WKB approximation for acoustics in combustion chambers with arbitrary steady-state heat release profiles

By O. M. Umurhan

1. Motivations and objectives

Advances made in simulation techniques and computing power have made it possible to study complex engineering related problems including those involving combustion induced acoustic instabilities. Much effort exists today towards developing large simulations of fully compressible three-dimensional turbulence and turbulent combustion to understand and study the problem. Large eddy simulation (LES) is a powerful tool that can lead to insights and provide real solutions to observed and unwelcome acoustic-combustive instabilities. To date, however, only low Mach number and largely incompressible codes have been developed and are inadequate at self-consistently modeling the physics of acoustic generation in combustion environments.

Current work at the Center for Turbulence Research is geared towards developing fully compressible turbulence with combustion. Tests and benchmark solutions are needed in order to measure the accuracy of a developed code and these should include, among others, comparing numerically generated acoustic spectra against those predicted theoretically. Even if the theoretical acoustic spectra are only roughly accurate, they serve as good signposts for the evaluation of the robustness of the compressible scheme.

There has been much recent work towards obtaining benchmark solutions beginning with the important work of Cummings (1977), who developed a one-dimensional WKB solution with no mean flow but with an arbitrary axial temperature gradient. Some exact solutions for combustion ducts with weak temperature gradients on top of a mean flow have been performed by Munjal & Prasad (1986) and Peat (1988, 1994, 1997). Other important work including an array of exact solutions for particular axial temperature gradients possessing no mean flow have been developed by Karthik *et al.* (1999), Kumar & Sujith (1997), and Sujith *et al.* (1995) and others.

We present here a general WKB solution for predicted acoustic eigenfunctions for a one-dimensional combustion chamber with an arbitrary (but positive) distributed heat release over the chamber domain. The solution, otherwise, makes no assumption about the asymptotic nature of the fluid flow velocity or the steepness of background thermodynamic axial gradients like previous authors have done. Our only restrictions are that these average quantities satisfy steady state continuity relationships and that the flow in the domain remains subsonic. Situations in which the flow becomes supersonic within the domain require special attention and are not dealt with here.

No analytical solutions yet exist in the literature in which the temperature gradient and the mean-flow are self consistent *and* non-trivial. More precisely, most previous work have assumed that either the temperature gradient is an order one quantity while the mean-flow is zero *or* the temperature gradient is weak and is embedded in a background axially uniform flow field. We present here a WKB acoustic solution in which the axial gradients for temperature and mean-flow are all order one quantities that derive exactly from an *a priori* heat release profile within the domain. The solution technique here is pedagogically demonstrated, and we show how, when the appropriate distinguished limit is taken, we recover the result of Cummings (1977).

The specific outline of this work is as follows: in the introduction of Section 2 we present the equations we plan to solve including steady-state configurations and perturbation equations, while in Section 2.1 we make our WKB ansatz and in a detailed manner present the perturbative solution method and obtain acoustic solutions; in Section 2.2 we briefly sketch out how the current solution reduces to the previously known result. The presentation is pedagogical in nature in order to provide a calculation template for future numerical investigation by others. In Section 3 we: (i) summarize the mathematical results, (ii) discuss the physical origins of the small parameter utilized in the WKB method employed, and (iii) suggest uses of these solutions in checking development of current compressible numerics and in subsequent acoustic analysis of combustion simulations. We conclude with some parting comments about current directions of this effort.

2. One-dimensional problem and solution

We begin by stating the dimensional, 1-d Euler equations

$$\partial_t \rho + \partial_x \rho u = 0 \tag{1a}$$

$$\rho \partial_t u + \rho u \partial_x u = -\partial_x P \tag{1b}$$

$$(\partial_t + u\partial_x)P + \gamma P \partial_x u = (\gamma - 1)Q \tag{1c}$$

where ρ, u and P are the density, velocity, and pressure respectively. The quantity γ is the ratio of specific heats and is defined by $\gamma = C_P/C_V$. Q is the heat release throughout the solution domain. We also assume an ideal gas equation of state

$$P = \mathcal{R}\rho T \tag{2}$$

where $\mathcal{R}$ is the gas constant and T is the temperature. We proceed in this analysis by searching for time independent steady-state configurations about which we will temporally perturb in order to study the resulting acoustic disturbances.

For one-dimensional *nondimensionalized* flows with a distributed nondimensionalized heat source $\bar{Q}(x)$, we find that the steady state configurations satisfy

$$\bar{\rho}\bar{u} = M \tag{3a}$$

$$\bar{P} + \gamma M \bar{u} = 1 + \gamma M^2 \tag{3b}$$

$$\bar{u}\bar{P}_x + \gamma \bar{P}\bar{u}_x = (\gamma - 1)\bar{Q}, \tag{3c}$$

where the constant parameter M is the Mach number of the incoming flow and γ is the ratio of specific heats of this ideal gas. The overbars refer to the steady state quantities for which we seek solutions. Furthermore, the subscript x denotes differentiation with respect to the x coordinate. The boundary conditions used are that the flow be consistent with values at some reference point we take to be $x = 0$, i.e. that $\bar{P}(0) = 1$, $\bar{u}(0) = M$. For details of the nondimensionalizations assumed see Umurhan (1999b).

Equations (3a-c) constitute self-consistent steady state relationships between mean flow quantities as was alluded to in the introduction. Finally, the solution for the velocity $\bar{u}(x)$ given the arbitrary heat release profile $\bar{Q}$ is the solution to the following quadratic equation

$$\bar{u}^2 - \frac{2\gamma(1+\gamma M^2)}{M(\gamma+1)}\bar{u} + \frac{2(\gamma-1)}{M(\gamma+1)}\Theta(x) + \frac{2\gamma(1+\gamma M^2)}{\gamma+1} - M^2 = 0, \qquad (4a)$$

which is obtained by combining Eqs. (3a-c) along with the boundary conditions of above. Furthermore we have defined,

$$\Theta(x) = \int_0^x \bar{Q}(x')dx'. \qquad (4b)$$

The velocity and pressure are fully determined once the heat release profile $\bar{Q}$ and its axially integrated aggregate Θ are provided. On a practical level, both $\bar{Q}$ and Θ are obtainable from both numerical simulations and real experiments (see, for example, Bogdan *et al.*, 1993). Taking x derivative of Eq. (3b) and inserting the result into Eq. (3c) and simplifying shows that, unless $\bar{Q}$ goes through zero somewhere on the domain, the mean flow remains monotonic throughout x. Finally, we note that there exists a maximum value of Θ for which the resulting velocity $\bar{u}$ becomes sonic (Umurhan, 1999a),

$$\Theta_{\max} = \frac{\gamma(1-M^2)^2}{2M(\gamma^2-1)}, \qquad (4c)$$

Equation (4c) is simply another restatement of the Rayleigh-Juegoet sonic condition.

Assuming a Fourier temporal solution form $\sim \mathrm{Exp}(\sigma t)$, we find that the equations governing perturbations off of steady state are

$$(\sigma + \bar{u}D)j - \sigma\bar{\rho}u = 0 \qquad (5a)$$

$$\bar{\rho}(\sigma + \bar{u}D)u + \gamma^{-1}DP = -j\bar{u}_x \qquad (5b)$$

$$(\sigma + \bar{u}D)P + \gamma\bar{P}Du = -u\bar{P}_x - \gamma P\bar{u}_x \qquad (5c)$$

where P, u, and j denote perturbations of pressure, velocity, and mass flux (or current) respectively. Note that we reserve the symbol D for x coordinate differentiation of *perturbation* quantities while the subscript x is reserved for *steady state* terms. The perturbed mass-current is defined by

$$j = \bar{\rho}u + \bar{u}\rho, \qquad (5d)$$

in which ρ is the density perturbation.

2.1 WKB ansatz and solution method

We assume that the large parameter in our problem is the frequency eigenvalue σ, and to ground this notion we formally write,

$$\sigma = \frac{1}{\epsilon}\sigma_0$$

where ϵ is small and σ_0 is an order 1 quantity. We adopt the following WKB ansatz,

$$\Phi = \left(\hat{\Phi}_0(x) + \epsilon\hat{\Phi}_1(x) + \cdots\right) e^{\frac{1}{\epsilon}\lambda(x)}, \tag{6}$$

where Φ is a generic designation for the mass-current fluctuation j and the pressure and velocity fluctuations are P and u. The function $\lambda(x)$ is undetermined at this stage.

The lowest order solution unfolds by inserting the above WKB ansatz into Eqs. (5a-c) and separating out and equating to zero each subsequent order of ϵ.

2.1.1 Order $\frac{1}{\epsilon}$ calculation

At the lowest non-trivial order of this expansion, we find the following *algebraic* system,

$$\mathcal{L}_0\hat{\mathbf{v}}_0 = \begin{pmatrix} \sigma_0 + \bar{u}\lambda_x & -\sigma_0\bar{\rho} & 0 \\ 0 & \bar{\rho}(\sigma_0 + \bar{u}\lambda_x) & \gamma^{-1}\lambda_x \\ 0 & \gamma\bar{P}\lambda_x & (\sigma_0 + \bar{u}\lambda_x) \end{pmatrix} \begin{pmatrix} \hat{j}_0 \\ \hat{u}_0 \\ \hat{P}_0 \end{pmatrix} = 0 \tag{7a}$$

Non-trivial solutions for the disturbance vector $\hat{\mathbf{v}}_0$ exist provided the determinant of $\mathcal{L}_0$ vanishes, or

$$\mathrm{Det}\mathcal{L}_0 = (\sigma_0 + \bar{u}\lambda_x)\left(\bar{\rho}(\sigma_0 + \bar{u}\lambda_x)^2 - \bar{P}\lambda_x^2\right) = 0 \tag{7b}$$

The solution to Eq. (7b) admits three eigen-functionals for $\lambda(x)$, namely the simple differential equations,

$$\lambda_x^{(0)} = -\frac{\sigma_0}{\bar{u}} \quad\longrightarrow\quad \lambda^{(0)} = -\sigma_0\int_0^x \frac{1}{\bar{u}}dx' \tag{8a}$$

$$\lambda_x^{(\pm)} = -\frac{\sigma_0}{\bar{u}\pm\bar{c}} \quad\longrightarrow\quad \lambda^{(\pm)} = -\sigma_0\int_0^x \frac{1}{\bar{u}\pm\bar{c}}dx' \tag{8b}$$

where $\bar{c}^2 = \bar{T}$. $\lambda^{(\pm)}$ corresponds to the usual left and right propagating acoustic modes, and $\lambda^{(0)}$ corresponds to a hydrodynamic mode (Peat, 1994), or equivalently an *entropy mode* (Bloxsidge *et al.* 1988, Dowling 1995, Umurhan 1999a).

From Eqs. (7a, 8a, 8b) the following relationships between perturbations are true

$$\hat{j}_0^{(\pm)} = \pm\frac{\bar{c}\pm\bar{u}}{\bar{c}^2}\hat{P}_0^{(\pm)} \tag{9a}$$

$$\hat{u}_0^{(\pm)} = \pm\frac{1}{\gamma\bar{\rho}\bar{c}}\hat{P}_0^{(\pm)} \tag{9b}$$

while

$$\hat{P}_0^{(0)} = \hat{u}_0^{(0)} = 0. \tag{9c}$$

Yet, at this stage the functional forms of $\hat{j}_0^{(0)}$ and $P_0^{(\pm)}$ are still unknown and they need to be determined by carrying out the analysis to the next order.

2.1.2 Order 1 *calculation and solution*

We find that the equations for the first order corrections, $\hat{\mathbf{v}}_1 = \{\hat{j}_1, \hat{u}_1, \hat{P}_1\}$, become

$$\mathcal{L}_0 \hat{\mathbf{v}}_1 = \begin{pmatrix} -\bar{u} D \hat{j}_0 \\ -\bar{\rho}\bar{u} D \hat{u}_0 - \gamma^{-1} D \hat{P}_0 - \hat{j}_0 \bar{u}_x \\ -\gamma \bar{P} D \hat{u}_0 - \bar{u} D \hat{P}_0 - \bar{P}_x \hat{u}_0 - \gamma \bar{u}_x \hat{P}_0 \end{pmatrix} = \hat{\mathbf{b}}_\mathbf{0} \tag{10}$$

Since by Eq. (7a) the determinant of the matrix $\mathcal{L}_0$ is zero, in order for $\hat{\mathbf{v}}_1$ to exist, we must require that the vector $\hat{\mathbf{b}}_\mathbf{0}$ lie in the null space of $\mathcal{L}_0$ for each of the eigenvectors of $\mathcal{L}_0$.

This is formally achieved in two steps. First, we determine the solution to the homogeneous adjoint problem of the LHS of Eq. (10), or,

$$\mathcal{L}_0^T \hat{\mathbf{v}}^\dagger = 0 \tag{11}$$

where $\mathcal{L}_0^T$ is the transpose of the matrix $\mathcal{L}_0$. Second, after obtaining the corresponding adjoint solutions $\mathbf{v}^\dagger$, we then require,

$$\mathbf{v}^\dagger \cdot \hat{\mathbf{b}}_\mathbf{0} = 0, \tag{12}$$

which enforces the condition that $\hat{\mathbf{b}}_\mathbf{0}$ lie in the proper null space. The relations that result from satisfying the *compatibility condition*, Eq. (12), yield equations for the functions $\hat{P}_0^{(\pm)}$ and $j_0^{(0)}$.

In our problem we have three adjoint solutions corresponding to the left- and right-going acoustic waves (the "±" solutions) and the hydrodynamic/entropy wave (the "0" solution). Using Eqs. (3a-b, 8a-b) we find that that the adjoint solution for the acoustic and hydrodynamic waves are,

$$\hat{\mathbf{v}}^{(\pm)\dagger} = \begin{pmatrix} 0 \\ \gamma \bar{c} \\ \pm 1 \end{pmatrix} \quad , \quad \hat{\mathbf{v}}^{(0)\dagger} = \begin{pmatrix} 1 \\ 0 \\ -\frac{M}{\gamma \bar{P}} \end{pmatrix} . \tag{13}$$

After performing the algebra we find that the compatibility conditions Eq. (12) for each solution type yields the following sets of relations:

- for the hydrodynamic mode we find simply,

$$\bar{u} D \hat{j}^{(0)} = 0 \tag{14a}$$

  implying that $\hat{j}_0^{(0)} = C_0^{(0)}$ where $C_0^{(0)}$ is just a constant,
- while for the acoustic modes we find

$$D \log \left[\frac{\hat{P}_0^{2(\pm)}}{\bar{\rho}\bar{c}} \right] = -\frac{\gamma + 1}{\bar{u} \pm \bar{c}} \bar{u}_x (\mp) \frac{\gamma - 1}{\bar{c}} \bar{u}_x \tag{14b}$$

Eq. (14b) may be integrated without any explicit knowledge of the spatial dependence of $\bar{u}$. In particular, since $\bar{u}$ is monotonic in the coordinate x, one can switch from x to $\bar{u}$ coordinates and perform the integration in the $\bar{u}$ coordinate. We save the reader the details and just present the final result,

$$\frac{\hat{P}_{0}^{(\pm)}}{\sqrt{\bar{\rho}\bar{c}}} = C_{0}^{(\pm)} e^{\pm\Psi} \frac{\sqrt{\bar{u}}}{\bar{c} \pm \bar{u}} \tag{15a}$$

in which

$$\Psi = \frac{1-2\gamma}{2\sqrt{\gamma}} \text{Arctan}\left[\frac{\bar{u}-\alpha}{\bar{c}}\sqrt{\gamma}\right] \tag{15b}$$

$$\alpha = \frac{1+\gamma M^2}{2\gamma M}. \tag{15c}$$

Where α is a constant dependent on the parameters of the system and where $C_{0}^{(\pm)}$ are the arbitrary integration constants which are chosen to satisfy boundary conditions for any particular problem.

Thus, the lowest order non-trivial solution to the acoustic perturbations is complete and is given by (for example) for the pressure as

$$P = C_{0}^{(+)} \frac{\sqrt{M\bar{c}}}{\bar{c}+\bar{u}} e^{\frac{1}{\epsilon}\lambda^{(+)}+\Psi} + C_{0}^{(-)} \frac{\sqrt{M\bar{c}}}{\bar{c}-\bar{u}} e^{\frac{1}{\epsilon}\lambda^{(-)}-\Psi} + \mathcal{O}(\epsilon) \tag{16a}$$

while for the mass-current j as

$$\begin{aligned} j =& C_{0}^{(0)} \exp\left[\frac{1}{\epsilon}\lambda^{(0)}\right] + C_{0}^{(+)} \sqrt{\frac{M}{\bar{c}^3}} \exp\left[\frac{1}{\epsilon}\lambda^{(-)} + \Psi\right] \\ & - C_{0}^{(-)} \sqrt{\frac{M}{\bar{c}^3}} \exp\left[\frac{1}{\epsilon}\lambda^{(-)} - \Psi\right] + \mathcal{O}(\epsilon) \end{aligned} \tag{16b}$$

2.2 Limiting form

We must verify whether or not Eq. (15a-c) recovers previously obtained results. In particular, Cummings (1977) developed a similar WKB solution in the limit where the mean flow velocity is zero while the axial temperature (and density) gradient still persists, yielding the envelope structure function of

$$\hat{P}_{0}^{(\pm)} \sim [\bar{\rho}(x)]^{1/4} \tag{17}$$

(Eq. (19) of Cummings, 1977). Before evaluating Eq. (15a) in the the zero mean flow limit, caution must be exercised. In particular, naively taking the $\bar{u} \to 0$ limit of Eq. (15a) predicts a zero amplitude for the perturbation, which is clearly incorrect. We show how to avoid this pitfall in the following two ways.

The first way is less rigorous but rather quickly demonstrates the recovery. In the limit that the mean flow velocity and its is derivative is negligibly small, we find that Eq. (14b) reduces to

$$\lim_{\bar{u}_x \to 0} D \log\left[\frac{\hat{P}_{0}^{2\,(\pm)}}{\bar{\rho}\bar{c}}\right] = 0. \tag{18}$$

In the limit where the mean flow velocity is negligibly small, the steady state pressure field $\bar{P}$ is a constant, which immediately yields $\bar{T} = 1/\bar{\rho}$. Consequently, integrating Eq. (18) and using the limiting relationship between $\bar{\rho}$ & $\bar{T}$ recovers the result Eq. (17).

A second more rigorous way proceeds by observing from Eq. (15c) that as $M \to 0$ the constant $\alpha \to \infty$. Thus, we alternatively rewrite the integration constants appearing in Eq. (16a) as

$$C_0^{(\pm)} = \frac{\tilde{C}_0^{(\pm)}}{\sqrt{M}} \mathrm{Exp}\left[\pm\left(\frac{1-2\gamma}{2\sqrt{\gamma}}\right)\frac{\pi}{2}\right]. \tag{19}$$

Inserting these re-expressed integration constants into Eq. (16a) then followed by taking the limits $M, \bar{u} \to 0$ properly recovers the Cummings amplitude form Eq. (17). We note that by defining the constants $C_0^{(\pm)}$ as in Eq. (19) we are enabling the distinguished limit (Bender & Orszag, 1978) to be taken of Eqs. (16a-b). By distinguished limit we mean that we are taking the constant $C_0^{(\pm)}$ to go infinite as $M \to 0$ in the prescribed manner above in order to recover the non-trivial result we seek.

3. Summary and discussion

In summary, we restate the WKB solutions of the perturbed pressure and mass-current accurate to $\mathcal{O}(\epsilon)$ in the formal ϵ expansion,

$$P_n = \sum_{\pm} C_n^{(\pm)} \frac{\sqrt{M\bar{c}}}{\bar{c} \pm \bar{u}} \mathrm{Exp}\left[\frac{1}{\epsilon}\lambda_n^{(\pm)} \pm \Psi\right] \mathrm{Exp}\left[\frac{\sigma_n}{\epsilon}t\right] + \mathcal{O}(\epsilon) \tag{20a}$$

$$\begin{aligned} j_n &= \sum_{\pm} \pm C_0^{(\pm)} \sqrt{\frac{M}{\bar{c}^3}} \exp\left[\frac{1}{\epsilon}\lambda_n^{(\pm)} \pm \Psi\right] \mathrm{Exp}\left[\frac{\sigma_n}{\epsilon}t\right] \\ &\quad + C_n^{(0)} \exp\left[\frac{1}{\epsilon}\lambda_n^{(0)}\right] \mathrm{Exp}\left[\frac{\sigma_n}{\epsilon}t\right] + \mathcal{O}(\epsilon) \end{aligned} \tag{20b}$$

in which the subscripted n quantities are meant to denote an acoustic wave of a given particular complex frequency σ_n. The integration constants $C_n^{(\pm)}$ $C_n^{(0)}$ and the eigenfrequencies are either already given or are to be obtained by solving a particular boundary value problem. We return to this below.

$$\Psi = \frac{1-2\gamma}{2\sqrt{\gamma}} \mathrm{Arctan}\left[\frac{\bar{u}-\alpha}{\bar{c}}\sqrt{\gamma}\right] \tag{20c}$$

$$\alpha = \frac{1+\gamma M^2}{2\gamma M}. \tag{20d}$$

The functions $\lambda(x)$ are given by

$$\lambda_n^{(0)} = -\sigma_n \int^x \frac{1}{\bar{u}} dx' \quad , \quad \lambda_n^{(\pm)} = -\sigma_n \int^x \frac{1}{\bar{u} \pm \bar{c}} dx' \tag{20e}$$

Along with the sound speed squared, $\bar{c}^2 = T$ where $\bar{P} = \bar{\rho}\bar{T}$ and where the subsonic nondimensionalized flow quantity $\bar{u}$ is a monotonic function of the x coordinate, provided the heat release $\bar{Q}$ is everywhere positive, and is given by the solution to Eq. (4a). A few comments about these results and their implications are in order:

- The small parameter, ϵ, originally evoked to formally carry out the preceding WKB analysis, must be properly defined in terms of familiar physical properties of the system. Here we may easily take the "small" regime to be defined as the ratio of the acoustic wavelength to the length scale associated with the distributed heat release:
$$\epsilon = \frac{\text{acoustic wavelength scale}}{\text{length scale of heat release}} \tag{21}$$
The asymptotic solution formally assumes $0 < \epsilon << 1$, which means that the WKB solutions developed are valid for high frequency acoustic waves. Yet, WKB approximations have historically proven to be rather good predictors away from their formal regimes of validity (see for example, Bender & Orszag, 1978). Setting $\epsilon = 1$ may predict relatively accurate eigenfunction profiles, though this would have to be tested against numerical solutions.
- There are conditions in which one may be interested in solving for the acoustic spectrum resulting from a combustion experiment or numerical simulation conducted in some simple geometries (combustion cans, ducted combustor, channel flow combustion). Obtaining a reasonably accurate modal decomposition of the generated acoustics is necessary and indispensable from the vantage point of understanding and controlling acoustic instabilities. As an applied example, one may be interested in performing an acoustic analysis of a combustion experiment (numerical or real) by spectrally decomposing the combustion generated acoustic field into an acoustic spectrum whose basis is comprised of these WKB eigenmodes, i.e.
$$P(\text{experimental}) = \sum_n A_n P_n \tag{22}$$
in which P is the sum total acoustic disturbances and where the sum is taken over all possible acoustic modes n permitted by the boundary conditions of the physical experiment. The constants A_n simply measure the power in each mode n. Once obtained, a plethora of analysis can be done to further understand the generation mechanisms which are still, to this day, unclear. This sort of analysis has been performed in other fields including, for example, the study of acoustic generation in turbulent compressible convection in stellar atmospheres by Bogdan *et al.* (1993). An analysis procedure analogous to the one performed by Bogdan *et al.* can now be applied to combustion simulations through the use of the WKB solutions obtained in this work.
- The use of these WKB solutions as benchmarks against which to test and help develop compressible numerical procedures is another obvious consequence. Sujith and Peat and their collaborators have elaborated upon this point and we simply add here that the WKB solutions developed here now permit numericists

to test their simulations in a multitude of differing laminar background test flows (e.g. mean flows $\bar{u}$ and mean temperature profiles $\bar{T}$) by comparing predicted acoustics against the acoustics generated in the simulations. This is in contrast to the limited class of background mean flows and states that can be tested by the exact benchmarks discovered and used thus far (i.e. Karthik *et al.*1999, and references therein).

We have presented a critical piece needed for proper acoustic analysis of combustion simulations and experiments. Immediate work to follow this effort will be motivated by a desire to qualitatively refine these WKB solutions. One of these courses is to include the effects of non-zero transverse wavenumber disturbances with the acoustics along with the effects of fluctuating heat release effects. Further work also entails checking the validity of these solutions against numerically generated ones and to perform some linear and weakly nonlinear stability analysis for these acoustic modes.

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REFERENCES

Bender, C. M. & Orszag, S. A. 1978 *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill, Inc.

Bloxsidge, G. J., Dowling, A. P. & Langhorne P. J. 1988 Reheat buzz: an acoustically coupled combustion instability. Part 2. Theory. *J. Fluid Mech.* **193,** 445-473.

Bogdan, T. J, Cattaneo, F, & Malagoli, A. 1993 On the generation of sound by turbulent convection. I - A numerical experiment. *Astrophys. J.* **407,** 316-329.

Dowling, A. P. 1995 The calculation of thermoacoustic oscillations. *J. Sound Vib.* **180,** 557-581.

Cummings, A. 1977 Ducts with axial temperature gradients: an approximate solution for sound transmission and generation. *J. Sound Vib.* **51,** 55-67.

Karthik, B., Mohanraj, R. K., Ramakrishnan, R., & Sujith, R. I. 1999 Exact solution for sound propagation in ducts with an axial mean temperature gradient and particulate damping. *J. Acoust. Soc. Am.* **106,** 2391-2395.

Kumar, M. B., & Sujith, R. I. 1997 Exact solution for one-dimensional acoustic fields in ducts with a quadratic mean temperature profile. *J. Acoust. Soc. Am.* **101,** 3798-3799.

Munjal, M. S. & Prasad, M. G. 1986 On plane-wave propagation in a uniform pipe in the presence of a mean flow and a temperature gradient. *J. Acoust. Soc. Am.* **80,** 1501-1506.

Peat, K. S. 1988 The transfer matrix of a uniform duct with a linear temperature gradient. *J. Sound Vib.* **123,** 43-53.

Peat, K. S. 1994 A first approximation to the effects of mean flow on sound propagation through cylindrical capillary tubes. *J. Sound Vib.* **175,** 475-489.

Peat, K. S. 1997 Convected acoustic wave motion along a capillary duct with an axial temperature gradient. *J. Sound Vib.* **203,** 855-866.

Rayleigh, J. W. S. 1896 *The Theory of Sound.* Dover.

Sujith, R. I., Waldherr, G. A. & Zinn, B. T. 1995 Exact solution for one-dimensional acoustic fields in ducts with an axial temperature gradient. *J. Sound Vib.* **184,** 389-402.

Umurhan, O. M. 1999a Exploration of fundamental matters of acoustic instabilities in combustion chambers. *(This volume.)*

Umurhan, O. M. 1999b An acoustic instability of a fast deflagration discontinuity in a tube. Submitted to *Phys. Fluids.*