Subgrid-scale models based on incremental unknowns for large eddy simulations

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1. Introduction

Turbulence modeling in the context of large eddy simulation (LES) is based on a decomposition of all flow variables into large (energy-containing) eddies and small scales carrying a small percentage of the total kinetic energy. The scale separation is achieved by applying a filter operation in physical space, based on a filter function, to flow fields. The net effect of the filter is to remove or at least to reduce the energy contained in scales of length smaller than the filter width Δ_f. The equation of motion of the large scales $\bar{u}_i$ are derived by applying the filter operation to the Navier-Stokes equations. The effect of the subgrid scales (SGS) on the dynamics of the large ones appears through a nonlinear interaction term, the SGS stress tensor $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$. This stress corresponds to two mechanical effects, i.e. an energy transfer from large to smaller scales, inducing a dissipative effect on the large scales, and an energy flux from the SGS to the resolved scales, called backscatter.

Among the most commonly used SGS models in LES are the eddy-viscosity models and their dynamic versions. The Smagorinsky model (Smagorinsky, 1963) is based on the assumption that the the SGS stress tensor is proportional to the strain-rate tensor S_{ij}. The traceless part of τ_{ij} is represented as

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} \equiv -2\nu_T \bar{S}_{ij},$$

where $\nu_T = (C\Delta_f)^2(2\bar{S}_{ij}\bar{S}_{ij})^{1/2}$, C is a non-dimensional constant. Lilly (1967) provided an estimate of C ($\simeq 0.18$) for homogeneous turbulent flows. However, an adjustment is necessary for wall-bounded flows, *viz.*, $C \simeq 0.1$ is more suited in this case. Major advances in eddy-viscosity models were accomplished by the introduction of dynamic modeling ideas (Germano *et al.* 1991, Lilly 1992, and Ghosal *et al.* 1995). Dynamic models provide an expression of the constant C in terms of resolved scales and is then computed as function of time (and space). They have been applied successfully to different kinds of flows, improving results obtained with other models in most cases.

Eddy viscosity models are able to properly predict the amount of SGS dissipation. As they are purely dissipative models, they are unable to account for backscatter effects which are of importance in some flows as transitional or non-equilibrium ones. Moreover, *a priori* analyses of DNS or experimental data (Clark *et al.* 1979,

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Kerr *et al.* 1996, and O'Neil and Meneveau 1997) have shown that the exact stress tensor correlates very poorly with the strain-rate tensor. Therefore, eddy viscosity models very poorly represent the local effects of the SGS scales on the resolved ones.

Scale similarity models (Bardina, Ferziger & Reynolds, 1983) provide a better physical representation of the SGS stresses. They assume that most of the SGS stress can be estimated from the smallest resolved scales. The stress tensor is expressed as $\tau_{ij} \equiv \overline{\bar{u}_i \bar{u}_i} - \bar{\bar{u}}_i \bar{\bar{u}}_j$. Similarity models underpredict the net SGS dissipation (Bardina, Ferziger & Reynolds 1983, Liu, Meneveau & Katz 1994, and Scotti & Meneveau 1998) and cannot be used to predict first order statistics of turbulent flows in actual LES. Mixed models (Bardina, Ferziger & Reynolds 1983, Zang, Street & Koseff 1993, and Sarghini & Piomelli 1998) combine the dissipative property of eddy viscosity models and the good representation of the SGS stress by a scale similarity expression.

In both eddy viscosity and similarity approaches, the SGS stress is parameterized in terms of the resolved scales. Recently, Domaradzki and collaborators (Domaradzki & Saiki 1997, and Domaradzki & Loh 1998) proposed a subgrid-scale estimation procedure. The aim of this model is to estimate from the resolved scales a range of smaller (SGS) scales of length $\Delta_f/2$. Schematically, the procedure consists of recovering the large scales by applying a deconvolution operation to the resolved (filtered) scales. Then, smaller scales are generated by nonlinear interactions among large scales. The estimated field, containing scales of length up to $\Delta_f/2$, is then used to evaluate the SGS stress tensor. Hence, the estimation model provides an approximation of the full velocity field as $u_i \simeq \bar{u}_i + \tilde{u}_i$, so that the spectral support of $\bar{u}_i + \tilde{u}_i$ is about two times larger than the support of the resolved scales. The increment $\tilde{u}_i$ is expressed in terms of the resolved scales,

$$\tilde{u}_i \;=\; \phi(\bar{u}_i).$$

Note that a similar point of view has been developed by Foias, Manley & Temam (1988) in the context of dynamical system approach of the Navier-Stokes equations.

The estimation models have been motivated by *a priori* analyses of energy transfer among different band of scales of DNS or experimental data (Kerr, Domaradzki & Barbier 1996, and Liu, Meneveau & Katz 1994). They have shown that energy transfers among resolved and SGS scales are dominated by local interactions, i.e. with modes within one octave of the cut-off wave number.

In the estimation procedure, the incremental components $\tilde{u}_i$ are generated by one nonlinear interaction. However, as pointed out in Domaradzki & Loh (1998), in turbulence the generation of small scales is much more complicated, involving nonlinear, viscous, and incompressibility effects among at least one eddy-turnover time. In this report, based on previous works (Dubois, Jauberteau & Temam (1998) and the references therein), we attempt to derive a more detailed procedure. The incremental unknowns (IU) $\tilde{u}_i$ are obtained by solving an approximated version of the SGS governing equations. Computing $\tilde{u}_i$ at each time iteration of the LES in such a way will require too much computational effort and will give results qualitatively similar to a coarse DNS on the grid $\Delta_f/2$. The increment components are evaluated every q ($q > 1$) iterations of the LES and are, therefore, frozen over time intervals of length $q\Delta t$. Then, the SGS stress dependent upon $\tilde{u}_i$ should be corrected during this period of time in order to preserve its dissipative properties. Two different corrections procedure are proposed leading to purely dissipative models. The IU models are implemented and evaluated for LES of incompressible forced and decaying homogeneous turbulent flows. The results are compared with filtered DNS data and with results obtained with dynamic eddy viscosity models.

2. Mathematical formulation

2.1 The large eddy simulation equations

In large eddy simulation (LES) of turbulent flows, the large and small scales are separated by applying a filter operation to the Navier-Stokes equations. For any flow variable $\boldsymbol{\psi}$, we define its resolved part as

$$\bar{\boldsymbol{\psi}}(\boldsymbol{x}) = \int_\Omega \boldsymbol{\psi}(\boldsymbol{x}')G(\boldsymbol{x},\boldsymbol{x}')\, d\boldsymbol{x}', \tag{1}$$

where Ω is the entire domain filled by the fluid and G is the filter (kernel) function. The net effect of the filtering operation (1) is to damp (or remove) the fluctuations with a characteristic length shorter than the filter width Δ_f.

By applying the filtering operation (Eq. 1) to the Navier-Stokes equations, we obtain the LES equations for incompressible flows

$$\begin{aligned} \frac{\partial \bar{u}_i}{\partial t} - \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\left(\bar{u}_i \bar{u}_j + \bar{p}\delta_{ij}\right) &= -\frac{\partial \tau_{ij}}{\partial x_j}, \\ \frac{\partial \bar{u}_i}{\partial x_i} &= 0, \end{aligned} \tag{2}$$

where ν is the kinematic viscosity and p is the pressure. The subgrid-scale (SGS) stress

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j, \tag{3}$$

represents the effect of the small scales on the resolved ones. This term must be modeled in terms of the resolved quantities in order to close the equations of motion (2).

The filtering operation (Eq. 1) induces a decomposition of the velocity field into large and small-scale components

$$\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{u}'(\boldsymbol{x}), \tag{4}$$

where $\boldsymbol{u}'(\boldsymbol{x})$ is the SGS velocity. By considering a filtering operation at scales $\Delta_f/2$, denoted by an overhat, the SGS velocity $\boldsymbol{u}'$ can be decomposed into

$$\boldsymbol{u}'(\boldsymbol{x}) = \tilde{\boldsymbol{u}}(\boldsymbol{x}) + \boldsymbol{u}''(\boldsymbol{x}),$$

where $\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \hat{\boldsymbol{u}}(\boldsymbol{x}) - \bar{\boldsymbol{u}}(\boldsymbol{x})$. Obviously, the filtered velocity field $\hat{\boldsymbol{u}}(\boldsymbol{x})$ satisfies an equation similar to Eq. (2). Therefore, the equation of motion for the velocity component $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ can be easily shown to be

$$\begin{aligned} \frac{\partial \tilde{u}_i}{\partial t} - \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\left(\widetilde{\hat{u}_i \hat{u}_j} + \tilde{p}\delta_{ij}\right) &= -\frac{\partial \tilde{\mathcal{T}}_{ij}}{\partial x_j}, \\ \frac{\partial \tilde{u}_i}{\partial x_i} &= 0, \end{aligned} \tag{5}$$

where $\mathcal{T}_{ij} = u_i u_j - \hat{u}_i \hat{u}_j$.

In Domaradzki *et al.* (1993), analysis of DNS data at low Reynolds number has shown that most of the energy transfers from large to small scales are dominated by local interactions, i.e. interactions of the resolved scales with wave numbers $k \in [k_f, 2k_f]$, with k_f being the cut-off wave number. A similar behavior has been noted by Liu, Meneveau & Katz (1994) by analyzing interactions among several separated bands of the fluctuating velocity. Based on these observations, Domaradzki and Saiki (1994) proposed to approximate the SGS stress tensor as follows:

$$\tau_{ij} \simeq \overline{\hat{u}_i \hat{u}_j} \; - \; \bar{\hat{u}}_i \bar{\hat{u}}_j. \tag{6}$$

With such an expression for the SGS tensor, the closure problem for Eq. (2) now consists in deriving an approximation of the incremental unknowns $\tilde{u}_i$. In Domaradzki and Saiki (1994), an SGS estimation procedure is proposed. Schematically, the filtered velocity $\hat{u}_i$ is determined solely in terms of the resolved velocity at larger scales $\bar{u}_i$. This is achieved in two steps. The first (kinematic) step consists in a deconvolution of $\bar{u}_i$. The second (dynamic) step generates scales of size two times smaller by nonlinear effects. Only this second step uses information from the Navier-Stokes equations. The unfiltered velocity obtained with this approach satisfies neither the incompressibility constraint nor the equation of motions (2 and 5). The procedure described in this report proposes a different approach. Our aim is to estimate the velocity increments $\tilde{u}_i$ as solutions of an approximation of the equation of motion (Eq. 5).

2.2 A multilevel scheme as a subgrid-scale estimation procedure

The aim of the proposed model is not to accurately evaluate the increment $\tilde{u}_i$, but to estimate a range of scales smaller than the resolved ones in order to obtain an approximation of the SGS stress tensor. Hence, as a first approximation, in Eq. (5) we neglect the nonlinear interactions with smaller scales, setting $\mathcal{T}_{ij} \equiv 0$. We then rewrite Eq. (5) as follows:

$$\begin{aligned} \frac{\partial \tilde{u}_i}{\partial t} - \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\left(\widetilde{\bar{u}_i \bar{u}_j} + \tilde{p}\delta_{ij}\right) &= -\frac{\partial \tilde{T}_{ij}}{\partial x_j}, \\ \frac{\partial \tilde{u}_i}{\partial x_i} &= 0, \end{aligned} \tag{7}$$

where $T_{ij} = \bar{u}_i\tilde{u}_j + \tilde{u}_i\bar{u}_j + \tilde{u}_i\tilde{u}_j$ represents the interactions among resolved and incremental scales. As previously mentioned, the computation at each time iteration of $\tilde{u}_i$ solving a time discretized version of Eq. (7) will require too much computational effort. Moreover, such results will be qualitatively similar to a coarse DNS on grid of mesh size $\Delta_f/2$. In Dubois *et al.* (1998a, 1998b), multilevel schemes have been developed and used to estimate the small scales of homogeneous turbulent flows. They have been applied in the context of DNS to scales with wave number $k \geq k_\eta/4$, with k_η the Kolmogorov wave number. The statistical properties, such as high-order moments of the velocity derivatives, are well reproduced by these procedures. They are based on a quasi-static (QS) approximation of the small scales, i.e. they are frozen over a few time iterations while the large scales are time advanced. Similarly, we apply a QS approximation to the velocity increments, i.e. $\tilde{u}_i$ are kept constant during q iterations. Therefore, we obtain a two-level scheme that can be summarized in the following two steps:

Step $nq + k,\ k \in [1, q]$**:** the resolved scale equation (2) is advanced according to the following time semi-discretized equations

$$
\begin{aligned}
\frac{\bar{u}_i^{nq+k} - \bar{u}_i^{nq+k-1}}{\Delta t} - \nu\frac{\partial^2 \bar{u}_i^{nq+k}}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i}\bar{p}^{nq+k} & \\
= -\frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j\right)^{nq+k-1} - \frac{\partial}{\partial x_j}\tau_{ij}^{nq+k-1}, & \qquad (8) \\
\frac{\partial}{\partial x_i}\bar{u}_i^{nq+k} = 0, &
\end{aligned}
$$

where we have set

$$
\tau_{ij}^{nq+l} = \overline{\bar{u}_i^{nq+l}\tilde{u}_j^m} + \overline{\tilde{u}_i^m\bar{u}_j^{nq+l}} + \overline{\tilde{u}_i^m\tilde{u}_j^m},\ m = nq,\ l \in [0, q-1].
$$

Step $m+1 = (n+1)q$**:** the incremental unknowns $\tilde{u}_i^{m+1}$ are computed according to

$$
\begin{aligned}
\frac{\tilde{u}_i^{m+1} - \tilde{u}_i^m}{\Delta t} - \nu\frac{\partial^2 \tilde{u}_i^{m+1}}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i}\tilde{p}^{m+1} &= -\frac{\partial}{\partial x_j}\widetilde{\bar{u}_i\bar{u}_j}^{\,m+1} - \frac{\partial T_{ij}^m}{\partial x_j}, \\
\frac{\partial}{\partial x_i}\tilde{u}_i^{m+1} &= 0,
\end{aligned}
\qquad (9)
$$

where

$$
T_{ij}^m = \widetilde{\bar{u}_i^{m+1}\tilde{u}_j^m} + \widetilde{\tilde{u}_i^m\bar{u}_j^{m+1}} + \widetilde{\tilde{u}_i^m\tilde{u}_j^m}.
$$

Note that the SGS stress $\mathcal{T}_{ij}$, representing the interactions between the velocity component $\hat{u}_i$ and u_i'' in Eq. (5), has been neglected in Eq. (9). The stress tensor T_{ij} corresponds to interactions between the resolved scales and the incremental unknowns $\tilde{u}_i$. Due to Eq. (9) the nonlinear term $\widetilde{\bar{u}_i\bar{u}_j}$ as well as the dissipative one act on the IU components over one time step only during the time interval $[m\Delta t, (m+1)\Delta t]$. The nonlinear term is essentially an energy transfer term, and the balance between dissipation and injection of energy present in the Navier-Stokes equations cannot be reproduced by Eq. (9). In fact, the net effect of this numerical treatment can be either to reduce or amplify artificially the kinetic energy of the IU scales $\tilde{u}_i$. Therefore, a discontinuity will appear near the cut-off wave number k_f on the distribution of kinetic energy in spectral space. In order to avoid such numerical artifact, the IU scales $\tilde{u}_i^m$ are modified so that the kinetic energy of their largest scales in a band of wave numbers close to k_f is equal to the kinetic energy contained in the smallest scales of $\bar{u}_i^m$. In the spectral space, the Fourier coefficients of $\tilde{u}_i^m$ are modified so that:

$$E_{\tilde{\boldsymbol{u}}^m}(k_f + j) \equiv E_{\tilde{\boldsymbol{u}}^m}(k_f + j)\frac{E_{\bar{\boldsymbol{u}}^m}(k_f)}{E_{\tilde{\boldsymbol{u}}^m}(k_f + 1)}, \tag{10}$$

for $j = 1, \dots, k_f$; $E_{\boldsymbol{\phi}}(k)$ denotes the energy spectrum function of the flow field $\boldsymbol{\phi}(\boldsymbol{x})$. A relation similar to Eq. (10) could be imposed in physical space by using filter functions with local support near k_f in the spectral space.

The above procedure of estimation of the IU components, consisting of Eqs. (9) and (10), does not insure *a priori* that the stress tensor τ_{ij} predicts the right amount of dissipation. In fact, the implementation of this scheme in LES of forced homogeneous turbulence has shown that the stress tensor underpredicts the SGS dissipation in the neighborhood of the cut-off wave number k_f. A more detailed analysis of the behavior of τ_{ij} revealed that the computation of the IU scales via Eqs. (9) and (10) at iteration m provides a stress tensor behaving similarly to the exact one (computed from DNS data) and predicting reasonably well the SGS dissipation. However, a decorrelation between the stress tensor τ_{ij}^{n+l-1} and the resolved scales $\bar{u}_i^{n+l}$ appears for $l > 1$. The SGS dissipation near k_f reduces from iteration to iteration, resulting in an increase of the kinetic energy of the smallest scales of $\bar{u}_i$. As a remedy we define hereafter two procedures to insure that the stress tensor has a dissipative effect on the resolved scales equations. In both cases, the spectral representation of the velocity field is used. Let us denote by $\hat{\boldsymbol{\phi}}(\boldsymbol{k})$ the Fourier coefficients of the flow variable $\boldsymbol{\phi}(\boldsymbol{x})$. The SGS force in the spectral form of Eq. (8) reads

$$\widehat{NL}_l(\boldsymbol{k}) \equiv ik_j\hat{\tau}_{lj}(\boldsymbol{k}),$$

so that in the energy equation we have $\widehat{NL}_i(\boldsymbol{k})\cdot\hat{\bar{u}}_i(\boldsymbol{k})$ as a source term. Once the IU components have been computed according to Eq. (9), we define a complex number of modulus equal to unity by

$$e^{i\theta^m(\boldsymbol{k})} = \frac{\widehat{NL}_i^m(\boldsymbol{k})\cdot\hat{\bar{u}}_i^m(-\boldsymbol{k})}{|\,\widehat{NL}_i^m(\boldsymbol{k})\cdot\hat{\bar{u}}_i^m(-\boldsymbol{k})\,|}.$$

The phase $\theta^m(\boldsymbol{k}) \in [-\pi, \pi]$ clearly represents the phase difference between $\widehat{NL}_i^m(\boldsymbol{k})$ and $\hat{\bar{u}}_i^m(\boldsymbol{k})$. The time decorrelation between the SGS force and the resolved scales can be avoided by keeping constant the phase $\theta^m(\boldsymbol{k})$ on the time interval $[m\Delta t, (m+1)\Delta t]$. However, such a procedure does not bring enough dissipation into the resolved scale equation.

The first procedure consists in modifying $\theta^m(\boldsymbol{k})$ as follows:

$$\tilde{\theta}^m(\boldsymbol{k}) \equiv \frac{\pi}{2} \quad \text{if} \quad \widehat{NL}_i^m(\boldsymbol{k}) \cdot \hat{\bar{u}}_i^m(-\boldsymbol{k}) \; < \; 0. \tag{11}$$

The net effect of Eq. (11) is to "turn off" the stress for wave numbers contributing to backscatter of energy. The phases of the SGS force in Eq. (8) are kept constant and set equal to the phase $\tilde{\theta}^m(\boldsymbol{k})$. Then, the SGS force is modified according to:

$$\widehat{NL}_i^{n+l}(\boldsymbol{k}) \equiv |\, \widehat{NL}_i^{n+l}(\boldsymbol{k})\,| \, \frac{\hat{\bar{u}}_i^{n+l}(\boldsymbol{k})}{|\, \hat{\bar{u}}_i^{n+l}(\boldsymbol{k})\,|} e^{i\tilde{\theta}^m(\boldsymbol{k})}, \tag{12}$$

for $l = 1, \ldots, q$. The model consisting of Eqs. (8-11) is, therefore, a dissipative SGS model and is denoted by IU_1 in the following section. Note that Liu, Meneveau & Katz (1994) proposed to modify in a similar way the stress tensor derived from scale similarity models.

Another version, denoted hereafter by IU_2, consists in modifying the phases of the SGS force in Eq. (8) at the temporal iteration $n + l$, according to:

$$\widehat{NL}_i^{n+l}(\boldsymbol{k}) \; \equiv |\, \widehat{NL}_i^{n+l}(\boldsymbol{k})\,| \, \frac{\hat{\bar{u}}_i^{n+l}(\boldsymbol{k})}{|\, \hat{\bar{u}}_i^{n+l}(\boldsymbol{k})\,|} e^{i\alpha\theta^m(\boldsymbol{k})}, \tag{13}$$

where α is a constant in the range $[0, 1]$. Hence, a dissipative LES model is obtained for values of $\alpha \leq 1/2$. For $\alpha = 0$, the dissipation induced by Eq. (13) is maximum. The use of this value in LES runs has shown that the smallest scales of the resolved field are excessively damped by an overprediction of the SGS dissipation. We have retained the value $\alpha = 1/2$ in the LES runs described in the following section.

3. Numerical implementation

3.1 Large eddy simulation of forced isotropic turbulence

The flow is forced in such a way that the energy injection rate $\langle \boldsymbol{f} \cdot \boldsymbol{u} \rangle$ is constant in time and equal to a given parameter ε. The Reynolds number is taken to be infinite so that $\nu = 0$. In the absence of a model for the SGS stress term, the energy spectrum function tends to reach an k^2 equipartition equilibrium and the total kinetic energy will grow constantly. In an idealistic situation, the model should provide the correct amount of net SGS dissipation at each scale so that the flow reaches a statistically steady state, with an energy spectrum of the form: $E(k) = C_K \; \varepsilon^{2/3} \; k^{-5/3}$, C_K being the Kolmogorov constant. The initial condition has an energy spectrum function of this form with C_K set to 1.6; the phases of the Fourier coefficients are randomly generated. The spectral coefficients $\hat{\boldsymbol{f}}(\boldsymbol{k})$ are nonzero only for wave numbers $|\boldsymbol{k}| \leq k_0$. Hence, by defining $N = \mathrm{Card}\{\boldsymbol{k} \in \mathcal{Z}^3; |\, \boldsymbol{k} \,| \leq k_0\}$ and $\theta(\boldsymbol{k})$ to be the phase of $\hat{\boldsymbol{u}}(\boldsymbol{k})$, $\hat{\boldsymbol{f}}(\boldsymbol{k})$ is given by

$$\hat{\boldsymbol{f}}(\boldsymbol{k}) \; = \; \frac{\varepsilon}{N} \, \frac{e^{i\theta(\boldsymbol{k})}}{|\, \hat{\boldsymbol{u}}(\boldsymbol{k})\,|}.$$

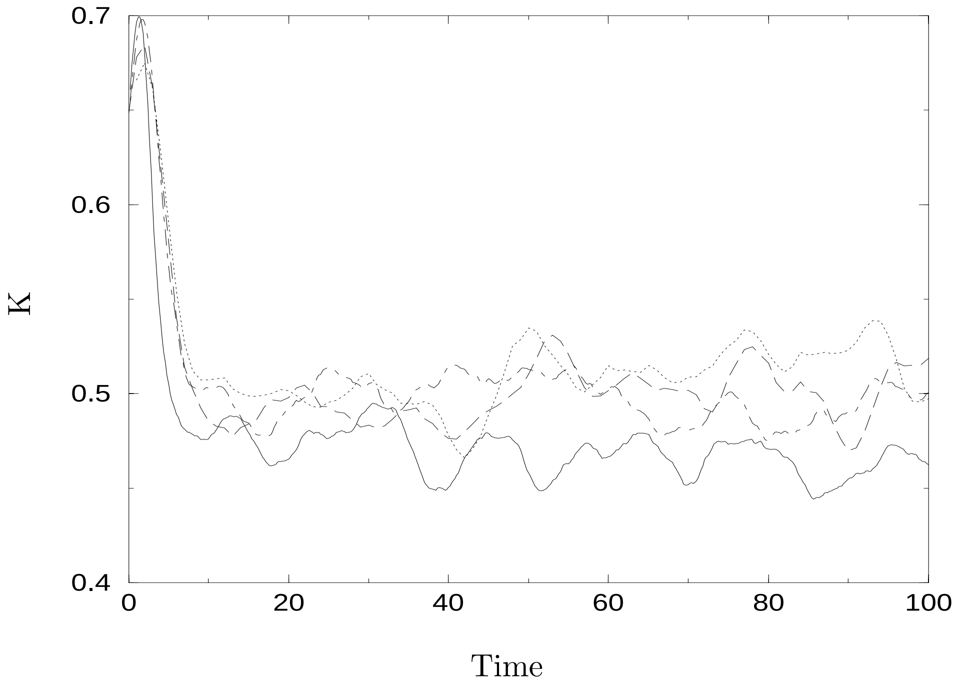

FIGURE 1. Time evolution of the kinetic energy $K = \frac{1}{2}\langle \bar{u}_i \bar{u}_i \rangle$. ——— : IU_1; –·–·–·– : IU_2; – – – – – : DM_1; ········ : DM_2.

In the simulations described hereafter, we have chosen $\varepsilon = 0.1$ and $k_0 = 2$. The same problem has been studied in Ghosal *et al.* (1995).

In order to compare results obtained with the IU models described in the previous section, runs with the same parameters and initial conditions have been performed with the dynamic model. Two versions are considered: DM_1 corresponding to the original form derived in Germano *et al.* (1991) and DM_2 the modified version of Lilly (1992). The LES runs are performed here on a 32^3 grid; the computation of the nonlinear terms are dealiased with the $3/2-$rule. The parameter q defining the frequency for the estimation of the IU scales was chosen equal to 5 for the simulations presented in this report. It was found that the results were weakly dependent of the value of q for $q \in [5, 20]$; larger values of q have not been tested.

The net SGS dissipation, well predicted by eddy-viscosity models, is known to be a quantity difficult to estimate accurately for other models, as scale similarity ones, in actual LES. For the considered problem, the prediction of

$$\epsilon_{\mathrm{SGS}} = \left\langle \tau_{ij} \bar{S}_{ij} \right\rangle ,$$

where $\langle - \rangle$ denotes volume average, determines the stability of the system. Indeed, if ϵ_{SGS} is underpredicted, the system will have a tendency to accumulate kinetic energy injected by the external force as no other dissipation than the SGS one is present in the equations of motion of the resolved scales. The resolved scales kinetic energy $K = 1/2 \langle \bar{u}_i \bar{u}_i \rangle$, represented as functions of time in Fig. 1, oscillates near a value in the range $[0.45, 0.5]$ for the IU and DM solutions. Therefore, the IU

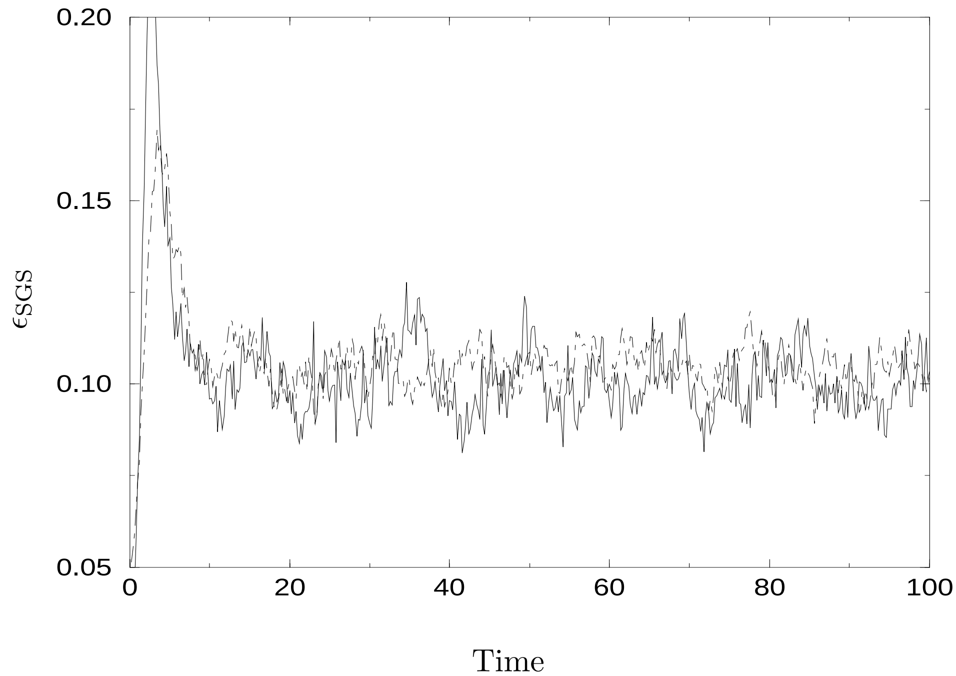

FIGURE 2. Time evolution of the SGS dissipation rate $\epsilon_{\mathrm{SGS}} = \langle \tau_{ij}\bar{S}_{ij} \rangle$. ——— : IU_1; - - - - - : IU_2.

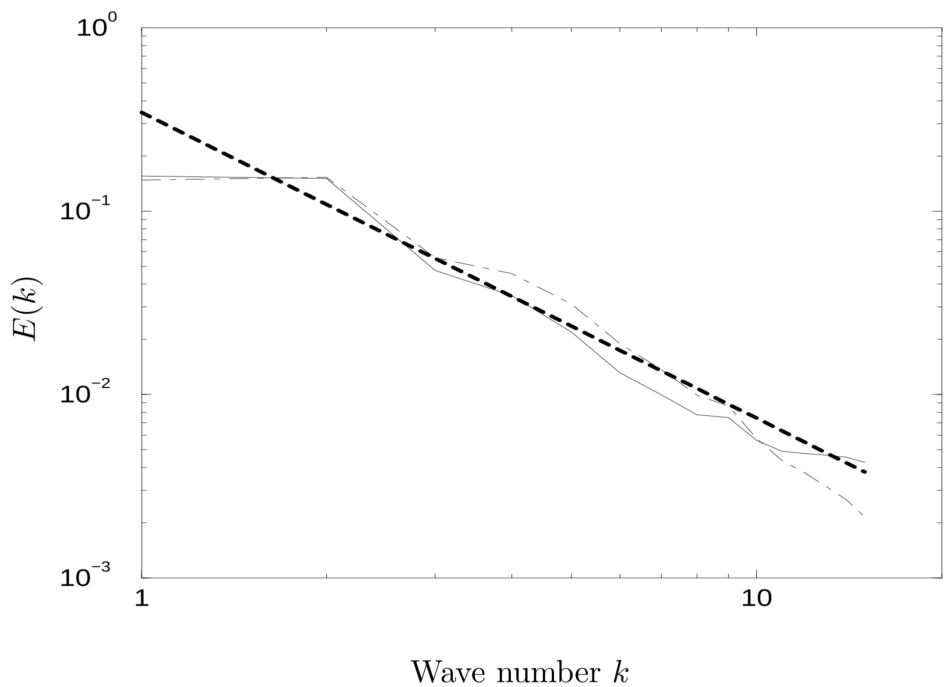

FIGURE 3. Time averaged energy spectrum functions $E(k)$. **- - - - - -** : $C_K\varepsilon^{2/3}k^{-5/3}$; ——— : IU_1; -·-·-·- : IU_2.

models are stable and predict the right amount of global SGS dissipation. This is confirmed by Fig. 2 showing that $\epsilon_{\rm SGS}$ obtained by the IU runs oscillate near the energy injection rate, i.e. $\varepsilon = 0.1$; $\rm IU_1$ and $\rm IU_2$ models, after the transient period, predict the same amount of global dissipation.

A requirement for LES models is the prediction of the SGS dissipation. Moreover, the spectral distribution of this quantity is of importance for an accurate representation of the distribution of kinetic energy in the spectral space, i.e. the energy spectrum function. For the considered problem, the models should ideally predict a $-5/3$ inertial range. Fig. 3 shows that $\rm IU_1$ model recovers a spectrum close to the Kolmogorov's 5/3 law; the $\rm IU_2$ seems to give a steeper decaying spectrum $E(k) \simeq k^{-2}$, very close to the $\rm DM_1$ spectrum. Compensated spectra $k^{5/3}\varepsilon^{-2/3}E(k)$, represented in Fig. 5, show a plateau for the $\rm IU_1$ and $\rm DM_2$ models with Kolmogorov constant $C_K \simeq 1.4$ for $\rm IU_1$ and $C_K \simeq 1.9$ for $\rm DM_2$. Values of this constant obtained by measurement of experiments are in the range $[1.3, 2.1]$ (Chasnov (1991)).

The kinetic energy contained in the IU component $K_{\rm SGS} = 1/2\,\langle \tilde{u}_i \tilde{u}_i \rangle$ is of the order of 0.125 for both IU models. A similar value has been found by Carati, Ghosal & Moin (1995) with a dynamic model carrying an equation for the SGS kinetic energy. Assuming a Kolmogorov law beyond the cut-off wave number k_f, we deduce that

$$K_{\rm SGS} = \frac{3}{2} C_K \left(\frac{\varepsilon}{k_f} \right)^{2/3}.$$

For C_K in the range $[1.4, 1.9]$, we obtain $K_{\rm SGS}$ in the range $[0.07, 0.1]$. This tends to show that the QS approximation is an efficient way to estimate the incremental scales.

3.2 Large eddy simulation of decaying turbulence

The flow is an incompressible time-decaying flow and is an analog of the grid-turbulence experiments of Comte-Bellot and Corrsin (1971). The reference test is the 512^3 DNS performed by Wray (1998). The initial condition for the LES runs is the 512^3 DNS velocity field at time $t \simeq 0.97$ filtered on a 32^3 grid. The Reynolds number based on the Taylor microscale is of the order of 100. As indicated by Fig. 6 representing energy spectra of DNS at various resolutions, a coarse DNS is unable to follow the decay of the energy spectrum function and, therefore, the decay of the total kinetic energy. Actual LES runs on a 32^3 grid corresponding to the resolved scales have been conducted with the $\rm IU_1$, $\rm IU_2$, $\rm DM_1$ and $\rm DM_2$ models.

The difficulty of this test is to accurately recover the decay of the kinetic energy $K = 1/2\,\langle \bar{u}_i \bar{u}_i \rangle$. This again depends on the capacity of the model to predict the net SGS dissipation. Moreover, the model should adjust this prediction as time evolves, as the SGS dissipation decays drastically in the first period of the run, $t \in [1, 2.5]$ (see Fig. 7). In this first period, $\rm DM_1$ and $\rm IU_2$ are very close to each other while the two other runs seem to overestimate K. In the second period of the decay, $t > 2.5$, the IU solutions are almost identical and accurately follow the curve corresponding to the filtered DNS. The DM models overpredict the kinetic energy by approximately 20% for $t > 2.5$.

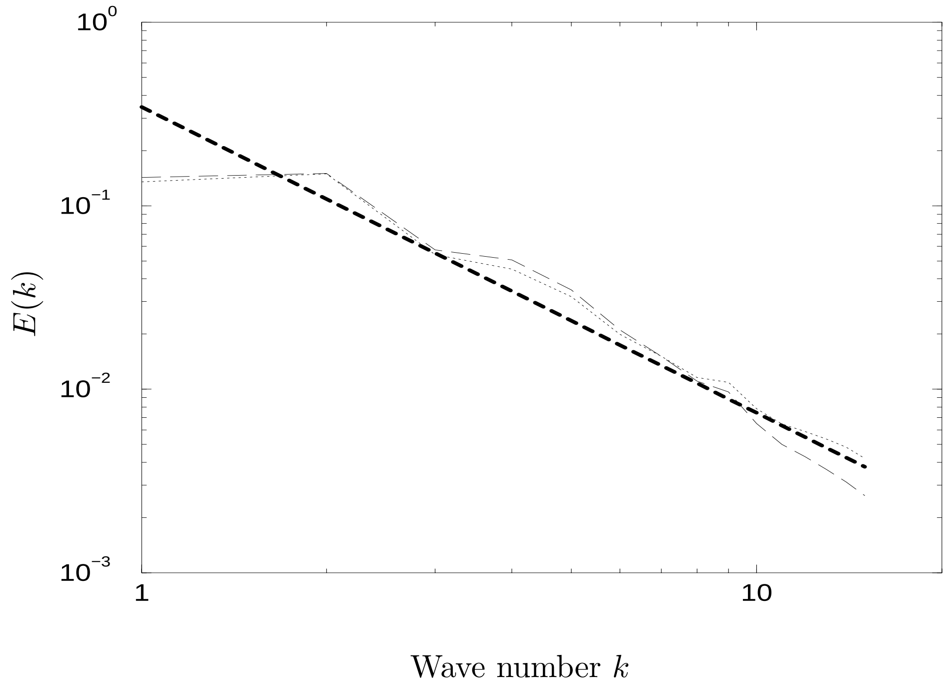

FIGURE 4. Time averaged energy spectrum functions $E(k)$. ▬ ▬ ▬ : $C_K \varepsilon^{2/3} k^{-5/3}$; – – – – – : DM_1; ········ : DM_2.

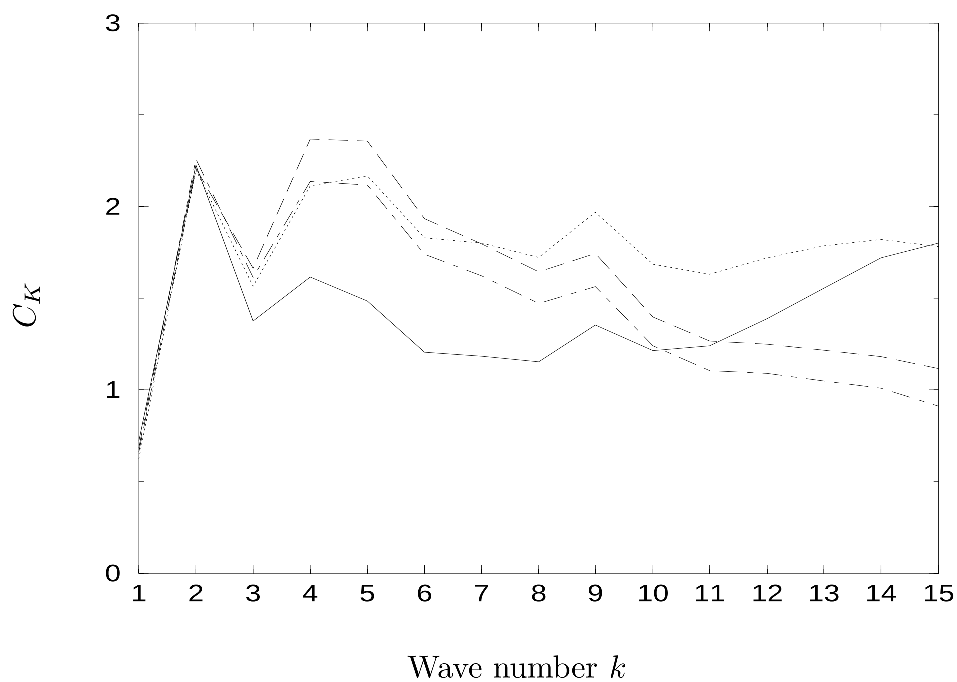

FIGURE 5. Prediction of the Kolmogorov constant $C_K = k^{5/3}\varepsilon^{-2/3}E(k)$. ——— : IU_1; –·–·–·– : IU_2; – – – – – : DM_1; ········ : DM_2.

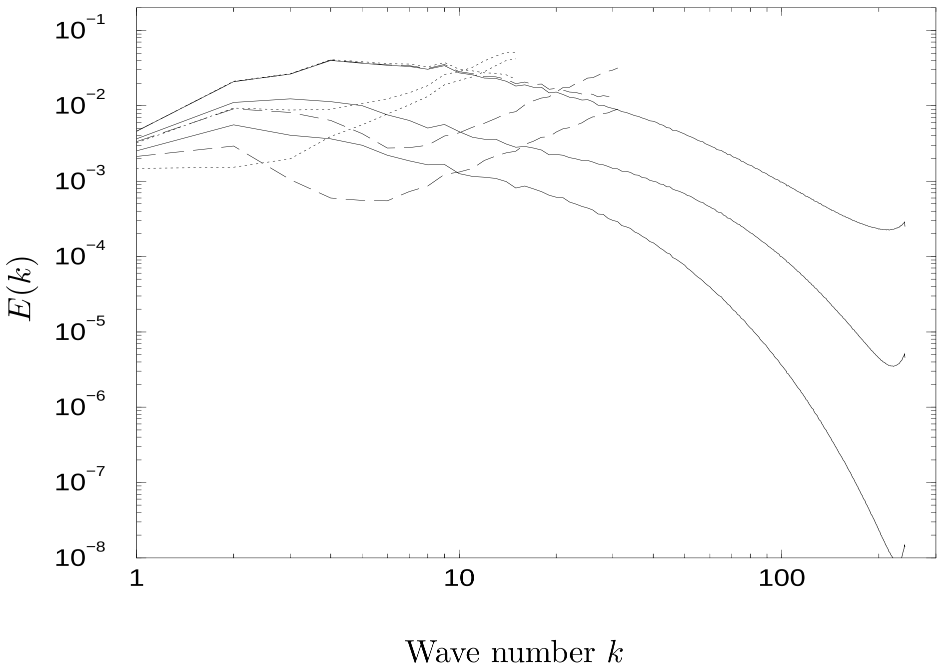

FIGURE 6. Energy spectrum functions at $t = 1.28,\ 2.45$ and $t = 5.47$. ——— : 256^3 DNS; ········ : 32^3 DNS; ------ 64^3 DNS.

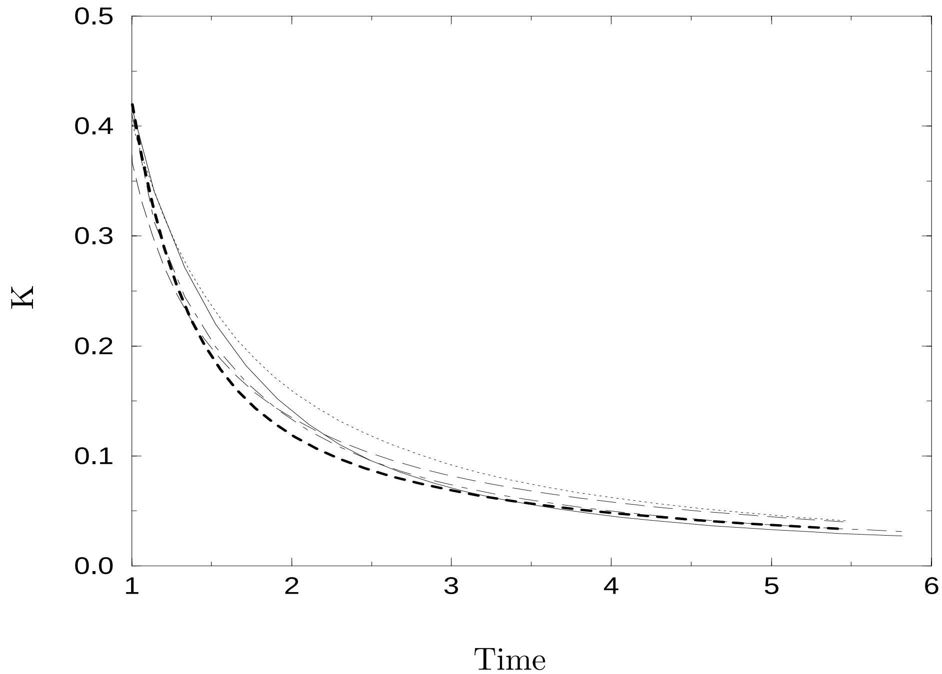

FIGURE 7. Decay of the kinetic energy $K = \frac{1}{2}\langle \bar{u}_i \bar{u}_i \rangle$. **– – – – –** : filtered DNS; ——— : IU_1; –·–·–·– : IU_2; – – – – – : DM_1; ········ : DM_2.

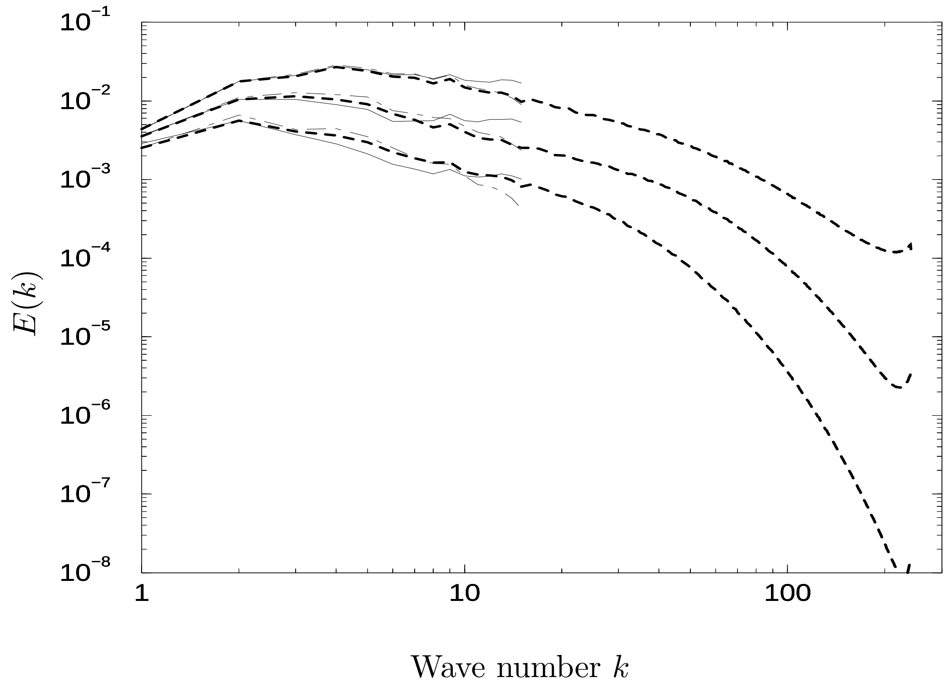

FIGURE 8. Energy spectrum functions at $t = 1.28,\ 2.45$ and $t = 5.47$. **- - - - - -** : filtered DNS; ——— : IU_1; -·-·-·- : IU_2.

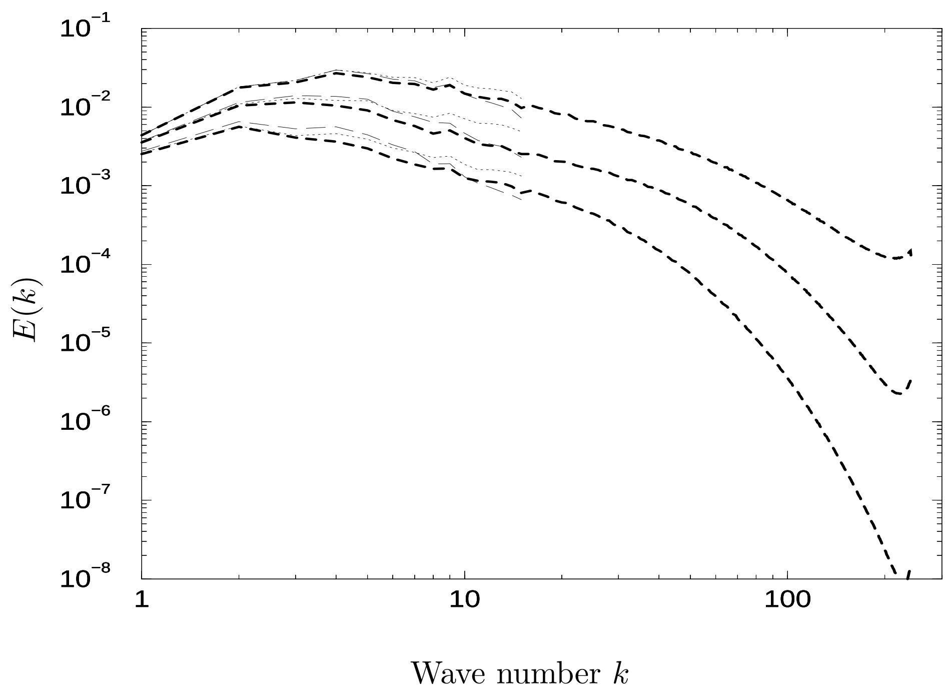

FIGURE 9. Energy spectrum functions at $t = 1.28,\ 2.45$ and $t = 5.47$. **- - - - - -** : filtered DNS; - - - - - : DM_1; ········ : DM_2.

The best fit of the DNS energy spectrum is obtained with the IU_2 model on the overall time of decay of the simulation as shown on Fig. 8. The IU_1 model has a tendency to slightly overestimate the last modes near the cut-off wave number at the intermediate time $t = 2.45$. The energy spectrum obtained with the DM models (Fig. 9) have a shape similar to the DNS spectrum but with slightly larger values. This test demonstrates the good dynamic property of the IU models proposed in Section 2.

4. Conclusion

Subgrid-scale modeling based on the concept of incremental unknowns (IU) has been introduced. The incremental velocity components correspond to scales beyond the cut-off wave number defining the resolved scales in LES. Schematically, the IU component have length scales two times smaller than the resolved ones. Therefore, IU modeling is similar to the estimation procedure recently introduced by Domaradzki and coworkers (1997, 1998). However, these approaches differ in the computation of the subgrid scales. The equations of motion are used to advance in time the IU components. The computation of IU components at each time iteration would very poorly resolve the SGS scales as in a coarse DNS. Rather, a quasi-static (QS) approximation is applied to the IU scales, i.e. they are not evaluated at each time iteration of the LES. With the QS approximation we attempt to mimic the dynamic of the subgrid scales. The aim is to develop a more detailed procedure than the estimation one, which generates small scales by only one nonlinear interaction. The QS approximation has been shown to generate small scales, providing SGS stresses with dissipative properties as expected. However, on the time period during which the IU scales are frozen, a decorrelation between IU and resolved scales occurs. This time decorrelation induces an underprediction of the net SGS dissipation. Therefore, the QS approximation cannot be used itself for SGS modeling, but it must be coupled with correction procedures of either the SGS stress tensor or the IU components. We have focused here on the development of phase correction procedures for the SGS tensor. Two procedures have been proposed and implemented in actual LES of forced and decaying homogeneous turbulence. The IU models obtained are fully dissipative ones. The LES runs have shown that both IU models provide the right amount of SGS dissipation. In the case of decaying turbulence, the dynamic property of the flow is well reproduced by the IU solutions, i.e. the decay of the kinetic energy of the resolved scales follows accurately the decay of the filtered DNS. In this test case, IU models perform better than the dynamic model. In the case of forced turbulence at infinite Reynolds number, IU models are able to predict an energy spectrum close to a $k^{-5/3}$ Kolmogorov inertial range. Therefore, we have proved that the IU approach can be used to derive efficient SGS model for LES of turbulent flows. However, the aim of these approaches, based on the evaluation of a range of subgrid scales, is to develop fully nonlinear models accurately representing the SGS stress tensor and its properties, i.e. dissipation and backscatter of energy. At this point, correction procedures have been used to insure such behavior of the SGS stresses. In the future, we should, therefore, concentrate our efforts on the development of models which are able to adjust the IU components instead of the SGS force. Moreover, IU modeling has been tested here on homogeneous turbulence. It should be applied to wall-bounded flows and even more complicated ones requiring the extension of the approach presented here to implementation in finite difference codes.

REFERENCES

Bardina, J. & Ferziger, J. H. & Reynolds, W. C. 1983 Improved turbulence models based on large eddy simulation of homogeneous incompressible turbulence. *Report TF-19.* Stanford University.

Carati, D. & Ghosal, S. & Moin, P. 1995 On the representation of backscatter in dynamic localization models. *Phys. Fluids.* **7,** 107.

Chasnov, J. R. 1991 Simulation of the Kolmogorov inertial subrange using an improved subgrid-scale model. *Phys. Fluids A.* **3,** 188-200.

Clark, R. A. & Ferziger, J. H. & Reynolds, W. C. 1979 Evaluation of subgrid-scale models using an accurately simulated turbulent flow. *J. Fluid Mech.* **91,** 1.

Comte-Bellot, G. & Corrsin, S. 1971 Simple Eulerian time correlations of full and narrow-band velocity signals in grid-generated isotropic turbulence. *J. Fluid Mech.* **48,** 273-337.

Domaradzki, J. A. & Liu, W. & Brachet, M. E. 1993 An analysis of subgrid-scale interactions in numerically simulated isotropic turbulence. *Phys. Fluids A 5.* **1747.**

Domaradzki, J. A. & Saiki, E. M. 1997 A subgrid-scale model based on the estimation of unresolved scales of turbulence. *Phys. Fluids.* **9**(7)**,** 2148-2164.

Domaradzki, J. A. & Loh, K. 1998 The subgrid-scale estimation model in the physical space representation. To appear in *Phys. Fluids.*

Dubois, T. & Jauberteau, F. & Temam, R. 1998 Incremental unknowns, multilevel methods and the numerical simulation of turbulence. *Comput. Methods Appl. Mech. Engrg.* **159,** 123-189.

Dubois, T. & Jauberteau, F. 1998 A dynamic multilevel model for the simulation of the small structures in homogeneous isotropic turbulence. *J. Sci. Comput.* **13**(3)**,** 323-367.

Foias, C. & Manley, O. P. & Temam R. 1988 Modelling of the interaction of small and large eddies in two-dimensional turbulent flows. *Math. Mod. and Num. Anal. (M2AN).* **22**(1)**,** 93-114.

Germano, M. & Piomelli, U. & Moin, P. & Cabot, W. H. 1991 A dynamic subgrid-scale eddy viscosity model. *Phys. Fluids A.* **3**(7)**,** 1760-1765.

Ghosal, S. & Lund, T. S. & Moin, P. & Akselvoll, K. 1995 A dynamic localization model for large-eddy simulation of turbulent flows. *J. Fluid Mech.* **286,** 229-255.

Kerr, R. M. & Domaradzki, J. A. & Barbier, G. 1996 Small-scale properties of nonlinear interactions and subgrid-scale energy transfer in isotropic turbulence. *Phys. Fluids.* **8,** 197.

Lilly, D. K. 1967 Proc. IBM Scientific Computing Symposium on Environmental Sciences. Yorktown Heights, N.Y., 195.

Lilly, D. K. 1992 A proposed modification of the Germano subgrid-scale closure method. *Phys. Fluids A.* **4**(3)**,** 633-635.

Liu, S. & Meneveau, C. & Katz, J. 1994 On the properties of similarity subgrid-scale models as deduced form measurements in a turbulent jet. *J. Fluid Mech.* **275,** 83.

O'Neil, J. & Meneveau, C. 1997 Subgrid-scale stresses and their modeling in a turbulent wake. *J. Fluid Mech.* **349,** 253.

Sarghini, F. & Piomelli, U. 1998 Scale-similar models for large-eddy simulations. *AIAA 98-0538.*

Scotti, A. & Meneveau, C. 1998 A fractal model for large-eddy simulation of turbulent flow. To appear in *Physica D.*

Smagorinsky, J. 1963 General circulation experiments with the primitive equations. *Mon. Weath. Rev.* **93,** 99.

Wray, A. A. 1998 in A selection of test cases for the evaluation of large-eddy simulations of turbulent flows. *AGARD Advisory report.* **345,** 63.

Zang, Y. & Street, R. L. & Koseff, J. R. 1993 A dynamic mixed subgrid-scale model and its applications to turbulent recirculating flows. *Phys. Fluids A.* **5,** 318.