On the construction of high order finite difference schemes on non-uniform meshes with good conservation properties

By Oleg V. Vasilyev[1]

1. Motivation and objectives

Numerical simulation of turbulent flows (DNS or LES) requires numerical methods that can accurately represent a wide range of spatial scales. One way to achieve a desired accuracy is to use high order finite difference schemes. However, additional constraints such as discrete conservation of mass, momentum, and kinetic energy should be enforced if one wants to ensure that unsteady flow simulations are both stable and free of numerical dissipation. In addition, both pressure and velocity fields must be physical. These requirements are usually achieved by using a staggered grid and enforcing continuity.

Until recently the standard second order accurate staggered grid finite difference scheme of Harlow and Welch (1965) was the only scheme that simultaneously conserved mass, momentum, and kinetic energy. It was observed by Ghosal (1996) that the accuracy of second order finite difference scheme is low and fine meshes are required to achieve acceptable results. For that reason Morinishi *et al.* (1998) derive the general family of fully conservative higher order accurate finite difference schemes for uniform staggered grids. Both the scheme of Harlow and Welch (1965) and that of Morinishi *et al.* (1998) conserve mass, momentum, and kinetic energy on a uniform mesh. However, generalizing these schemes to non-uniform meshes and preserving the conservation properties is not straightforward. For example, the generalization of the fourth order accurate finite difference scheme, suggested in (Morinishi *et al.*, 1998), does not even conserve momentum. Furthermore, Morinishi *et al.* (1998) mistakenly concluded that in order to construct conservative schemes, one should choose between the accuracy and conservation. One of the reasons why the authors came to this conclusion may be the fact that they tried to generalize the scheme by changing the weights in the difference operators as a function of local grid spacings and preserving the order of local truncation error. As a consequence of this generalization, the resulting scheme does not preserve symmetries of the uniform mesh case. Veldman and Versappen (1998), in their analysis of convective-diffusion equation on non-uniform meshes, showed that in order for the scheme to be conservative, it should preserve symmetries of the underlying operator, *i.e.* the convective derivative should be approximated by skew-symmetric operator.

1 Present address: Department of Mechanical and Aerospace Engineering, University of Missouri, Columbia, MO 65211

This report is an attempt to generalize the high order schemes of Morinishi *et al.* (1998) to non-uniform meshes by preserving the symmetries of the uniform mesh case.

2. Accomplishments

2.1 Analytical requirements

In this section, we briefly outline the analytical requirements for conservation of mass, momentum, and energy for incompressible flow. For further details we refer the reader to (Morinishi *et al.*, 1998).

An equation of the form:

$$\frac{\partial \phi}{\partial t} + {}^1Q(\phi) + {}^2Q(\phi) + {}^3Q(\phi) + \cdots = 0, \tag{1}$$

is said to be written in conservative form if all the terms ${}^kQ(\phi)$ can be written in divergence form:

$${}^kQ(\phi) = \nabla \cdot ({}^k\mathbf{F}(\phi)) = \frac{\partial({}^kF_j(\phi))}{\partial x_j}. \tag{2}$$

In this report we use bold letters to denote a vector function, *e.g.* $\mathbf{F} = (F_1, F_2, F_3)^{\mathrm{T}}$. The requirement (2) follows from Gauss' divergence theorem. In particular, if we integrate Eq. (1) over a volume, we obtain:

$$\frac{\partial}{\partial t}\int\int\int_V \phi \, dV = -\int\int_S ({}^1\mathbf{F}(\phi) +{}^2 \mathbf{F}(\phi) +{}^3 \mathbf{F}(\phi) + \cdots) \cdot d\mathbf{S}. \tag{3}$$

From this equation it is easy to see that the integral never changes in the periodic case if ${}^kQ(\phi)$ has a conservative form for all k. Following this definition of conservation, it is easy to show that mass, pressure, and viscous terms are conserved *a priori* since these terms appear in divergence form. The convective term is also conservative *a priori* if it is written in divergence form, which is not always the case. There are four commonly used forms of the convective term. These forms are referred to as *divergence*, *advective*, *skew-symmetric*, and *rotational* forms and are defined as follows:

$$(Div.)_i \equiv \frac{\partial u_j u_i}{\partial x_j}, \tag{4a}$$

$$(Adv.)_i \equiv u_j \frac{\partial u_i}{\partial x_j}, \tag{4b}$$

$$(Skew.)_i \equiv \frac{1}{2}\frac{\partial u_j u_i}{\partial x_j} + \frac{1}{2} u_j \frac{\partial u_i}{\partial x_j}, \tag{4c}$$

$$(Rot.)_i \equiv u_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\frac{\partial u_j u_j}{\partial x_i}. \tag{4d}$$

The four forms are connected with each other through the following analytical relations:

$$(Adv.)_i = (Div.)_i - u_i \cdot (Cont.), \tag{5a}$$

$$(Skew.)_i = \frac{1}{2}(Div.)_i + \frac{1}{2}(Adv.)_i, \tag{5b}$$

$$(Rot.)_i = (Adv.)_i, \tag{5c}$$

where $(Cont.) \equiv \frac{\partial u_i}{\partial x_i}$. Note that the advective, skew-symmetric, and rotational forms are conservative as long as the continuity equation is satisfied.

The transport equation of the square of a velocity component, for instance, ${u_1}^2/2$, can be written as

$$\frac{\partial {u_1}^2/2}{\partial t} + u_1 \cdot (Conv.)_1 + u_1 \cdot (Pres.)_1 + u_1 \cdot (Visc.)_1 = 0, \tag{6}$$

where $(Conv.)_i$ is a generic form of the convective term, and $(Pres.)_i$ and $(Visc.)_i$ are the pressure and viscous terms respectively. The convective term in Eq. (6) can be written for each of the forms as

$$u_1 \cdot (Div.)_1 = \frac{\partial u_j {u_1}^2/2}{\partial x_j} + \frac{1}{2}{u_1}^2 \cdot (Cont.), \tag{7a}$$

$$u_1 \cdot (Adv.)_1 = \frac{\partial u_j {u_1}^2/2}{\partial x_j} - \frac{1}{2}{u_1}^2 \cdot (Cont.), \tag{7b}$$

$$u_1 \cdot (Skew.)_1 = \frac{\partial u_j {u_1}^2/2}{\partial x_j}. \tag{7c}$$

Note that the skew-symmetric form is conservative *a priori* in the velocity square equation. Since the rotational form is equivalent to the advective form, the four convective forms are energy conservative if the continuity equation is satisfied.

The transport equation of the kinetic energy, $K \equiv u_i u_i/2$ can be written as

$$\frac{\partial K}{\partial t} + u_i \cdot (Conv.)_i + u_i \cdot (Pres.)_i + u_i \cdot (Visc.)_i = 0. \tag{8}$$

The conservation property of the convective term can be determined in the same manner as for ${u_1}^2/2$. The terms involving pressure and viscous stress in Eq. (8) can be written as

$$u_i \cdot (Pres.)_i = \frac{\partial p u_i}{\partial x_i} - p \cdot (Cont.), \tag{9a}$$

$$u_i \cdot (Visc.)_i = \frac{\partial \tau_{ij} u_i}{\partial x_j} - \tau_{ij}\frac{\partial u_i}{\partial x_j}. \tag{9b}$$

The pressure term is conservative if the continuity equation is satisfied. The viscous term is not conservative because the second term on the right-hand side of Eq. (9b) is the kinetic energy dissipation.

Morinishi *et al.* (1998) derived a class of high order schemes for a uniform staggered grid which satisfy the conservation properties in a discrete sense. The objective of this work is to generalize the higher order schemes of Morinishi *et al.* (1998) to the non-uniform meshes while preserving discrete conservation as much as possible.

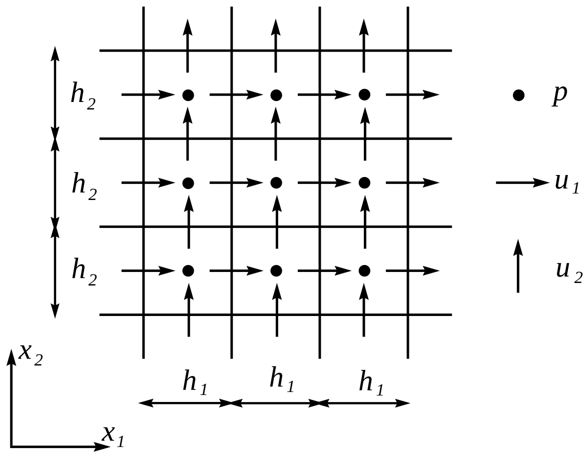

FIGURE 1. Staggered grid arrangement.

2.2 Discrete operators

In order to simplify the analysis, we limit our consideration to the rectangular algebraic non-uniform meshes with non-uniform grid spacing in each x_1, x_2, and x_3 direction. By algebraic grid we imply that the computational grid in physical domain is obtained by mapping a uniform computational grid in the computational domain to physical domain. Let $D = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ and $\Omega = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_3, \beta_3]$ be respectively the physical and computational domains, $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}}$ be coordinates in physical and computational domains, $\boldsymbol{\xi} = \mathbf{f}(\mathbf{x})$ be a nonlinear map of physical domain D into computational domain, and Δ_1, Δ_2, Δ_3 be uniform grid spacings in the respective directions in computational domain Ω. In this report we limit our consideration to the case when mapping $\boldsymbol{\xi} = \mathbf{f}(\mathbf{x})$ can be written in the form

$$\xi_i = f_i(x_i), \qquad i = 1, \ldots, 3. \tag{10}$$

In other words, we consider only uni-directional mappings, and the computational grid in physical space can be constructed as a tensor product of one-dimensional computational grids.

Let us briefly describe the staggered grid arrangement. An example of a uniform staggered grid is shown in Fig. 1. In the case of uniform grid spacings, the choice for location of velocity and pressure points is natural: the velocity components U_i $(i = 1, 2, 3)$ are distributed around the pressure points. The continuity equation is centered at the pressure points while the momentum equations corresponding to each velocity component are centered at the respective velocity points. In the case of a non-uniform staggered grid, the location of pressure and velocity points are ambiguous: these points can be determined as geometrical volume and edge centers either in physical or computational spaces. Morinishi *et al.* (1998) followed the first approach. However, the generalization to non-uniform meshes suggested in (Morinishi *et al.*, 1998) preserves the conservation properties only in the case of the second order scheme. The reason is that for the higher order schemes

(4th order and higher) the resulting discrete operators do not preserve symmetries of the uniform mesh case. Veldman and Rinzema (1992) and Veldman and Versappen (1998) showed that in order for the scheme to be conservative, it should preserve symmetries of the underlying operator. The basic idea behind Veldman and Versappen's generalization is that the differentiation operation is performed in computational space. The derivative in physical space is calculated using the local Jacobian, which can be found numerically using the same stencil and the same order accuracy as finite differencing operator in the computational space. To illustrate this idea let us consider one dimensional case. First, we approximate derivative in computational space

$$\frac{\delta\phi}{\delta\xi} = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta},$$

where Δ is uniform grid spacing. The derivative in physical space is found as

$$\frac{\delta\phi}{\delta x} = \frac{1}{J}\frac{\delta\phi}{\delta\xi}, \tag{11}$$

where J is the Jacobian of the transformation $x \to \xi$, which can be found numerically by substituting x for ϕ

$$J = \frac{\delta x}{\delta\xi} = \frac{x_{i+1} - x_{i-1}}{2\Delta}.$$

Substitution of this equation into Eq. (11) gives us the following approximation of the derivative in physical space:

$$\frac{\delta\phi}{\delta x} = \frac{\phi_{i+1} - \phi_{i-1}}{x_{i+1} - x_{i-1}},$$

This seemingly simple idea is the key which enables us to generalize the higher order schemes of Morinishi *et al.* (1998) to non-uniform meshes.

Let the finite difference operator in computational domain with stencil n acting on ϕ with respect to ξ_1 be defined as

$$\left.\frac{\delta_n\phi}{\delta_n\xi_1}\right|_{\xi_1,\,\xi_2,\,\xi_3} \equiv \frac{\phi(\xi_1 + n\Delta_1/2,\ \xi_2,\ \xi_3) - \phi(\xi_1 - n\Delta_1/2,\ \xi_2,\ \xi_3)}{n\Delta_1}. \tag{12a}$$

The interpolation operator with stencil n acting on ϕ in the ξ_1 direction is given by

$$\left.\overline{\phi}^{n\xi_1}\right|_{\xi_1,\,\xi_2,\,\xi_3} \equiv \frac{\phi(\xi_1 + n\Delta_1/2,\ \xi_2,\ \xi_3) + \phi(\xi_1 - n\Delta_1/2,\ \xi_2,\ \xi_3)}{2}. \tag{12b}$$

In addition, we define a special interpolation operator with stencil n of the product of ϕ and ψ in the ξ_1 direction,

$$\begin{aligned}\left.\widetilde{\phi\psi}^{n\xi_1}\right|_{\xi_1,\,\xi_2,\,\xi_3} \equiv\ & \frac{1}{2}\phi(\xi_1 + n\Delta_1/2,\ \xi_2,\ \xi_3)\ \psi(\xi_1 - n\Delta_1/2,\ \xi_2,\ \xi_3) \\ &+ \frac{1}{2}\psi(\xi_1 + n\Delta_1/2,\ \xi_2,\ \xi_3)\ \phi(\xi_1 - n\Delta_1/2,\ \xi_2,\ \xi_3).\end{aligned} \tag{12c}$$

Discrete operators in the ξ_2 and ξ_3 directions are defined in the same way as for the ξ_1 direction.

The following identities will be needed to derive some relations later in the paper:

$$\frac{\delta_n \widehat{\phi\psi}^{n\xi_j}}{\delta_n \xi_j} = \phi \frac{\delta_{2n}\psi}{\delta_{2n}\xi_j} + \psi \frac{\delta_{2n}\phi}{\delta_{2n}\xi_j}, \tag{13a}$$

$$\widehat{(\phi\psi)\cdot\psi}^{n\xi_j} = \overline{\phi}^{n\xi_j}\,\widehat{\psi\psi}^{n\xi_j}, \tag{13b}$$

$$\overline{\phi}^{n\xi_j}\,\overline{\psi}^{n\xi_j} = \frac{1}{2}\overline{\phi\psi}^{n\xi_j} + \frac{1}{2}\widehat{\phi\psi}^{n\xi_j}, \tag{13c}$$

$$\frac{\delta_n \overline{\phi}^{n\xi_j}}{\delta_n \xi_j} = \frac{\delta_{2n}\phi}{\delta_{2n}\xi_j}, \tag{13d}$$

$$\frac{\delta_n \overline{\phi}^{m\xi_i}}{\delta_n \xi_j} = \overline{\frac{\delta_n \phi}{\delta_n \xi_j}}^{m\xi_i}, \tag{13e}$$

$$\overline{\psi\frac{\delta_n \phi}{\delta_n \xi_j}}^{n\xi_j} = \frac{\delta_n\ \psi\cdot\overline{\phi}^{n\xi_j}}{\delta_n \xi_j} - \phi\frac{\delta_n \psi}{\delta_n \xi_j}, \tag{13f}$$

$$\phi\frac{\delta_n\ \psi\cdot\overline{\phi}^{n\xi_j}}{\delta_n \xi_j} = \frac{1}{2}\frac{\delta_n\ \psi\cdot\widehat{\phi\phi}^{n\xi_j}}{\delta_n \xi_j} + \frac{1}{2}\phi\phi\frac{\delta_n \psi}{\delta_n \xi_j}. \tag{13i}$$

Note that ξ_i appearing as a superscript does not follow the summation convention.

For notational convenience let us introduce the discrete finite difference operator in the physical domain:

$$\left.\frac{\delta_n \phi}{\delta_n x_i}\right|_{x_1,\,x_2,\,x_3} \equiv \frac{1}{J(\xi_i)}\left.\frac{\delta_n \phi}{\delta_n \xi_i}\right|_{\xi_1,\,\xi_2,\,\xi_3}, \tag{14a}$$

where $J(\xi_i)$ is local Jacobian of the transformation $x_i \to \xi_i$. Note that the subscript i appearing in $J(\xi_i)$ in Eq. (14a) and all subsequent equations does not follow the summation convention. We emphasize that it is the form of Eq. (14a) which allows the construction of higher order schemes on non-uniform meshes with good conservation properties.

The averaging operators (12b) and (12c) use only functional values at grid points and do not use any information about grid spacing. Consequently, these operations can be performed in both physical and computational spaces. For clarity of the notation, we define the following operators in physical space:

$$\left.\overline{\phi}^{nx_i}\right|_{x_1,\,x_2,\,x_3} \equiv \left.\overline{\phi}^{n\xi_1}\right|_{\xi_1,\,\xi_2,\,\xi_3}, \tag{14b}$$

$$\left.\widehat{\phi\psi}^{nx_i}\right|_{x_1,\,x_2,\,x_3} \equiv \left.\widehat{\phi\psi}^{n\xi_i}\right|_{\xi_1,\,\xi_2,\,\xi_3}. \tag{14c}$$

We define two types of conservative forms in the discrete systems. ${}^kQ(\phi)$ in Eq. (1) is (*locally*) *conservative* if the term can be written as

$$ {}^kQ(\phi) = \frac{\delta_1({}^kF_j^1(\phi))}{\delta_1 x_j} + \frac{\delta_2({}^kF_j^2(\phi))}{\delta_2 x_j} + \frac{\delta_3({}^kF_j^3(\phi))}{\delta_3 x_j} + \cdots. \tag{15} $$

This definition corresponds to the analytical conservative form of Eq. (2).

We call ${}^kQ(\phi)$ to be *globally conservative* if the following relation holds in a periodic field:

$$ \sum_{x_1}\sum_{x_2}\sum_{x_3} {}^kQ(\phi)\,\Delta V(\mathbf{x}) = 0, \tag{16} $$

where the sums that appear in Eq. (16) are taken in the respective directions, $\Delta V(\mathbf{x}) \equiv J(\boldsymbol{\xi})\Delta V(\boldsymbol{\xi})$, $J(\boldsymbol{\xi}) = \prod_{k=1}^3 J(\xi_k)$ is the Jacobian of the transformation $\mathbf{x} \to \boldsymbol{\xi}$, and $\Delta V(\boldsymbol{\xi}) = \prod_{k=1}^3 \Delta_k$ is a constant volume in the computational domain. Note that in the periodic case local conservation (15) also implies global conservation. Also note that the definition (16) is a discrete analogue of Eq. (3).

2.3 Finite difference schemes on a non-uniform staggered grid

2.3.1 Continuity and pressure terms

We define the discrete continuity and pressure terms as

$$ (Cont. - NS2) \equiv \frac{\delta_1 U_i}{\delta_1 x_i} = 0, \tag{17} $$

$$ (Pres. - NS2)_i \equiv \frac{\delta_1 p}{\delta_1 x_i}, \tag{18} $$

where the $NS2$ denotes the second order accurate finite difference scheme on a non-uniform staggered grid. Analogously, fourth order approximations are

$$ (Cont. - NS4) \equiv \frac{9}{8}\frac{\delta_1 U_i}{\delta_1 x_i} - \frac{1}{8}\frac{\delta_3 U_i}{\delta_3 x_i} = 0, \tag{19} $$

$$ (Pres. - NS4)_i \equiv \frac{9}{8}\frac{\delta_1 p}{\delta_1 x_i} - \frac{1}{8}\frac{\delta_3 p}{\delta_3 x_i}. \tag{20} $$

Local kinetic energy is an ambiguous quantity in a staggered grid arrangement since the individual velocity components are defined at different locations in space. Some sort of interpolation must be used in order to obtain the kinetic energy at the same point. The required interpolations for the pressure terms in the K equations are

$$ \frac{1}{J(\xi_i)}\overline{U_i\frac{\delta_1 p}{\delta_1 \xi_i}}^{1\xi_i} = \frac{\delta_1 U_i \overline{p}^{1x_i}}{\delta_1 x_i} - p\cdot(Cont - NS2), \tag{21} $$

$$ \frac{9}{8}\frac{1}{J(\xi_i)}\overline{U_i\frac{\delta_1 p}{\delta_1 \xi_i}}^{1\xi_i} - \frac{1}{8}\frac{1}{J(\xi_i)}\overline{U_i\frac{\delta_3 p}{\delta_3 \xi_i}}^{3\xi_i} = \frac{9}{8}\frac{\delta_1 U_i \overline{p}^{1x_i}}{\delta_1 x_i} - \frac{1}{8}\frac{\delta_3 U_i \overline{p}^{3x_i}}{\delta_3 x_i} - p\cdot(Cont - NS4). \tag{22} $$

Therefore, Eqs. (18) and (20) are globally conservative if the corresponding discrete continuity equations are satisfied.

2.3.2 Second order accurate convective schemes

As we have already mentioned, local kinetic energy $K \equiv U_i U_i/2$ can not be defined uniquely on a staggered grid. Let us assume that a term is (locally) conservative in the transport equation of K if the term is (locally) conservative in the transport equations of ${U_1}^2/2$, ${U_2}^2/2$ and ${U_3}^2/2$. Since the conservation properties of ${U_2}^2/2$ and ${U_3}^2/2$ are estimated in the same manner as for ${U_1}^2/2$, only the conservation properties of the convective schemes in the momentum and ${U_1}^2/2$ equations need to be considered.

Let us define second order accurate convective schemes for non-uniform staggered grids as follows:

$$(Div. - NS2)_i \equiv \frac{\delta_1 \, \overline{U_j}^{1x_i} \, \overline{U_i}^{1x_j}}{\delta_1 x_j}, \tag{23}$$

$$(Adv. - NS2)_i \equiv \frac{1}{J(\xi_j)} \overline{\overline{U_j}^{1\xi_i} \frac{\delta_1 U_i}{\delta_1 \xi_j}}^{1\xi_j}, \tag{24}$$

$$(Skew. - NS2)_i \equiv \frac{1}{2}(Div. - NS2)_i + \frac{1}{2}(Adv. - NS2)_i. \tag{25}$$

Using Eqs. (13e), (13f), (14a), and (14b) the advective $(Adv.-NS2)_i$ and divergence $(Div. - NS2)_i$ forms of the convective term are connected via

$$(Adv. - NS2)_i = (Div - NS2)_i - U_i \frac{\delta_1 \overline{U_j}^{1x_i}}{\delta_1 x_j}. \tag{26}$$

Using (13e), Eq. (26) can be further simplified as follows:

$$\begin{aligned}(Adv. - NS2)_i = (Div - NS2)_i - U_i \cdot \overline{(Cont. - NS2)}^{1x_i} \\ + U_i \cdot \left[\overline{\frac{\delta_1 U_i}{\delta_1 x_i}}^{1x_i} - \frac{\delta_1 \overline{U_i}^{1x_i}}{\delta_1 x_i} \right],\end{aligned} \tag{27}$$

where there is no summation over i. Note that the term in square brackets is the commutation error between finite differencing (14a) and averaging (14b) operators and in general is not zero, unless the grid is uniform in x_i direction.

Equations (23) and (27) are the discrete analogs of the Eqs. (4a) and (5a) respectively. Clearly, Eqs. (4a) and (23) have the same structure while Eq. (27) has an additional term in it when compared to Eq. (5a). For that reason the discrete conservation properties for both advective and skew-symmetric forms of the convective term are different from analytical ones. In other words, the divergence $(Div. - NS2)_i$ form of the convective term is conservative *a priori* in the momentum equation while enforcing the discrete continuity is not enough to make both advective $(Adv. - NS2)_i$ and skew-symmetric $(Skew. - NS2)_i$ forms conserve the momentum. This is due to the presence of commutation error term which, in general, is non-zero for non-uniform meshes.

Using Eqs. (13f), (13i), and (14) the product between U_1 and $(Skew. - NS2)_1$ can be rewritten as

$$U_1 \cdot (Skew. - NS2)_1 = \frac{1}{2} \frac{\delta_1 \, \overline{U_j}^{1x_1} \widehat{U_1 U_1}^{1x_j} / 2}{\delta_1 x_j}. \tag{28}$$

Therefore, $(Skew. - NS2)_1$ is conservative *a priori* in the transport equation of ${U_1}^2/2$. Note that in the case of the non-uniform staggered grid, the commutation error term is non-zero and neither divergence $(Div. - NS2)_i$ nor advective $(Adv. - NS2)_i$ forms of the convective term conserve kinetic energy. We also note that in the case of a uniform mesh, the commutation error is zero, and we fully recover the conservation properties described in (Morinishi *et al.*, 1998).

2.3.3 Higher order accurate convective schemes

In this section we will generalize the higher order accurate convective schemes of Morinishi *et al.* (1998) for non-uniform meshes. The fourth order accurate convective schemes on a non-uniform staggered grid are defined as

$$\begin{aligned}(Div. - NS4)_i \equiv & \frac{9}{8} \frac{\delta_1}{\delta_1 x_j} \left\{ \left(\frac{9}{8} \overline{U_j}^{1x_i} - \frac{1}{8} \overline{U_j}^{3x_i} \right) \overline{U_i}^{1x_j} \right\} \\ & - \frac{1}{8} \frac{\delta_3}{\delta_3 x_j} \left\{ \left(\frac{9}{8} \overline{U_j}^{1x_i} - \frac{1}{8} \overline{U_j}^{3x_i} \right) \overline{U_i}^{3x_j} \right\},\end{aligned} \tag{29}$$

$$\begin{aligned}(Adv. - NS4)_i \equiv & \frac{9}{8} \frac{1}{J(\xi_j)} \overline{\left(\frac{9}{8} \overline{U_j}^{1\xi_i} - \frac{1}{8} \overline{U_j}^{3\xi_i} \right) \frac{\delta_1 U_i}{\delta_1 \xi_j}}^{1\xi_j} \\ & - \frac{1}{8} \frac{1}{J(\xi_j)} \overline{\left(\frac{9}{8} \overline{U_j}^{1\xi_i} - \frac{1}{8} \overline{U_j}^{3\xi_i} \right) \frac{\delta_3 U_i}{\delta_3 \xi_j}}^{3\xi_j},\end{aligned} \tag{30}$$

$$(Skew. - NS4)_i \equiv \frac{1}{2} (Div. - NS4)_i + \frac{1}{2} (Adv. - NS4)_i. \tag{31}$$

Using Eqs. (13e), (13f), (14a), and (14b), the advective $(Adv. - NS4)_i$ and divergence $(Div. - NS4)_i$ forms of the convective term are connected via

$$\begin{aligned}(Adv. - NS4)_i = & (Div. - NS4)_i \\ & - U_i \cdot \left(\frac{9}{8} \overline{(Cont. - NS4)}^{1x_i} - \frac{1}{8} \overline{(Cont. - NS4)}^{3x_i} \right) \\ & + \frac{9}{8} U_i \cdot \left(\frac{9}{8} \left[\overline{\frac{\delta_1 U_i}{\delta_1 x_i}}^{1x_i} - \frac{\delta_1 \overline{U_i}^{1x_i}}{\delta_1 x_i} \right] - \frac{1}{8} \left[\overline{\frac{\delta_3 U_i}{\delta_3 x_i}}^{1x_i} - \frac{\delta_3 \overline{U_i}^{1x_i}}{\delta_3 x_i} \right] \right) \\ & - \frac{1}{8} U_i \cdot \left(\frac{9}{8} \left[\overline{\frac{\delta_1 U_i}{\delta_1 x_i}}^{3x_i} - \frac{\delta_1 \overline{U_i}^{3x_i}}{\delta_1 x_i} \right] - \frac{1}{8} \left[\overline{\frac{\delta_3 U_i}{\delta_3 x_i}}^{3x_i} - \frac{\delta_3 \overline{U_i}^{3x_i}}{\delta_3 x_i} \right] \right),\end{aligned} \tag{32}$$

where there is no summation over i. Fourth order convective schemes exhibit the same pattern as second order schemes: only the divergence form $(Div. - NS4)_i$ of the convective term is conservative *a priori* in the momentum equation. The presence of commutation error in both advective $(Adv. - NS4)_i$ and skew-symmetric $(Skew. - NS4)_i$ forms of the convective term results in non-conservation of momentum on a non-uniform mesh.

The conservation properties for $U_1^2/2$ can be estimated exactly the same way as in previous section. Using Eqs. (13f), (13i), and (14), the following relation can be obtained:

$$\begin{aligned} U_1 \cdot (Skew. - NS4)_1 = & \frac{9}{8}\frac{\delta_1}{\delta_1 x_j}\left\{\left(\frac{9}{8}\overline{U_j}^{1x_1} - \frac{1}{8}\overline{U_j}^{3x_1}\right)\widehat{\frac{U_1U_1}{2}}^{1x_j}\right\} \\ & - \frac{1}{8}\frac{\delta_3}{\delta_3 x_j}\left\{\left(\frac{9}{8}\overline{U_j}^{1x_1} - \frac{1}{8}\overline{U_j}^{3x_1}\right)\widehat{\frac{U_1U_1}{2}}^{3x_j}\right\}. \end{aligned} \tag{33}$$

Thus, $(Skew. - NS4)_i$ is conservative *a priori* in the transport equation of ${U_1}^2/2$ while both the divergence $(Div. - NS4)_i$ and advective $(Adv. - NS4)_i$ forms of the convective term do not conserve kinetic energy when the staggered grid is non-uniform.

Higher order finite difference schemes on non-uniform meshes can be constructed in the same way as for the fourth order schemes. The nth order accurate convective schemes on a non-uniform staggered grid are defined as

$$(Div. - NSn)_i \equiv \sum_{k=1}^{n/2} \alpha_k \frac{\delta_{(2k-1)}}{\delta_{(2k-1)}x_j}\left\{\left(\sum_{l=1}^{n/2}\alpha_l \overline{U_j}^{(2l-1)x_i}\right)\overline{U_i}^{(2k-1)x_j}\right\}, \tag{34}$$

$$(Adv. - NSn)_i \equiv \sum_{k=1}^{n/2} \frac{\alpha_k}{J(\xi_j)} \overline{\left(\sum_{l=1}^{n/2}\alpha_l \overline{U_j}^{(2l-1)\xi_i}\right)\frac{\delta_{(2k-1)}U_i}{\delta_{(2k-1)}\xi_j}}^{(2k-1)\xi_j}, \tag{35}$$

where the α_k are the interpolation weights. The continuity and pressure terms involve straightforward applications of the higher order interpolation operators and can be written as

$$(Cont. - NSn) \equiv \sum_{k=1}^{n/2} \alpha_k \frac{\delta_{(2k-1)}U_i}{\delta_{(2k-1)}x_i} = 0, \tag{36}$$

$$(Pres. - NSn)_i \equiv \sum_{k=1}^{n/2} \alpha_k \frac{\delta_{(2k-1)}p}{\delta_{(2k-1)}x_i}. \tag{37}$$

As an example, the sixth order accurate finite difference schemes on a staggered non-uniform grid are given by

$$(Cont. - NS6) \equiv \frac{150}{128}\frac{\delta_1 U_i}{\delta_1 x_i} - \frac{25}{128}\frac{\delta_3 U_i}{\delta_3 x_i} + \frac{3}{128}\frac{\delta_5 U_i}{\delta_5 x_i} = 0, \tag{38}$$

$$(Pres. - NS6)_i \equiv \frac{150}{128}\frac{\delta_1 p}{\delta_1 x_i} - \frac{25}{128}\frac{\delta_3 p}{\delta_3 x_i} + \frac{3}{128}\frac{\delta_5 p}{\delta_5 x_i}, \tag{40}$$

$$\begin{aligned}(Div. - NS6)_i \equiv\ & \frac{150}{128}\frac{\delta_1}{\delta_1 x_j}\left\{\left(\frac{150}{128}\overline{U_j}^{1x_i} - \frac{25}{128}\overline{U_j}^{3x_i} + \frac{3}{128}\overline{U_j}^{5x_i}\right)\overline{U_i}^{1x_j}\right\} \\ & - \frac{25}{128}\frac{\delta_3}{\delta_3 x_j}\left\{\left(\frac{150}{128}\overline{U_j}^{1x_i} - \frac{25}{128}\overline{U_j}^{3x_i} + \frac{3}{128}\overline{U_j}^{5x_i}\right)\overline{U_i}^{3x_j}\right\} \\ & + \frac{3}{128}\frac{\delta_5}{\delta_5 x_j}\left\{\left(\frac{150}{128}\overline{U_j}^{1x_i} - \frac{25}{128}\overline{U_j}^{3x_i} + \frac{3}{128}\overline{U_j}^{5x_i}\right)\overline{U_i}^{5x_j}\right\},\end{aligned} \tag{41}$$

$$\begin{aligned}(Adv. - NS6)_i \equiv\ & \frac{150}{128}\frac{1}{J(\xi_j)}\overline{\left(\frac{150}{128}\overline{U_j}^{1\xi_i} - \frac{25}{128}\overline{U_j}^{3\xi_i} + \frac{3}{128}\overline{U_j}^{5\xi_i}\right)\frac{\delta_1 U_i}{\delta_1 \xi_j}}^{1\xi_j} \\ & - \frac{25}{128}\frac{1}{J(\xi_j)}\overline{\left(\frac{150}{128}\overline{U_j}^{1\xi_i} - \frac{25}{128}\overline{U_j}^{3\xi_i} + \frac{3}{128}\overline{U_j}^{5\xi_i}\right)\frac{\delta_3 U_i}{\delta_3 \xi_j}}^{3\xi_j} \\ & + \frac{3}{128}\frac{1}{J(\xi_j)}\overline{\left(\frac{150}{128}\overline{U_j}^{1\xi_i} - \frac{25}{128}\overline{U_j}^{3\xi_i} + \frac{3}{128}\overline{U_j}^{5\xi_i}\right)\frac{\delta_5 U_i}{\delta_5 \xi_j}}^{5\xi_j},\end{aligned} \tag{42}$$

$$(Skew. - NS6)_i \equiv \frac{1}{2}(Div. - NS6)_i + \frac{1}{2}(Adv. - NS6)_i. \tag{43}$$

2.4 Periodic inviscid flow simulations

To confirm the results of the previous sections numerically, three-dimensional inviscid channel flow simulations are performed. The flow field is assumed to be periodic in the streamwise (x_1) and spanwise (x_3) directions. The fourth order accurate finite difference scheme is used for the convective term. The zero-normal velocity boundary conditions are assumed along the walls. Solenoidal initial velocity fields are generated using homogeneous random numbers. A third order Runge-Kutta (RK3) method of Spalart *et al.* (1991) is used for time integration. The splitting method by Dukowicz and Dvinsky (1992) is used to enforce the solenoidal condition. The resulting discrete Poisson's equation for the pressure is solved using a discrete Fourier transform in the periodic directions and a penta-diagonal direct matrix solver in the wall normal direction. The computational box is $2\pi \times 2 \times 2\pi$ and $16 \times 16 \times 16$ mesh points are used. The grid spacings in the periodic directions are uniform. The wall normal grid is stretched using a hyperbolic-tangent function

$$x_2(j) = \frac{\tanh\left(\gamma\left(2j/N_2 - 1\right)\right)}{\tanh(\gamma)}, \qquad j = 0, \ldots, N_2. \tag{44}$$

Numerical tests are performed for $\gamma = 3$.

The analytical conservation requirements dictate that the total momentum, $\langle u_i \rangle$, and total kinetic energy, $\langle K \rangle \equiv \frac{1}{2}\left\langle {u_1}^2 + {u_2}^2 + {u_3}^2 \right\rangle$, should be conserved in time. We normalize the initial velocity field in such a way that $\langle u_1|_{t=0} \rangle = \langle u_3|_{t=0} \rangle = 0$ and $\langle K|_{t=0} \rangle = 1$. Due to the fact that grid spacing is uniform in streamwise and spanwise directions, the convective schemes have much better conservation

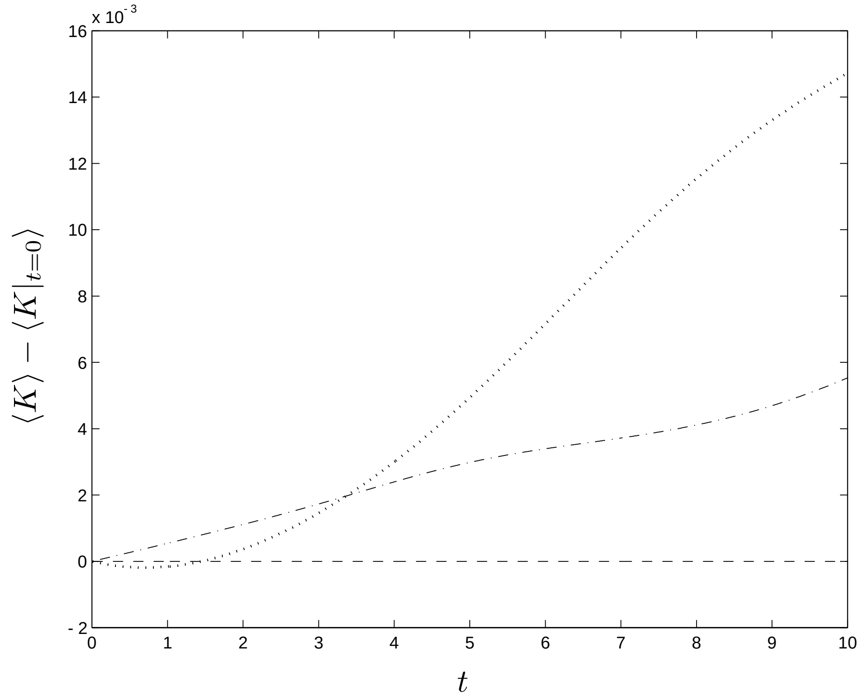

FIGURE 2. Evolution of the kinetic energy conservation error for $(Div. - NS4)$ (········), $(Adv. - NS4)$ (—·—), and $(Skew. - NS4)$ (----) convective schemes.

properties. Since commutation error in Eq. (32) is zero for $i = 1, 3$, both advective and skew-symmetric forms of the convective term conserve momentum in x_1 and x_3 directions. However, the commutation error between averaging and differencing operators in wall normal direction is not zero. Consequently, the kinetic energy is still conserved only for the skew-symmetric form of the convective term.

The conservation of momentum is confirmed numerically up to machine accuracy. Surprisingly, the momentum is conserved for all three forms of the convective term in all three directions even though the grid in wall normal direction is not uniform. We attribute this to the specific properties of the inviscid flow between parallel plates.

As we have already mentioned, the total kinetic energy is also an ambiguous quantity since it can not be defined uniquely on a staggered grid. In this report we used the following norm for the total kinetic energy:

$$K = \sum_{i=1}^{3} \sum_{x_1} \sum_{x_2} \sum_{x_3} U_i^2(\mathbf{x}) \, \Delta V(\mathbf{x}). \tag{45}$$

where the sums that appear in Eq. (45) are taken in the respective directions, $\Delta V(\mathbf{x}) \equiv J(\xi_2)\Delta V_\xi$, $J(\xi_2)$ is the Jacobian of the transformation $x_2 \to \xi_2$, and $\Delta V_\xi = \prod_{k=1}^{3} \Delta_k$ is a constant volume in the computational domain. The energy norm (45) is not conserved for both divergence and advective forms of the convection

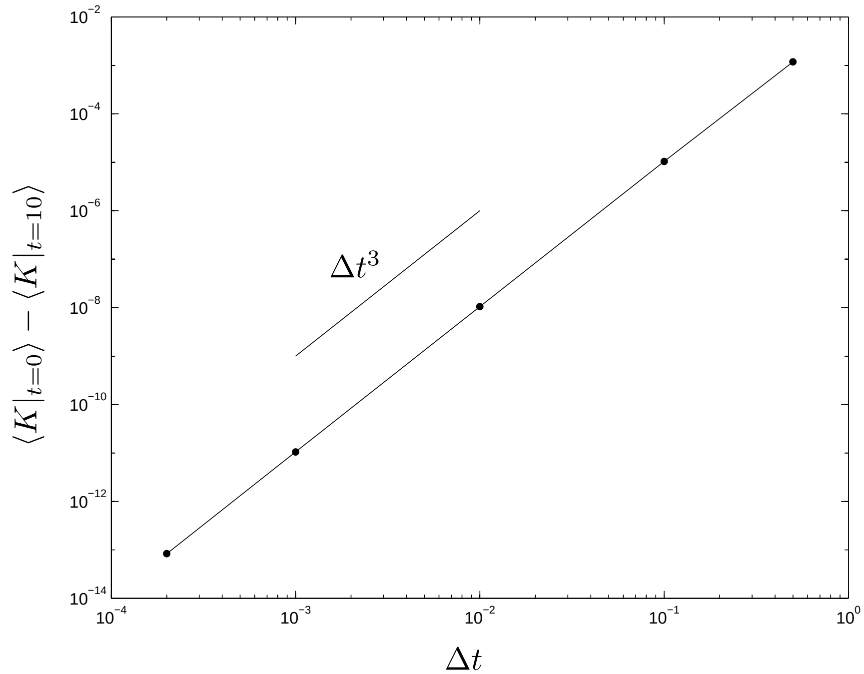

FIGURE 3. Kinetic energy conservation error at $t = 10$ as a function of time step Δt for $(Skew. - NS4)$ convective scheme.

term. However, an alternative energy norm may be conserved. For that reason further investigation is needed to confirm or deny the existence of such a norm.

The time evolution of the total kinetic energy defined by Eq. (45) is shown in Fig. 2. It can be easily seen that for both divergence and advective forms of the convective term the energy is not conserved. Also it should be noticed that the sign of the conservation energy is not defined since the conservation error is given by the nonlinear term, which can be either positive or negative.

The conservation of the kinetic energy for the skew-symmetric form is confirmed in Fig. 3. Kinetic energy is not conserved exactly since the third order Runge-Kutta time stepping method introduces a slight dissipative error. To demonstrate that the skew-symmetric scheme is conservative, the time step is decreased and the error is compared against the time step. As expected, the time stepping error decreases with the cube of Δt (see Fig. 3), and we observe no violation of kinetic energy conservation due to the spatial scheme.

2.4 Conclusions

The class of high order staggered grid finite difference schemes proposed by Morinishi *et al.* (1998) is generalized to non-uniform meshes. The proposed schemes do not simultaneously conserve mass, momentum, and kinetic energy. However, depending on the form of the convective term, conservation of either momentum or energy in addition to mass can be achieved. Furthermore, the non-conservation is weak; it is a function of the commutation error, which is very small for smoothly varying meshes. Certainly, experience has shown that schemes that are fully conservative on uniform meshes perform considerably better on non-uniform meshes when compared to the schemes which are not fully conservative even on uniform meshes. The results presented in this report are not discouraging at all: the same kind of analysis for the standard generalization to a non-uniform grid of the second order scheme of Harlow and Welch (1965) would lead to similar conclusions. Thus, the generalized schemes developed in this report will enable us to perform numerical simulations with greater accuracy while preserving the conservation properties of the second order scheme of Harlow and Welch.

3. Future plans

The new higher order schemes for non-uniform staggered grids will be tested in high Reynolds number channel flow to demonstrate that they have an advantage over the non-conservative formulation of Morinishi *et al.* (1998). In addition, the issue of conservation of kinetic energy will be investigated further to see whether there exists an alternative kinetic energy norm which would be conserved in divergence form of the convective term.

REFERENCES

Dukowicz, J. K. & Dvinsky, A. S. 1992 Approximation as a higher order splitting for the implicit incompressible flow equations. *J. Comp. Phys.* **102,** 336-347.

Ghosal, S. 1996 An analysis of numerical errors in large-eddy simulations of turbulence. *J. Comp. Phys.* **125,** 187-206.

Harlow, F. H. & Welch, J. E. 1965 Numerical calculation of time-dependent viscous Incompressible Flow of Fluid with Free Surface. *Phys. of Fluids.* **8,** 2182-2189.

Morinishi, Y, Lund, T. S., Vasilyev, O. V., & Moin, P. 1998 Fully conservative higher order finite difference schemes for incompressible flow. *J. Comp. Phys.* **142,** 1-35.

Spalart, P., Moser, R., & Rogers, M. 1991 Spectral methods for the Navier-Stokes equations with one infinite and two periodic direction. *J. Comp. Phys.* **96,** 297-324.

Veldman, A. E. P. & Rinzema, K. 1992 Playing with nonuniform grids. *J. Eng. Mech.* **26,** 119-130.

Veldman, A. E. P. & Verstappen, R. W. C. P. 1998 Symmetry-conserving discretization with application to the simulation of turbulent flow. In *Proceedings of ICFD, Conference on Numerical Methods for Fluid Dynamics*, University of Oxford, March 31-April 3.