Structure-based modeling for homogeneous MHD turbulence

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1. Motivation and objectives

The impact of a strong magnetic field on the turbulent flow of an electrically conductive fluid is encountered in diverse technological applications. Examples include liquid-metal cooling systems for fusion reactors, electromechanical brakes in continuous steel casting, solar wind turbulence and coronal heating, and the optimization process of semiconductor crystal growth. Because magnetohydrodynamic (MHD) effects are encountered in technologically important flows, it is desirable to incorporate the relevant effects in mainstream turbulence models. Current modeling of MHD turbulence tends to rely on models developed for specific applications, which by their nature are of limited scope and applicability. Incorporating the relevant physics in more general modern turbulence models has the advantage that all physics, not just the interaction of the turbulence with the magnetic field, can be captured for a wide range of conditions.

One would expect structure-based models to be well suited for capturing MHD effects. To understand why, it is useful to consider the limit of low magnetic Reynolds numbers and strong magnetic fields, conditions that are relevant to technological flows. In this limit, the Lorentz force can be treated in a quasi-static approximation and expressed as a linear function of the velocity fluctuations. This simplified picture of the interaction between the magnetic field and homogeneous turbulence highlights the key role played by the turbulence structure in MHD flows. The magnetic field has a strong effect on the angular distribution of turbulent kinetic energy in spectral space and, hence, on the anisotropy of the componentality and dimensionality of the turbulence. The Lorentz force preferentially counteracts velocity fluctuations perpendicular to the direction of the magnetic field, in the process causing a net dissipation of turbulent kinetic energy called the Joule dissipation. Joule dissipation is highly anisotropic, being largest for those modes that have their wavenumber aligned with the magnetic field and smallest for those modes that have their wavenumber in the plane normal to the magnetic field. Overall, the magnetic field tends to eliminate gradients of the turbulence in the direction of the magnetic lines and in the process lengthen turbulent eddies in that direction. Joule damping tends to produce structurally two-dimensional, (but nevertheless three-component) turbulence, where the velocity depends only on the coordinates in the plane perpendicular to the magnetic field. It is important to note that turbulence produced this way has a larger fraction of the turbulent kinetic energy residing in velocity fluctuations along the direction of the magnetic field. Thus MHD turbulence seems to remain three-component even though it is often approaching a two-dimensional state. Despite its angular anisotropy, Joule dissipation acts equally at all scales and, hence, modifies the standard Kolmogorov phenomenology of the turbulent spectra, which assumes only viscous dissipation at small scales. This simplified picture of homogeneous MHD turbulence shows the key role played by the structure of the turbulence during the interaction with the magnetic field. In turn this suggests that the structure-based family of models is particularly well suited to capture the main aspects of MHD turbulence and stands a good chance of bringing these physics into more mainstream turbulence models.

Here we present models of MHD turbulence in the context of the Interacting Particle Representation Model (IPRM) (Kassinos & Reynolds, 1994 and 1996) and the one-point *R-D* model (Kassinos & Reynolds, 1997). The IPRM is effectively a simplified two-point theory that is exact for Rapid Distortion Theory (RDT) and quite robust for general deformations of homogeneous turbulence. The *R-D* model is a one-point structure-based model which is currently limited to general irrotational deformations of homogeneous turbulence.

2. Accomplishments

2.1 Background

The effects of a uniform magnetic field applied on the turbulent flow of an electrically conductive fluid is characterized by two dimensionless parameters, the first being the magnetic Reynolds number

$$R_{m_L} = \frac{vL}{\eta} = (\frac{v}{L})(\frac{L^2}{\eta}). \tag{1}$$

Here v is the r.m.s. fluctuating velocity

$$v = \sqrt{\tfrac{1}{3} R_{ii}} \qquad R_{ij} = \overline{u_i u_j}\,, \tag{2}$$

where u_i is the fluctuating velocity, and L is the integral length scale. η is the magnetic diffusivity

$$\eta = 1/(\sigma\mu^*)\,, \tag{3}$$

where σ is the electric conductivity of the fluid, and μ^* is the fluid magnetic permeability (here we use μ^* for the magnetic permeability and reserve μ for the dynamic viscosity). Thus the magnetic Reynolds number represents the ratio of the characteristic time scale for diffusion of the magnetic field to the time scale of the turbulence. Here we are interested in the limit of low magnetic Reynolds number,

$$R_{m_L} \ll 1. \tag{4}$$

For a wide range of applications, (4) is an excellent approximation. For example, for liquid metals the magnetic Prandtl number,

$$P_m \equiv \frac{\nu}{\eta} = \frac{R_{m_L}}{Re_L} \qquad Re_L = \frac{vL}{\nu} \tag{5}$$

is quite small ($\approx 10^{-7}$ for mercury, and $\approx 10^{-5}$ for sodium). The physical interpretation of (4) is that the distortion of the magnetic field lines by the fluid turbulence is sufficiently small so that the induced magnetic fluctuations $\mathbf{b}$ around the mean (imposed) magnetic field $\mathbf{B}$ are small and can be computed by a quasi-static theory (Roberts, 1967). In this case, the Lorentz force induced by the magnetic field $\mathbf{B}$ becomes a linear function of the velocity $\mathbf{u}$.

The second relevant dimensionless parameter is the magnetic interaction number (or Stuart number),

$$N \equiv \frac{\sigma B^2 L}{\rho v} = \frac{\tau}{\tau_m} \tag{6}$$

where B is the magnitude of the magnetic field and ρ is the fluid density. N represents the ratio of the large-eddy turnover time τ to the Joule time τ_m, i.e. the characteristic time scale for dissipation of turbulent kinetic energy by the action of the Lorentz force. The ability of an imposed magnetic field to drive the turbulence to a two-dimensional three-component state depends on N. Under the continuous action of the Lorentz force, energy becomes increasingly concentrated in modes independent of the coordinate direction aligned with $\mathbf{B}$. As a two-dimensional state is approached, Joule dissipation decreases because fewer and fewer modes with gradients in the direction of $\mathbf{B}$ are left available. In addition, the tendency towards two-dimensionalization and anisotropy is continuously opposed by non-linear angular energy transfer from modes perpendicular to $\mathbf{B}$ to other modes, which tends to restore isotropy. If N is larger than some critical value N_c, the Lorentz force is able to drive the turbulence to a state of complete two-dimensionalization. For smaller N, the Joule dissipation is balanced by non-linear transfer before a complete two-dimensionalization is reached. For very small N ($N \lesssim 1$), the anisotropy induced by the Joule dissipation is negligible. Here we consider N in the range $1 - 50$.

2.2 Governing equations

We consider homogeneous turbulence in an incompressible electrically conductive fluid with uniform fluid density ρ, kinematic viscosity ν, and magnetic diffusivity η. A uniform magnetic field $\tilde{\mathbf{B}}$ is applied to the fluid. The governing magnetohydrodynamic equations can be summarized as (Roberts, 1967)

$$\tilde{u}_{i,i} = 0 \tag{7}$$

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_z \; \tilde{u}_{i,z} = -\frac{1}{\rho}\tilde{p}_{,i} + \frac{1}{\mu^* \rho}\tilde{B}_z \; \tilde{B}_{i,z} + \nu \; \tilde{u}_{i,zz} \tag{8}$$

$$\tilde{B}_{i,i} = 0 \tag{9}$$

$$\frac{\partial \tilde{B}_i}{\partial t} + \tilde{u}_z \; \tilde{B}_{i,z} = \tilde{B}_z \; \tilde{u}_{i,z} + \eta \; \tilde{B}_{i,zz} \, . \tag{10}$$

Here $\tilde{(\ )}$ denotes the instantaneous value of the variable ( ). At this point it is convenient to decompose the hydrodynamic and magnetic field variables into mean and fluctuating parts according to the scheme

$$\tilde{u}_i = U_i + u_i, \qquad \tilde{p} = P + p \tag{11}$$

and

$$\tilde{B}_i = B_i + b_i \,, \tag{12}$$

where U_i and P are the mean values of the fluid velocity and pressure, and u_i and p are the corresponding fluctuating values. B_i is the mean (externally applied) value of the magnetic field, and b_i is the turbulence-induced magnetic fluctuation. Applying Eqs. (11) and (12) to Eqs. (7)-(10) and making use of the homogeneity assumption, we obtain for the mean fields

$$\frac{DU_i}{Dt} = -\frac{1}{\rho}P_{,i} + \frac{1}{\mu^* \rho} B_z B_{i,z} \tag{13}$$

$$\frac{DB_i}{Dt} = U_{i,z} B_z \,. \tag{14}$$

Hereafter we employ the notation

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + U_z \frac{\partial(\ )}{\partial x_z} \qquad \text{and} \qquad G_{ij} = U_{i,j} \,. \tag{15}$$

Note that under the homogeneity assumption the equations for the mean fields are decoupled from the turbulence. Also note that in this case the mean magnetic lines are transported by the flow as mean material lines regardless of the value of the magnetic diffusivity. This occurs because under the homogeneity assumption the spatial uniformity of B_i is preserved and the value of the magnetic diffusivity is irrelevant to the evolution of the mean field. Finally, in decaying turbulence with no mean deformation B_i remains uniform in space and constant in time. The continuity and transport equations for the fluctuating fields, assuming uniform B_i, are

$$u_{i,i} = 0 \tag{16}$$

$$\frac{Du_i}{Dt} = -G_{iz}u_z - (u_i u_z)_{,z} - \frac{1}{\rho}p_{,i} + \frac{1}{\mu^* \rho}(B_z b_{i,z} + b_z b_{i,z}) + \nu u_{i,zz} \tag{17}$$

$$b_{i,i} = 0 \tag{18}$$

$$\frac{Db_i}{Dt} = G_{iz} b_z + B_z u_{i,z} - u_z b_{i,z} + b_z u_{i,z} + \eta b_{i,zz} \,. \tag{19}$$

We consider the limit of low magnetic Reynolds number ($R_{m_L} \ll 1$) described above, namely

$$R_{m_L} = (\frac{v}{L})(\frac{L^2}{\eta}) \ll 1 \,. \tag{20}$$

When the magnetic diffusion time scale is much smaller than the other time scales in the problem, the magnetic fluctuations induced by the fluid turbulence are much smaller than the mean magnetic field (Roberts, 1967), and as a consequence we take

$$\beta \ll B, \quad \text{where} \quad \beta = \sqrt{\langle b_i b_i \rangle / 3}, \qquad B = \sqrt{B_i B_i}\,. \tag{21}$$

It is easily seen from (21) that in this limit one can simplify (17) to

$$\frac{Du_i}{Dt} = -G_{iz}u_z - (u_i u_z)_{,z} - \frac{1}{\rho}p_{,i} + \frac{1}{\mu^* \rho}Bb_{i,z} + \nu u_{i,zz}\,. \tag{22}$$

Somewhat more care is needed in simplifying the equation (19) for b_i. For this purpose we use the non-dimensionalization

$$t = t^* \,\frac{L}{v} \qquad u_i = u_i^* \; v \qquad U_i = U_i^* \; SL \qquad x_i = x_i^* \; L \qquad b_i = b_i^* \; \beta \tag{23}$$

where v and L are the rms turbulent velocity and integral scale defined above and here evaluated at some appropriate time $t = t_1$. $S = \sqrt{(S_{ij}S_{ij})}$ is the rms mean strain-rate. Using this non-dimensionalization scheme, one finds

$$\frac{\partial b_i^*}{\partial t^*}R_{m_L} + \frac{SL}{v}R_{m_L}U_z^* b_{i,z}^* = -R_{m_L}u_z^* b_{i,z}^* + R_{m_L}\frac{B}{\beta}\frac{B_z}{B}u_{i,z}^* + R_{m_L}\frac{SL}{v}U_{i,z}^* b_z^* + b_{i,zz}^*\,. \tag{24}$$

It is understood that spatial derivatives in (24) are with respect to $\mathbf{x}^*$. The implicit assumption here is that the convective terms on the LHS of (24) scale on the characteristic time of the turbulence. The factor SL/v multiplying the second term represents the ratio of the time scale of the turbulence to that of the mean deformation. Here we assume mild deformation rates with $SL/v \approx 1$. This allows us to drop the second term, but clearly, this assumption is invalid under RDT. Under the assumption of low magnetic Reynolds number, (24) reduces to

$$0 = R_{m_L}\frac{B}{\beta}\frac{B_z}{B}u_{i,z}^* + b_{i,zz}^*\,. \tag{25}$$

The interpretation of (25) is that when the time scale of the turbulence is much larger than the time needed for magnetic diffusion, the magnetic fluctuation field adjusts instantaneously to changes induced by the velocity field. This simplification corresponds to the quasi-static approximation of Roberts (1967). In dimensional form (25) becomes

$$\eta b_{i,kk} = -B_z u_{i,z}\,. \tag{26}$$

Note that the quasi-static approximation (26) allows the fluctuating magnetic field to be obtained from knowledge of the velocity fluctuation field and $\mathbf{B}$.

Here we consider homogeneous turbulence, for which the hydromagnetic variables can be expanded in Fourier series. Substitution of these series in (22) and (26) leads to the evolution equations for the velocity Fourier coefficients. In the next section, we present an equivalent formulation in terms of the IPRM.

2.3 An Interacting Particle Representation Model (IPRM)

After a brief review of the original IPRM formulation proposed by Kassinos & Reynolds (1994, 1996), we present a modified IPRM that includes MHD effects for the case of low magnetic Reynolds number ($R_{m_L} << 1$). At present we focus on building an IPRM formulation that captures the basic physics of homogeneous MHD turbulence and for this reason our discussion is biased towards homogeneous *decaying* MHD turbulence (no mean deformation). This bias in the discussion stems from the fact that most of the available experiments and simulations reported data on one-point statistics only for this special case. Our long term objective is to formulate a unified IPRM that can handle homogeneous MHD turbulence in the presence of general mean deformation, and, therefore, modifications proposed here and evaluated for the case of decaying homogeneous MHD turbulence may have to be revised.

The basic IPRM formulation

The basic idea in the IPRM is to follow an ensemble of "particles", determine the statistics of the ensemble, and use those as the representation for the one-point statistics of the corresponding field. The IPRM particles are not physical elements of fluid, rather they represent a convenient conceptual construction. Details of the IPRM formulation are described in Kassinos & Reynolds (1996, 1997), and the interested reader is referred there. Each of the hypothetical particles is assigned a set of properties:

- **V** velocity vector
- **W** vorticity vector
- **S** stream function vector
- **N** gradient vector
- P pressure.

In the IPRM we follow the evolution of "clusters" of particles, each cluster representing a collection of particles having the same unit gradient vector $n_i = N_i/N$. Averaging over the particles of a given cluster produces conditional moments. For example, the conditional stresses are defined by

$$R_{ij}^{|\mathbf{n}} \equiv \langle V_i V_j | \mathbf{n} \rangle \, . \tag{27}$$

In a similar fashion, one can define the conditioned dimensionality and circulicity tensors

$$D_{ij}^{|\mathbf{n}} \equiv \langle V^2 n_i n_j | \mathbf{n} \rangle = \langle V^2 | \mathbf{n} \rangle n_i n_j$$

and

$$F_{ij}^{|\mathbf{n}} \equiv \langle V^2 s_i s_j | \mathbf{n} \rangle . \tag{28}$$

The physical interpretation of these conditional moments is discussed in Kassinos & Reynolds (1994, 1996). Further averaging of the conditional moments over all possible clusters yields the one-point statistics for the turbulent field, for example

$$R_{ij} \equiv \overline{u_i u_j} = \langle R_{ij}^{|\mathbf{n}} \rangle \qquad D_{ij} \equiv \overline{\Psi'_{n,i} \Psi'_{n,j}} = \langle D_{ij}^{|\mathbf{n}} \rangle \qquad F_{ij} \equiv \overline{\Psi'_{i,n} \Psi'_{j,n}} = \langle F_{ij}^{|\mathbf{n}} \rangle \tag{29}$$

where the brackets in (29) represent averaging over all clusters and Ψ_i' is turbulence stream function (see Kassinos & Reynolds, 1994).

The relevant equations for a given cluster are the conditionally-averaged stress evolution equation

$$\begin{aligned}\dot{R}_{ij}^{|\mathbf{n}} &= -G_{ik}^{v} R_{kj}^{|\mathbf{n}} - G_{jk}^{v} R_{ki}^{|\mathbf{n}} - C_r [2R_{ij}^{|\mathbf{n}} - R_{kk}^{|\mathbf{n}} (\delta_{ij} - n_i n_j)] \\ &\quad + [G_{km}^{n} + G_{km}^{v}](R_{im}^{|\mathbf{n}} n_k n_j + R_{jm}^{|\mathbf{n}} n_k n_i)\end{aligned} \tag{30}$$

and the unit gradient vector evolution equation

$$\dot{n}_i = -G_{ki}^{n} n_k + G_{kr}^{n} n_k n_r n_i \,. \tag{31}$$

Note that (30) and (31) are *closed* for the conditional stress tensor $R_{ij}^{|\mathbf{n}}$ and n_i. That is, they can be solved without reference to other conditioned moments.
The *effective* gradient tensors G_{ij}^{v} and G_{ij}^{n} are defined by

$$G_{ij}^{n} = G_{ij} + C_n \; G_{ij}^{e} \qquad G_{ij}^{v} = G_{ij} + C_v \; G_{ij}^{e} \qquad G_{ij}^{e} = \frac{1}{\tau^*} r_{ik} d_{kj} \,. \tag{32}$$

Here $r_{ij} = R_{ij}/q^2$ and $d_{ij} = D_{ij}/q^2$ where $q^2 = 2k = R_{ii}$. The two constants are taken to be $C_n = 2.2C_v = 2.2$. The different values for these two constants account for the different rates of return to isotropy of $\mathbf{D}$ and $\mathbf{R}$. The time scale τ^* is evaluated so that the dissipation rate in the IPRM

$$\epsilon^{\mathrm{PRM}} = q^2 \frac{C_v}{\tau^*} r_{ik} d_{km} r_{mi} \tag{33}$$

matches that obtained from a model equation for the dissipation rate,

$$\dot{\epsilon} = -C_0(\epsilon^2/q^2) - C_s S_{pq} r_{pq} \epsilon - C_\Omega \sqrt{\Omega_n \Omega_m d_{nm}}\, \epsilon \,. \tag{34}$$

The last term in (34) accounts for the suppression of ϵ by mean rotation. Here Ω_i is the mean vorticity vector, and the constants are taken to be

$$C_0 = 3.67 \qquad C_s = 3.0 \qquad \text{and} \qquad C_\Omega = 0.01 \,. \tag{35}$$

Mean rotation acting on the particles tends to produce rotational randomization of the $\mathbf{V}$ vectors around the $\mathbf{n}$ vectors (Mansour *et al.* 1991, Kassinos & Reynolds 1994). The third (bracketed) term on the RHS of (30) is the *slow rotational randomization model*, which assumes that the effective rotation due to nonlinear particle-particle interactions, $\Omega_i^* = \epsilon_{ipq} G_{qp}^{e}$, should induce a similar randomization effect while leaving the conditional energy unmodified. Based on dimensional considerations and requirements for material indifference to rotation (Speziale 1981, 1985), we take

$$C_r = 8.5\, \Omega^* \, f_{pq} n_p n_q, \quad \Omega^* = \sqrt{\Omega_k^* \Omega_k^*}, \quad \Omega_i^* = \epsilon_{ipq} G_{qp}^{e} \,. \tag{36}$$

Here $f_{ij} = F_{ij}/q^2$. The rotational randomization coefficient C_r is sensitized to the orientation of the $\mathbf{n}$ vector so that the slow rotational randomization vanishes whenever the large-scale circulation is confined in the plane normal to $\mathbf{n}$.

The pressure P is determined by the requirement that $R_{ik}^{|\mathbf{n}} n_k = 0$ is maintained by (30) and (31). This determines the effects of the slow pressure strain−rate-term without the need for further modeling assumptions

$$P = \underbrace{-2G_{mk}\frac{V_k N_m}{N^2}}_{\text{rapid}} - \underbrace{\frac{(C_v + C_n)}{\tau^*} r_{mt} d_{tk} \frac{V_k N_m}{N^2}}_{\text{slow}} \,. \tag{37}$$

2.3.1 Modifications for MHD turbulence

In the case of homogeneous turbulence in a conductive fluid at low R_{m_L}, MHD effects can be incorporated in the basic IPRM formulation with minimal additional modeling. The effects of Joule dissipation as described by (22) and (26) can be incorporated in the evolution equation for the particle velocity (see Kassinos & Reynolds, 1994, 1996) through the addition of the term

$$\dot{V}_i = \ldots \ - \frac{1}{\tau_m} \ m_z m_k n_z n_k V_i \,. \tag{38}$$

Here the ellipsis stands for the standard terms as given by Kassinos & Reynolds (1996), and $1/\tau_m$ is the inverse magnetic time scale defined by [see (6)]

$$\frac{1}{\tau_m} = \frac{\sigma B^2}{\rho} \,.$$

In (38) we have also used m_i to denote the unit mean applied magnetic field

$$m_i = B_i/B \qquad B = \sqrt{B_k B_k} \,. \tag{39}$$

The corresponding contribution in the cluster averaged Reynolds stress equation is

$$\dot{R}_{ij}^{|\mathbf{n}} = \ldots - \frac{2}{\tau_m} \ m_z m_k n_z n_k R_{ij}^{|\mathbf{n}} \,. \tag{40}$$

Here the ellipsis stands for the RHS of (30). In addition, because the Lorentz force modifies the basic phenomenology of the non-linear interactions and the turbulent cascade, we have found that improved predictions can be obtained if the effective-gradients model is replaced by the formulation

$$G_{ij}^n = G_{ij} + C_n G_{ij}^e \qquad G_{ij}^v = G_{ij} + C_v G_{ij}^e \qquad G_{ij}^e = \frac{1}{\tau^*} r_{ik} f_{kj} \,. \tag{41}$$

Here $r_{ij} = R_{ij}/q^2$ and $f_{ij} = f_{ij}/q^2$ where $q^2 = 2k = R_{ii}$. The two constants are taken to be $C_n = C_v = 1.0$. Note that the effective gradients model (41) differs from the model (32) suggested by Kassinos & Reynolds (1996). The present model seems to be better suited for modeling *decaying* MHD turbulence than the one previously reported, but additional evaluation in *deformed* homogeneous turbulence (with and without MHD effects) is needed to assess its performance in more general flows.

As before, the time scale τ^* is evaluated so as to ensure that the dissipation in the IPRM matches the one obtained from a model equation for the evolution of the dissipation rate, which now also includes a term accounting for the effects of the Lorentz force,

$$\dot{\epsilon} = -C_0(\epsilon^2/q^2) - C_s S_{pq} r_{pq} \epsilon - C_\Omega \sqrt{\Omega_n \Omega_m d_{nm}}\, \epsilon - \frac{C_m}{\tau_m} m_z m_k d_{kz}\ \epsilon\,. \tag{42}$$

The value of the model constant C_m is taken to be $C_m = 2.9$. The total magnetic (or Joule) dissipation is determined from the trace of (40) and is

$$\epsilon_\mu = \frac{2}{\tau_m}\ m_z m_k d_{zk}\,. \tag{43}$$

One-point R-D formulation

A one-point model for the irrotational deformation of homogeneous turbulence was formulated by Kassinos & Reynolds (1997). This model uses the evolution equations for the normalized Reynolds stress **r** and dimensionality **d** as obtained directly from the IPRM formulation. Additional modeling assumptions are introduced in order to deal with the non-locality of the pressure fluctuations and of the magnetic effects.

Modifications to the *R-D* formulation for the case of homogeneous MHD turbulence follow directly from the IPRM modifications suggested above. Averaging the modified IPRM equations (31), (38), and (40) over all clusters and using group theory to put higher-rank tensors (like the familiar M_{ijpq} in the rapid pressure–strain-rate term) in a convenient form, we obtain

$$\begin{aligned}\dot{d}_{ij} = &-d_{jk} G^n_{ki} - d_{ik} G^n_{kj} + 2G^v_{km} r_{km}(d_{ij} - \tfrac{2}{3}\delta_{ij}) \\ &- \tfrac{2}{3} G^v_{km} d_{mk}\delta_{ij} + G^v_{kk}(\delta_{ij} - \tfrac{4}{3} d_{ij} - \tfrac{2}{3} r_{ij}) \\ &+ (2G^n_{km} + G^v_{km}) Z^d_{kmij} + G^v_{km} Z^r_{mkij} - G^v_{km} Z^f_{mkij} - \frac{2}{\tau_m}\ m_p m_q (Z^d_{pqij} - d_{pq} d_{ij})\end{aligned} \tag{44}$$

and

$$
\begin{aligned}
\dot{r}_{ij} = & \tfrac{1}{3}\big[(G^v_{mj} + G^n_{mj})(2d_{mi} + r_{mi}) + (G^v_{mi} + G^n_{mi})(2d_{mj} + r_{mj}) \\
& + G^v_{jm}(d_{mi} - r_{mi}) + G^v_{im}(d_{mj} - r_{mj}) + G^n_{jm}(d_{mi} + 2r_{mi}) + G^n_{im}(d_{mj} + 2r_{mj})\big] \\
& + 2G^v_{km} r_{km} r_{ij} - \tfrac{1}{2}(G^v_{ij} + G^v_{ji} + G^n_{ij} + G^n_{ji}) \\
& + (G^v_{mk} + G^n_{mk})(Z^f_{ikmj} - Z^r_{ikmj} - Z^d_{ikmj}) \\
& - \hat{C}_r f_{pq}[Z^f_{ijpq} - Z^r_{ijpq} + \tfrac{2}{3}\delta_{pq}(r_{ij} - f_{ij}) + \tfrac{1}{3}\delta_{ij}(r_{pq} - f_{pq})] \\
& - \frac{1}{\tau_m}[m_p m_q (Z^f_{pqij} - Z^r_{pqij} - Z^d_{pqij}) - \delta_{ij} + \tfrac{2}{3}(2r_{ij} + d_{ij}) + \tfrac{2}{3} m_p m_q (2d_{pq} + r_{pq})\delta_{ij}] \\
& + \frac{2}{\tau_m} m_p m_q d_{pq} r_{ij} \,.
\end{aligned}
\tag{45}
$$

Here G^n_{ij} and G^v_{ij} are as defined for the IPRM in (41), and $\hat{C}_r = 8.5\Omega^*$ where Ω^* is given in (36). The fourth-rank tensors

$$
Z^r_{ijkm} = \langle V^2 v_i v_j v_k v_m \rangle / q^2, \quad Z^d_{ijkm} = \langle V^2 n_i n_j n_k n_m \rangle / q^2, \quad Z^f_{ijkm} = \langle V^2 s_i s_j s_k s_m \rangle / q^2 \tag{46}
$$

must be modeled (**v**, **n**, and **s** denote unit vectors). We have constructed a model for the energy-weighted fourth moment of any vector t_i in terms of its second moment t_{ij} that allows the successful closure of (45) and (46) while maintaining full realizability of the fourth-order moments. The same model can be used for each of the three vectors v_i, n_i, and s_i and their moments and has the general form

$$
\begin{aligned}
Z^t_{ijpq} = & \langle V^2 t_i t_j t_p t_q \rangle / q^2 = C_1\, \mathbf{i} \circ \mathbf{i} + C_2\, \mathbf{i} \circ \mathbf{t} \\
& + C_3\, \mathbf{t} \circ \mathbf{t} + C_4\, \mathbf{i} \circ \mathbf{t^2} + C_5\, \mathbf{t} \circ \mathbf{t^2} + C_6\, \mathbf{t^2} \circ \mathbf{t^2}.
\end{aligned}
\tag{47}
$$

Here **i** and **t** stand for δ_{ij} and $t_{ij} = \langle V^2 t_i t_j \rangle / q^2$ respectively. Extended tensor notation is used in (47) where the fully symmetric product of two second-rank tensors **a** and **b** is denoted by

$$
\mathbf{a} \circ \mathbf{b} \equiv a_{ij} b_{pq} + a_{ip} b_{jq} + a_{jp} b_{iq} + a_{iq} b_{jp} + a_{jq} b_{ip} + a_{pq} b_{ij} \,. \tag{48}
$$

The coefficients C_1-C_6 are functions of the invariants of t_{ij} and are determined by enforcing the trace condition $Z^t_{ijkk} = t_{ij}$ and 2D realizability conditions for the case when the vectors t_i lie in a plane.

For high magnetic numbers the applied field can drive decaying homogeneous turbulence into a 2D-3C state. In this case, the magnetic contribution in (45) dominates, and careful modeling of the **Z** terms therein is important in order to maintain realizability. In the limit when the turbulence becomes 2D-3C, independent of the direction of the magnetic field **m**, the magnetic contribution from the fourth-moments reduces exactly to the simple form

$$
m_p m_q (Z^f_{pqij} - Z^r_{pqij} - Z^d_{pqij}) = \tfrac{1}{2}[m_i m_k (f_{kj} - r_{kj}) + m_j m_k (f_{ki} - r_{ki})] \,. \tag{49}
$$

To maintain realizability it is important to satisfy (49) in the limit of 2D-3C turbulence, especially in the dominant magnetic contribution term. At this stage we have used rather simple **Z** models, which satisfy individual realizability constraints but do not satisfy (49) exactly. In order to test the basic ideas without worrying at this early stage about fine-tuning the **Z** model (which can nevertheless be done), we maintain realizability by using a simple interpolation of the exact forms of the magnetic term for (nearly) isotropic turbulence and for 2D-3C turbulence (49). The resulting form of the **r** equation is

$$\begin{aligned}
\dot{r}_{ij} = {} & \tfrac{1}{3}\big[(G^v_{mj}+G^n_{mj})(2d_{mi}+r_{mi})+(G^v_{mi}+G^n_{mi})(2d_{mj}+r_{mj}) \\
& + G^v_{jm}(d_{mi}-r_{mi})+G^v_{im}(d_{mj}-r_{mj})+G^n_{jm}(d_{mi}+2r_{mi})+G^n_{im}(d_{mj}+2r_{mj})\big] \\
& + 2G^v_{km}r_{km}r_{ij} - \tfrac{1}{2}(G^v_{ij}+G^v_{ji}+G^n_{ij}+G^n_{ji}) \\
& + (G^v_{mk}+G^n_{mk})(Z^f_{ikmj}-Z^r_{ikmj}-Z^d_{ikmj}) \\
& - \hat{C}_r f_{pq}[Z^f_{ijpq}-Z^r_{ijpq}+\tfrac{2}{3}\delta_{pq}(r_{ij}-f_{ij})+\tfrac{1}{3}\delta_{ij}(r_{pq}-f_{pq})] \\
& - \frac{1}{\tau_m}\Big[\phi\; m_p m_q(Z^f_{pqij}-Z^r_{pqij}-Z^d_{pqij}) \\
& + (1-\phi)\tfrac{1}{2}[m_i m_k(r_{kj}-f_{kj})+m_j m_k(r_{ki}-f_{ki})] \\
& - \delta_{ij}+\tfrac{2}{3}(2r_{ij}+d_{ij})+\tfrac{2}{3}m_p m_q(2d_{pq}+r_{pq})\delta_{ij}\Big] + \frac{2}{\tau_m}m_p m_q\; d_{pq}r_{ij}\,.
\end{aligned} \tag{50}$$

The parameter

$$\phi = (3m_k m_z d_{kz})^{0.02\log(1+N)}.$$

is unity for isotropic turbulence and becomes zero in the limit of 2D turbulence independent of the direction of the magnetic field.

In the future we plan to ensure realizability by using an improved formulation of **Z** model that ensures that the proper relations among $\mathbf{Z^r}$, $\mathbf{Z^f}$ and $\mathbf{Z^d}$ are satisfied in the limit of 2D turbulence.

2.4 Evaluation for homogeneous MHD turbulence

The modified IPRM and one-point *R-D* models have been evaluated for the case of decaying homogeneous turbulence in a conductive fluid which at time $t = t_1$ is exposed to a uniform magnetic field $B_i = B\delta_{i1}$.

The test case and conditions have been chosen to match as closely as possible those used in the DNS of Schumann (1976). The simulations of Schumann suffer from the limitations in computer power of that era, but they are one of the few numerical experiments that report detailed information on one-point statistics.

In Schumann's simulations the spatial resolution was 32^3, the initial Reynolds number based on the integral length scale L was

$$Re_L = \frac{vL}{\nu} = 60 \qquad v = \sqrt{R_{ii}/3}. \tag{51}$$

The initial condition was a realization of a Gaussian random velocity field. The turbulence was allowed to decay for a period $t \le t_1$, during which the magnetic field was switched off. At time $t = t_1$, a uniform magnetic field $B_i = B\delta_{i1}$ was switched on and maintained constant until a later time, $t = t_2$, when it was again switched off. The simulations were terminated at a later time $t = t_3$. Results from these simulations were scaled with the integral length scale $L_1 = L(t_1)$ and the r.m.s. velocity $v_1 = v(t_1)$, which were evaluated at the activation time $t = t_1$. In these normalized units, the times at which the magnetic field was switched on or off and when the computations were terminated are

$$\{t_1, t_2, t_3\} = \{0.4, 1.2, 2.0\}\frac{L_1}{v_1}\,. \tag{52}$$

Simulations were carried out with magnetic fields corresponding to magnetic interaction numbers $N = 0, 1, 5$, and 50. Schumann used the initial conditions at $t = 0$ to extract the inverse magnetic time scale $\sigma(B)^2/\rho$ for the time interval (t_1, t_2).

For the purpose of the IPRM and *R-D* models, the inverse magnetic time scale was estimated from the initial conditions according to [see (6)]

$$\frac{1}{\tau_m} = \sigma(B)^2/\rho = N\,\frac{v_o}{L_o} = N\,v_o\frac{\epsilon_o}{(v_o)^3} = N\,\frac{\epsilon_o}{(v_o)^2} = 3N\,\frac{\epsilon_o}{q_o^2}\,, \tag{53}$$

where the subscript $_o$ in (53) denotes variables at $t = 0$ and we have used

$$L_o = (v_o)^3/\epsilon_o\,. \tag{54}$$

Figure 1a shows the evolution of the turbulent kinetic energy normalized with its value $k(t_1)$ at activation time t_1. Following Schumann's choice, evolutions are shown in terms of the dimensionless time

$$t^* = t\,\frac{L_1}{v_1}\,. \tag{55}$$

Results are shown for four different magnetic interaction numbers ($N = 0, 1, 5$, and 50). DNS results are shown as symbols, the predictions of the IPRM are shown as solid lines, and those of the *R-D* model are shown as dashed lines. The case $N = 0$ corresponds to pure decay of homogeneous isotropic turbulence for the entire time interval $t = (0, t_3)$. The discrepancy between the DNS and model predictions for the rate of decay at large times (for $N = 0$) can partly be attributed to the rather limited resolution of the DNS. For the remaining cases the magnetic field is active in the time interval $t = (t_1, t_2)$, and this is reflected in the enhanced rate of decay of the turbulent kinetic energy due to Joule dissipation. The predictions of the IPRM and one-point model are in good agreement with each other and in relatively good

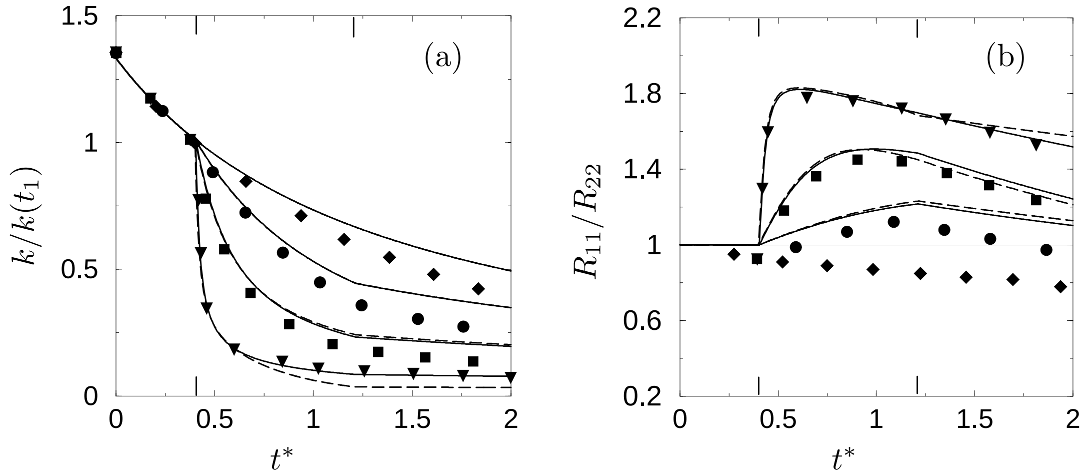

FIGURE 1. (a) Normalized turbulent kinetic energy as a function of time for various values of the magnetic number N. (b) Ratio of R_{11} (Reynolds stress component parallel to applied magnetic field) to R_{22} (Reynolds stress component normal to applied magnetic field) as a function of time for various values of the magnetic number N. Solid lines represent predictions of the IPRM and dashed lines represent those of the one-point *R-D* model. Symbols are from the 1976 DNS of Schumann (♦: $N = 0$, ● : $N = 1$, ■ : $N = 5$, ▼: $N = 50$).

agreement with the DNS results. Except for $N = 50$, the models tend to predict a somewhat lower decay rate than what is found in the DNS.

Figure 1b shows the time evolution of the ratio R_{11}/R_{22}, i.e. the ratio between the Reynolds stress components in the directions parallel and normal to the magnetic field $\mathbf{m}$. In general the development of Reynolds stress anisotropy is captured well by both models. It is worth noting that, in the absence of a magnetic field, the DNS data tends to develop anisotropy in the opposite direction. It is not clear how much results at nonzero N are affected by this trend, but it is evident that at activation time t_1 the DNS results exhibit a reverse anisotropy that has to be overcome by the action of the magnetic field.

The evolution of the magnetic dissipation ϵ_μ [see (43)] is shown in Fig. 2. Following Schumann's work, ϵ_μ is plotted for the entire time interval $t = 0$ to $t = t_3$ as if the magnetic field was constant at all times. Both the IPRM and the one-point *R-D* model are in excellent agreement with DNS predictions. Upon activation of the magnetic field at $t = t_1$, Joule dissipation, ϵ_μ, decreases quickly as the eddies become elongated in the direction of the magnetic lines, and the turbulence becomes independent of the direction along $\mathbf{m}$. At large magnetic interaction numbers ($N \approx 50$), these adjustments take place almost instantaneously upon activation of the magnetic field.

The evolution of the components of the normalized Reynolds stress and dimensionality tensors are shown in Fig. 3 for the case $N = 5$. Here the comparison is only between the IPRM and the one-point *R-D* model. Both models maintain isotropy until the activation time $t = t_1$. Activation of the magnetic field is followed by

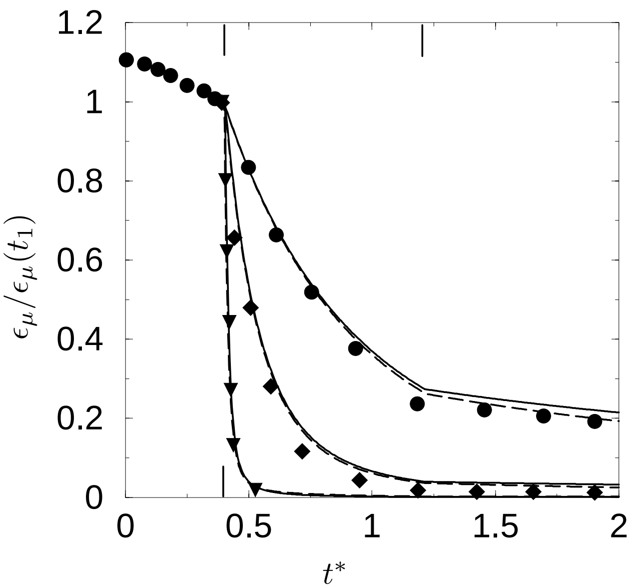

FIGURE 2. Evolution of the total Joule dissipation for various values of the magnetic number N . The results are plotted as if the magnetic field was active at all times. Solid lines represent predictions of the IPRM and dashed lines represent those of the one-point *R-D* model. Symbols are from the 1976 DNS of Schumann (● : $N = 1$, ♦: $N = 5$, ▼: $N = 50$).

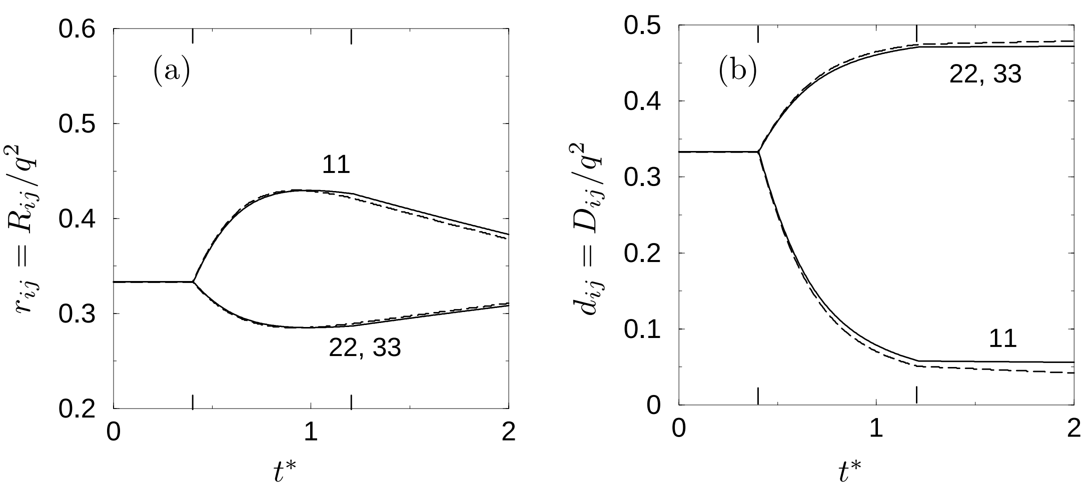

FIGURE 3. (a) Evolution of the normalized Reynolds stress tensor for the case $N = 5$. (b) Evolution of the normalized dimensionality tensor for the case $N = 5$. Solid lines represent predictions of the IPRM and dashed lines represent those of the one-point *R-D* model.

a rapid decrease in the dimensionality component in the direction of the magnetic field, d_{11}, and corresponding increase in r_{11}. As shown in Fig. 3, by the deactivation time $t = t_2$ the turbulence has become almost two-dimensional (2D) independent of the x_1, but clearly the turbulence is still in a three-component (3C) state.

3. Future plans

Here we developed extensions to the IPRM and one-point *R-D* models for homogeneous MHD turbulence. In principle these extensions should be valid not only for the case of decaying MHD turbulence (the only test case considered here), but in the presence of mean deformation as well. We plan to evaluate these models for more general cases where a mean deformation is applied to homogeneous turbulence in addition to the magnetic field. Our aim is to evaluate the performance of the modified effective gradients for general deformations, and to improve the model for the fourth-moment tensors $\mathbf{Z}$, so that important relations among fourth-moments are satisfied when the turbulence reaches important limiting states.

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