at the free-surface we obtain

$$\hat{t}\cdot\frac{d\mathbf{u}_1}{dt} = -\frac{1}{\rho}\frac{\partial p_1}{\partial s} + \nu\hat{n}\cdot\nabla\boldsymbol{\omega} - g\hat{j}\cdot\hat{t} \tag{10}$$

where g is the gravitational constant; $\hat{j}$ is upward.

For our purposes this equation may be put in a more tractable form by further manipulation. First we observe that

$$\hat{t}\cdot\frac{d\mathbf{u}_1}{dt}=\frac{d\mathbf{u}_1\cdot\hat{t}}{dt}+\mathbf{u}_1\cdot\frac{d\hat{t}}{dt}$$

and

$$\mathbf{u}_1\cdot\frac{d\hat{t}}{dt}=\mathbf{u}_1\cdot\ \hat{n}\ \hat{n}\cdot\frac{d\hat{t}}{dt}\ .$$

Then using the fact that the free-surface is a material surface we obtain the kinematic identity

$$\hat{n}\cdot\frac{d\hat{t}}{dt}=\hat{t}\cdot\nabla\mathbf{u}\cdot\hat{n}$$

$$=\frac{\partial\mathbf{u}_1\cdot\hat{n}}{\partial s}-\mathbf{u}_1\cdot\hat{t}\ \kappa\ .$$

Using this identity we find

$$\frac{d\mathbf{u}_1\cdot\hat{t}}{dt}=\mathbf{u}_1\cdot\hat{n}\ \frac{\partial\mathbf{u}_1\cdot\hat{n}}{\partial s}-\mathbf{u}_1\cdot\hat{t}\ \mathbf{u}_1\cdot\hat{n}\ \kappa-\frac{1}{\rho}\frac{\partial p_1}{\partial s}+\nu\hat{n}\cdot\nabla\omega-g\hat{\jmath}\cdot\hat{t}\ . \tag{11}$$

We emphasize that the material derivative here is taken following a fluid particle on side 1 of the interface.

With p_1 substituted from (9), this formula may be regarded as equivalent to the pressure boundary condition. Except for the flux term all the terms on the right-hand-side of the equation are quantities defined on the surface and derivatives of these along the surface. We prefer to think of the role of the vorticity flux in this equation as a term which modifies the surface acceleration, rather than consider that the equation determines the flux.

Using a strategy analogous to Lighthill's for a solid wall, we propose a fractional step algorithm that enforces the pressure boundary condition in a vorticity-velocity framework. This strategy allows us to gain insight into the development and generation of vorticity at a viscous free surface and can be used as a building tool for a numerical method.

3. A fractional step algorithm

In order to show that the free-surface boundary conditions are satisfied in a velocity-vorticity formulation, we consider the evolution of the flow field during a single time step. In a manner similar to Lighthill's approach for a solid boundary, a vortex sheet is employed to enforce the boundary conditions. The vortex sheet becomes part of the vorticity field of the flow. The *difference* between the solid wall and the free surface is the role of the surface vortex sheet in adjusting the velocity field of the flow. In the case of the *solid wall*, the vortex sheet is *eliminated* from the boundary (so that the no-slip boundary condition is enforced) and enters the flow diffusively, resulting in the flux of vorticity into the flowfield. In the case of a *free* *surface*, the vortex sheet *remains at the surface* to enforce the pressure boundary condition and constitutes a part of the vorticity field of the flow. The task is to determine the strength of the vortex sheet at the free surface so as to satisfy the boundary conditions.

For the purpose of describing this process, we assume that the velocity and the vorticity field are known at time t^n throughout the flow field and at the free surface, and we wish to obtain the flow field at time $t^{n+1} (\equiv t^n + \delta t)$.

Step 1. Given the velocity and vorticity at time t^n we update the positions of the vorticity carrying elements and the surface markers by solving $d\mathbf{x}_p/dt = \mathbf{u}(\mathbf{x}_p, t)$;

$$\mathbf{x}_p^{n+1} = \mathbf{x}_p^n + \delta t \mathbf{u}^n(\mathbf{x}_p^n) \ .$$

We update the vorticity field by solving

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \nu \nabla^2 \omega$$

with initial condition $\omega = \omega^n$ at $t = t^n$ and boundary condition $\omega = \omega^n(\mathbf{x}_f)$ at $\mathbf{x} = \mathbf{x_f}$. The solution to this equation, which we denote by $\omega^{n+1/2}$, is still incomplete. It does not satisfy the correct vorticity boundary condition at the end of the time step and must be corrected in step 2. The boundary condition which we have imposed ensures (rather arbitrarily) that the vorticity on the boundary is purely convected. The correction which is needed will be a vortical layer along the free-surface with vorticity of order δt and with thickness of order $(\delta t)^{1/2}$. We reason that the additional *velocity* field induced across this layer can be neglected since its variation is only of order $(\delta t)^{3/2}$.

For an incompressible flow the velocity may be expressed in terms of a stream function ψ by

$$\mathbf{u} = -\hat{k} \times \nabla \psi \tag{12}$$

and the vorticity itself is related to ψ by

$$\omega \equiv \hat{k} \cdot \nabla \times \mathbf{u} = -\nabla^2 \psi. \tag{13}$$

We use the convention that $\hat{n}$ is always outward from the fluid, $\hat{t}$ is the direction of integration along the surface, and $\hat{k} = \hat{n} \times \hat{t}$ is a unit vector out of the page. The solution of this equation gives

$$\psi = \psi_\omega + \psi_\gamma \tag{14}$$

where

$$\psi_\omega(\mathbf{x}) = -\frac{1}{2\pi} \int_{\text{fluid}} \omega(\mathbf{x}_a, t) \ln|\mathbf{x} - \mathbf{x}_a| d\mathbf{x}_a \tag{15}$$

and ψ_γ represents an irrotational flow selected to satisfy boundary conditions. It is consistent with vortex dynamics to take this irrotational part as the flow induced by a vortex sheet along the boundary of the fluid, i.e. by

$$\psi_\gamma(\mathbf{x}, t) = -\frac{1}{2\pi} \int_{\text{intfc}} \gamma(\mathbf{x}_f(s'), t) \ln|\mathbf{x} - \mathbf{x}_f(s')| ds' \ , \tag{16}$$

but it must be shown that this can be done in such a way as to satisfy the boundary conditions. In this formulation the boundary can be either solid or free or a mix of these, but in this paper we are specifically interested in free boundaries which separate an incompressible fluid from a fluid of negligible mass density. The velocity field is obtained by applying (12), giving the Biot-Savart law;

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_\omega(\mathbf{x},t) + \mathbf{u}_\gamma(\mathbf{x},t) \tag{17}$$

where

$$\mathbf{u}_\omega(\mathbf{x}) \;=\; \frac{1}{2\pi}\int\limits_{\text{fluid}} \boldsymbol{\omega}(\mathbf{x}_a,t)\;\hat{k}\times\nabla\ln|\mathbf{x}-\mathbf{x}_a|d\mathbf{x}_a \tag{18}$$

and

$$\mathbf{u}_\gamma(\mathbf{x}) = \frac{1}{2\pi}\int\limits_{\text{intfc}} \gamma(\mathbf{x}_f(s),t)\;\hat{k}\times\nabla\ln|\mathbf{x}-\mathbf{x}_f(s)|ds\;. \tag{19}$$

The velocity field is also defined by these integrals for points outside the fluid; $\mathbf{u}_\omega$ is continuous across the interface, and $\mathbf{u}_\gamma$ has a jump discontinuity. As the position vector $\mathbf{x}$ tends to a point on the interface from inside the fluid, which we will indicate with a subscript "1", we get

$$(\mathbf{u}_\gamma\,\cdot\,\hat{t})_1 = -\frac{\gamma(s)}{2} - \text{P.V.}\frac{1}{2\pi}\int\limits_{\text{intfc}} \gamma(s',t)\;\hat{n}\,\cdot\,\nabla\ln|\mathbf{x}_f(s)-\mathbf{x}_f(s')|ds' \tag{20}$$

while as the point is approached from the outside, indicated by "2",

$$(\mathbf{u}_\gamma\,\cdot\,\hat{t})_2 = +\frac{\gamma(s)}{2} - \text{P.V.}\frac{1}{2\pi}\int\limits_{\text{intfc}} \gamma(s',t)\;\hat{n}\,\cdot\,\nabla\ln|\mathbf{x}_f(s)-\mathbf{x}_f(s')|ds'\;. \qquad 21$$

Here P.V. indicates the principal value of these singular integrals. By subtracting these equations it is clear that the vortex sheet strength is the jump in tangential velocity across the interface; since $\mathbf{u}_\omega\cdot\hat{t}$ is continuous, we have

$$\gamma = \mathbf{u}_2\cdot\,\hat{t} - \mathbf{u}_1\cdot\hat{t}\;. \tag{22}$$

By (17) and (20) the tangential component of the surface velocity is

$$-\frac{\gamma(s)}{2} - \text{P.V.}\frac{1}{2\pi}\int\limits_{\text{intfc}} \gamma(s',t)\;\hat{n}\cdot\nabla\ln|\mathbf{x}_f(s)-\mathbf{x}_f(s')|ds' = \mathbf{u}_1\cdot\hat{t} - (\mathbf{u}_\omega\cdot\hat{t})_1\;\;. \tag{23}$$

Equation (23) is a Fredholm integral equation of the second kind the solution of which determines the strength (γ) of the free surface vortex sheet when the right-hand side is given. In the case of multiply connected domains the equation needs to be supplemented with m constraints for the strength of the vortex sheet, where *m+1* is the multiplicity of the domain (Prager, 1928). For example, in the case of a free surface extending to infinity, no additional constraint needs to be imposed as the problem involves integration over a singly connected domain. However, in the case of a bubble, an additional constraint such as the conservation of total circulation in the domain needs to be imposed in order to obtain a unique solution.

The right-hand side of the equation may be determined from the quantities which have been updated. In particular $\mathbf{u}_\omega$ can be computed via the Biot-Savart integral (18) from the known vorticity field $\omega^{n+1/2}$ with order δt accuracy. The tangential component of the velocity of the free surface can be computed using (11) in the form

$$(\mathbf{u}_1 \cdot \hat{t})^{n+1} = (\mathbf{u}_1 \cdot \hat{t})^n + \delta t Q^n(\mathbf{u}_1, \hat{n}, \hat{t}, \nu \frac{\partial \omega}{\partial n}, p_1)$$

where Q^n signifies the right-hand side of (11) evaluated at time t^n. The pressure boundary condition enters the formulation of the problem at this stage. Upon solving (23) the strength of the vortex sheet is determined such that the pressure boundary condition is satisfied, justifying the previous assertion. We should add that (23) admits more than one solution in multiply connected domains, such as a two-dimensional bubble configuration, but unique solutions may be obtained by using Fredholm's alternative.

Note that the present method of enforcing the pressure boundary condition is equivalent to previous *irrotational* formulations (Lundgren & Mansour, 1988, 1991) which employ a velocity potential.

At the end of this step the points of the free-surface, the velocity field, and the strength of the vortex sheet have been updated (${\mathbf{x}_p}^{n+1}$, $\mathbf{u}^{n+1}$ and γ^{n+1}). The vorticity field ($\omega^{n+1/2}$) still needs to be corrected near the free surface.

Step 2. At this step we consider generation of vorticity at the free surface. Having determined the strength of the vortex sheet from Step 1, we can compute the normal and tangential components of the velocity field at the free surface in order to determine the free surface vorticity and enforce the zero-shear stress boundary condition.

Using (14,15,16) we can compute an updated value of the stream function on the surface and from this compute $\mathbf{u}_1 \cdot \hat{n} = \partial\psi/\partial s$. Since the surface shape and $\mathbf{u}_1 \cdot \hat{t}$ have already been updated, we have all the ingredients necessary to compute an updated value of ω_1 from (3). The next step in this process is to solve the vorticity transport equation for the vorticity field using ω_1 as boundary condition. For the final partial step we need to solve the heat equation,

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega \ , \qquad (24)$$

with initial condition $\omega = 0$ at $t = t^n$, and with the boundary condition

$$\omega(\mathbf{x}_f) = (\omega_1^{n+1} - \omega_1^n)(t - t^n)/\delta t$$

assuming a linear time variation of the surface vorticity between the two time levels. The solution of this partial step is to be added to $\omega^{n+1/2}$, thus yielding the completely updated vorticity field ω^{n+1}.

An analytical solution for this diffusion equation can be obtained using the method of heat potentials (Friedman, 1964). For a two-dimensional flow the solution to the above equation may be expressed in terms of double-layer heat potentials as

$$\omega(\mathbf{x}, t+\delta t) = \int_t^{t+\delta t} \int_{\text{intfc}} \frac{\partial G}{\partial n'}(\mathbf{x}-\mathbf{x}_f(s'), t-t')\mu(s',t')\, ds'\, dt'$$

where G is the fundamental solution of the heat equation and the function $\mu(s,t)$ is determined by the solution of the following second order Fredholm integral equation:

$$-\frac{1}{2}\mu(s,t) + \int_t^{t+\delta t} \int_{\text{intfc}} \mu(s',t)\, \frac{\partial G}{\partial n'}(\mathbf{x}_f(s)-\mathbf{x}_f(s'), t-t')\, ds'\, dt' = \omega(\mathbf{x}_f(s), t)\ .$$

Following Greengard and Strain (1990) and Koumoutsakos, Leonard and Pepin (1994), we can obtain asymptotic formulas for the above integrals. Similar formulas could help in the development of a numerical method based on the proposed algorithm.

This update strategy was posed without requiring any particular numerical methods for the computational steps. We have particular methods in mind, however, for using this strategy for future numerical work. We will use a boundary integral method similar to that used by Lundgren & Mansour (1988, 1991) for the surface computations. That work was for irrotational inviscid flow. Instead of the pressure boundary condition in the form of (11), an unsteady Bernoulli equation was used to access the pressure.

For the vortical part of the flow we propose to use the point vortex method employed by Koumoutsakos *et al* (1994, 1995) for viscous flow problems with solid boundaries. In these problems the Lighthill strategy provides a vorticity flux boundary condition for the second step in the vorticity update, a Neuman condition. In the proposed free-surface strategy, a Dirichlet condition is required for the second vorticity step. This modification can be accomplished by using double layer heat potentials (as suggested above) where single layer potentials were used in the solid boundary work.

4. Conservation of vorticity

We will show that vorticity is conserved in two-dimensional free-surface problems; vorticity which flows through the free-surface doesn't disappear, but resides in the vortex sheet along the surface. (This is shown for general three-dimensional flows in the Lundgren and Koumoutsakos (1998).)

In the interior of the fluid it is easy to show from Helmholtz's equation that

$$\frac{d}{dt}\int_{A_1} \omega dA = \int_{S_1} \nu \frac{\partial \omega}{\partial n} ds \tag{25}$$

where A_1 is a material "volume" and S_1 its "surface", n is outward from the region, and $-\nu\partial\omega/\partial n$ is the vorticity flux in the outward direction. This says that the vorticity in A_1 increases because of viscous vorticity flux into the region; there are no vorticity sources in the interior of the fluid.

Everything we need to know about the velocity on side 2 is contained in (17,18,19). We will only use the fact that, because the velocity on side 2 is irrotational, there must be a velocity potential($\mathbf{u}_2 = \nabla\Phi$). We use d/dt to mean the material derivative along side 1, and note that $\mathbf{u}_2 - \mathbf{u}_1 = \gamma\hat{t}$, then by some simple manipulations

$$\frac{d\mathbf{u}_2}{dt} = \frac{\partial \mathbf{u}_2}{\partial t} + \mathbf{u}_1 \cdot \nabla \mathbf{u}_2 \tag{26}$$

$$= \nabla\Big(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\mathbf{u}_2 \cdot \mathbf{u}_2\Big) - \gamma\hat{t} \cdot \nabla\gamma\hat{t} - \gamma\hat{t} \cdot \nabla\mathbf{u}_1 \ .$$

Then

$$\hat{t} \cdot \frac{d\mathbf{u}_2}{dt} = \frac{\partial}{\partial s}\Big(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\mathbf{u}_2 \cdot \mathbf{u}_2 - \frac{1}{2}\gamma^2\Big) - \gamma\hat{t} \cdot \nabla\mathbf{u}_1 \cdot \hat{t} \ . \tag{27}$$

The last term in this equation is the strain-rate of a surface element and may be expressed as

$$\hat{t} \cdot \nabla\mathbf{u}_1 \cdot \hat{t} = \frac{1}{ds}\frac{d}{dt}ds \ , \tag{28}$$

where ds is a material line element on side 1. Subtracting (10) from (27) then gives

$$\frac{d\gamma}{dt} + \frac{\gamma}{ds}\frac{d}{dt}ds = \frac{\partial}{\partial s}\Big(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\mathbf{u}_2 \cdot \mathbf{u}_2 - \frac{1}{2}\gamma^2 + \frac{p_1}{\rho} + gy\Big) - \nu\hat{n} \cdot \nabla\omega \ . \tag{29}$$

This may be written

$$\frac{d}{dt}\gamma ds = -\nu\frac{\partial \omega}{\partial n}ds - \frac{\partial \Phi}{\partial s}ds \tag{30}$$

with Φ given by

$$\Phi = -\left[\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\mathbf{u}_2 \cdot \mathbf{u}_2 - \frac{1}{2}\gamma^2 + gy\right] - \frac{p_1}{\rho} \ . \tag{31}$$

If we integrate (30) over a material segment along the interface we obtain

$$\frac{d}{dt}\int_a^b \gamma ds = -\int_a^b \nu\frac{\partial \omega}{\partial n}ds - \int_a^b \frac{\partial \Phi}{\partial s}ds \ . \tag{32}$$

From this form we see that Φ should be interpreted as a surface-vorticity flux. Since γ is a density (circulation density or surface-vorticity density) the last term in (32), which may be written $\Phi_a - \Phi_b$, is the flux of surface-vorticity into the interval at a minus the flux out at b, while the first term on the right is the flux of vorticity into the interval through the surface.

If the interval is extended over the entire interface, by extending it to infinity for an "ocean" or continuing b around to a for a closed interface like a bubble, we get

$$\frac{d}{dt}\int_{\text{intfc}} \gamma ds = -\int_{\text{intfc}} \nu\frac{\partial\omega}{\partial n}ds. \tag{33}$$

Now letting A_1 in (25) be the entire fluid, we get

$$\frac{d}{dt}\int_{\text{fluid}} \omega dA = \int_{\text{intfc}} \nu\frac{\partial\omega}{\partial n}ds \ . \tag{34}$$

Adding (34) and (33) gives

$$\frac{d}{dt}\int_{\text{fluid}} \boldsymbol{\omega} dA + \frac{d}{dt}\int_{\text{intfc}} \gamma ds = 0 \ . \tag{35}$$

It is in this sense that vorticity is conserved.

We began this approach as an attempt to obtain an evolution equation for γ which would eliminate solving an integral equation, (23), to update γ. Equation (29) or (30) might appear to play such a role, but the occurrence of the velocity potential ϕ_2 in the equation makes it unuseable for this purpose. Since ϕ_2 could be expressed by an integration over the surface involving γ, the time derivative of ϕ_2 would involve a surface integral of $d\gamma/dt$ therefore an integral equation for $d\gamma/dt$ would result, defeating the purpose.

A similar result can be shown for the conservation of vorticity in three-dimensional flows (Lundgren and Koumoutsakos, 1998).

4.1 Pedley problem

A problem solved by Pedley (1968) as part of a study on the stability of swirling torroidal bubbles gives an example which illustrates some concepts discussed here. One can describe the flow as a potential vortex of circulation Γ swirling around a bubble cavity of radius R. The flow is induced by a vortex sheet of strength $\gamma_0 = \Gamma/2\pi R$ at the bubble interface. At some initial time one turns on the viscosity and vorticity begins to leak from the vortex sheet into the fluid. The circulation at infinity remains constant; therefore, the strength of the vortex sheet must decrease with time.

We pose this problem in the form described in Section 2. Since the flow is axially symmetric the vorticity satisfies

$$\frac{\partial\omega}{\partial t} = \nu\left(\frac{\partial^2\boldsymbol{\omega}}{\partial r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r}\right) \ . \tag{36}$$

The vorticity boundary condition (8) is

$$\omega_1 = -2V_1/R \ , \tag{37}$$

where $V_1 = \mathbf{u}_1 \cdot \hat{t}$ is the tangential component of the velocity at the interface (with the tangent convention used earlier V_1 is negative for positive swirl), and R is the constant radius of curvature of the surface. The pressure boundary condition (11) is

$$\frac{\partial V_1}{\partial t} = -\nu \left(\frac{\partial \omega}{\partial r} \right)_1 \quad . \tag{38}$$

The velocity inside the bubble is zero so $\mathbf{u}_2 \cdot \hat{t} = 0$. The strength of the vortex sheet is therefore $\gamma = -V_1$, a positive quantity. The sense of the problem is that since ω_1 is required to be non-zero, a layer of positive vorticity must develop in the fluid. The resulting flux of vorticity out of the interface causes γ to decrease with time.

Equations (37) and (38) may be combined into a single boundary condition

$$\frac{\partial \omega_1}{\partial t} = \frac{2\nu}{R} \left(\frac{\partial \omega}{\partial r} \right)_1 \ . \tag{39}$$

Therefore, the problem is to solve (36) with this boundary condition and with initial conditions $\omega = 0$ for all $r > R$ and $\omega = 2\gamma_0/R$ for $r = R$. This last condition prevents the trivial solution.

For large $\tau (\equiv \nu t/R^2)$ Pedley gives an approximate solution;

$$\omega = \frac{\pi \gamma_0}{2R\tau} \exp\left(- \frac{r^2}{4R^2\tau} \right) \quad . \tag{40}$$

This satisfies (36) exactly but has a relative error of order τ^{-1} in the boundary condition. For small τ another approximate similarity solution is

$$\omega = \frac{2\gamma_0}{R} \exp(2x + 4\tau) \ \mathrm{Erfc}\left(\frac{x}{2\sqrt{\tau}} + 2\sqrt{\tau} \right) \tag{41}$$

where $x = (r - R)/R$. This solution satisfies the boundary condition exactly but neglects the last term in (36), requiring that τ be small enough that the vortical layer is thin compared to the radius of the bubble.

Further details of the solution are unimportant here. This problem illustrates both conservation of vorticity and generation of vorticity when there is flow along a curved free-surface.

5. Conclusion

In this paper we have presented a strategy for solving free surface viscous flow problems in a vortex dynamics formulation. This strategy centers on determining suitable boundary conditions for the vorticity in analogy with Lighthill's strategy for solid boundary flows. The two free surface boundary conditions play distinct roles in determining free surface viscous flows. We have shown that the pressure boundary condition determines the strength of a vortex sheet at the free surface, which determines the irrotational part of the flow. The pressure force modifies the surface velocity, from which the vortex sheet strength is found by solving an integral equation. The zero shear stress boundary condition, on the other hand, determines the *value* of the vorticity at the surface, providing a Dirichlet condition for the vorticity equation.

We have shown that vorticity is conserved for both two- and three-dimensional free surface flows, the vortex sheet being considered part of the vorticity field. It follows that vorticity which might appear to be lost by flux across the free surface now resides in the vortex sheet. It was shown in the appendix that vorticity is conserved for two *viscous* fluids in contact across an interface. It is physically clear that, in the limit as the density and viscosity of one of the fluids tend to zero, the vorticity in that fluid would be confined to a thin surface layer. Vorticity would then be conserved in the remaining fluid plus a contribution in the surface layer. Therefore, the conclusions we draw for free surface flows are physically reasonable for real fluids.

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