Modeling of the subgrid-scale term of the filtered magnetic field transport equation

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Accurate subgrid-scale turbulence models are needed to perform realistic numerical magnetohydrodynamic (MHD) simulations of the subsurface flows of the Sun. To perform large-eddy simulations (LES) of turbulent MHD flows, three unknown terms have to be modeled. As a first step, we propose to use a priori tests to measure the accuracy of various models proposed to predict the SGS term appearing in the transport equation of the filtered magnetic field. We then evaluate the SGS model accuracy in term of structural and functional performance, i.e. the model capacity to locally approximate the unknown term and to reproduce its energetic action, respectively. From our tests, it appears that a mixed model based on the scale-similarity model has better performance.

1. Introduction

Significant progress toward quantitative understanding of the Sun and predictive capabilities for solar activity and space weather requires a large-scale, integrated modeling of the physical conditions in subsurface layers of the Sun. Realistic numerical magnetohydrodynamic (MHD) simulations of the subsurface flows and magnetic structures have become achievable because of the development of fast supercomputer systems and efficient parallel computer codes. These simulations are extremely important for understanding the complicated physics of the upper turbulent convective boundary layer of the Sun. The dynamics of this layer is critical for understanding of the formation of magnetic regions on the Sun and their activity. This layer is also a source of solar oscillations. The wave excitation and propagation properties change dramatically in strong field regions. Their modeling and investigation are important for local helioseismology and helioseismic data analysis. Accurate large-eddy simulations (LES) of the subsurface dynamics depend on the development of specific and accurate subgrid-scale turbulence models. These models have to provide a realistic description of effects of small-scale unresolved turbulence, which are particularly important for studying wave excitation.

In LES of MHD flows, to know the filtered velocity, $\bar{u}_i$, and magnetic, $\bar{b}_i$, fields, the filtered MHD equations, expressed in Alfven-speed units, have to be solved,

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j} - \tau_{ij}^u + \tau_{ij}^b\right) + \frac{\partial \bar{b}_i \bar{b}_j}{\partial x_j}, \tag{1.1}$$

$$\frac{\partial \bar{b}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{b}_i \bar{u}_j - \bar{b}_j \bar{u}_i) = \frac{\partial}{\partial x_j}\left(D\frac{\partial \bar{b}_i}{\partial x_j} - \tau_{ij}^{ub}\right) \text{ and} \tag{1.2}$$

$$\frac{\partial \bar{b}_i}{\partial x_i} = \frac{\partial \bar{u}_i}{\partial x_i} = 0, \tag{1.3}$$

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where D is magnetic diffusivity, ν is kinematic viscosity and $\bar{p}$ is filtered total pressure. In these equations, $\tau_{ij}^u = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$, $\tau_{ij}^b = \overline{b_i b_j} - \bar{b}_j \bar{b}_i$, and $\tau_{ij}^{ub} = \overline{b_i u_j} - \bar{b}_i \bar{u}_j - \left(\overline{u_i b_j} - \bar{u}_i \bar{b}_j\right)$ are subgrid-scale (SGS) tensors. In a LES, these SGS tensors cannot explicitly be determined but are estimated via SGS models assuming relationships with resolved quantities. Various works have addressed the modeling of the three SGS tensors of LES of MHD flows (Yoshizawa 1990; Theobald *et al.* 1994; Müller & Carati 2002; Miki & Menon 2008). In this work, we investigate the performance of the models proposed for τ_{ij}^{ub} from a priori tests. The goal is to distinguish the modeling error due to the models of τ_{ij}^{ub} from the other modeling errors. The modeling performances are evaluated as structural and functional performances.

2. Modeling of τ_{ij}^{ub} and performance measurement

2.1. *Available SGS models*

The modeling of the unknown SGS term, τ_{ij}^{ub}, appearing in the filtered transport equation of the magnetic field has been addressed in various ways in the past. First, an approach based on the definition of an eddy magnetic diffusivity, D_t, has been proposed, leading to the general gradient-diffusion model expression

$$\tau_{ij}^{ub} = -2D_t \bar{J}_{ij}, \tag{2.1}$$

with $\bar{J}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{b}_j}{\partial x_i} - \frac{\partial \bar{b}_i}{\partial x_i}\right)$, the filtered magnetic rotation tensor. Numerous definitions of D_t are available in the literature. Various extensions of the Smagorinsky model (Smagorinsky 1963) have been proposed. First, Yoshizawa (1987) defined the eddy magnetic diffusivity as

$$D_t = C_\lambda \Delta^2 \left(\frac{1}{2}C_\nu \bar{S}_{ij}^2 + C_\lambda \bar{j}_i^2\right)^{1/2}, \tag{2.2}$$

with $\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_i}\right)$, the resolved velocity rate-of-strain tensor and $\vec{j} = \vec{\nabla} \times \vec{b}$, the current density. Assuming a local equilibrium between production and dissipation, Hamba & Tsuchiya (2010) determined the constant values as $C_\lambda = \frac{5}{7}C_\nu$ and $C_\nu = 0.046$. Theobald *et al.* (1994) defined D_t only with the current density norm,

$$D_t = C_1 \Delta^2 |\bar{\vec{j}}|. \tag{2.3}$$

Müller & Carati (2002) used this model by computing dynamically the model coefficient as usually done for LES of hydrodynamic turbulence(Lilly 1992; Germano *et al.* 1991). In the same paper, Müller & Carati (2002) defined a new eddy magnetic diffusivity based on the cross helicity dissipation,

$$D_t = C_2 \Delta^2 |\bar{\vec{j}}.\bar{\vec{\omega}}|^{1/2}, \tag{2.4}$$

with $\vec{\omega}$ the vorticity vector. The model coefficient is also computed dynamically.

Another approach to model τ_{ij}^{ub} is based on the filtering operation itself. For example, a Taylor series expansion of a filtered product, $\overline{fg}$ (where f and g are quantities describing magnetic or flow fields), can be given for a Gaussian filter, [for details, see Balarac *et al.* (2008)]

$$\overline{fg} = \bar{f}\bar{g} + \frac{\Delta^2}{12}\frac{\partial \bar{f}}{\partial x_i}\frac{\partial \bar{g}}{\partial x_i} + \frac{\Delta^4}{288}\frac{\partial^2 \bar{f}}{\partial x_i \partial x_j}\frac{\partial^2 \bar{g}}{\partial x_i \partial x_j} + \frac{\Delta^6}{10368}\frac{\partial^3 \bar{f}}{\partial x_i \partial x_j \partial x_k}\frac{\partial^3 \bar{g}}{\partial x_i \partial x_j \partial x_k} + ... \tag{2.5}$$

Keeping only the first term, a gradient model for τ_{ij}^{ub} can be written as

$$\tau_{ij}^{ub} = \frac{\Delta^2}{12}\left(\frac{\partial \bar{b}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} - \frac{\partial \bar{b}_j}{\partial x_k}\frac{\partial \bar{u}_i}{\partial x_k}\right). \tag{2.6}$$

Another model can be constructed from the scale-similarity hypothesis proposed by Bardina *et al.* (1980). The main idea is to assume that the statistical structure of the tensors constructed on the basis of the subgrid scales is similar to that of their equivalents evaluated on the basis of the smallest resolved scales. From this assumption, the scale-similarity model for τ_{ij}^{ub} can be written as

$$\tau_{ij}^{ub} = \overline{\bar{b}_i\bar{u}_j} - \overline{\bar{b}_j\bar{u}_i} - \left(\bar{\bar{b}}_i\bar{\bar{u}}_j - \bar{\bar{b}}_j\bar{\bar{u}}_i\right). \tag{2.7}$$

However, these types of models are known to not provide enough energy transfer between grid scale (GS) and subgrid-scale (SGS), leading to unstable simulations when they are used to close the filtered Navier-Stokes equations (Vreman *et al.* 1997). To avoid this unstable behavior, a mixed model has been proposed. As posited by Clark *et al.* (1979) for the Navier-Stokes equations, the mixed model adds an eddy magnetic diffusivity approach to the gradient or to the scale-similarity model. Using the model (2.3), the mixed-gradient model is thus written as

$$\tau_{ij}^{ub} = C_3\Delta^2|\bar{\bar{\vec{j}}}|\bar{J}_{ij} + \frac{\Delta^2}{12}\left(\frac{\partial \bar{b}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} - \frac{\partial \bar{b}_j}{\partial x_k}\frac{\partial \bar{u}_i}{\partial x_k}\right), \tag{2.8}$$

whereas the mixed-scale-similarity model is written as

$$\tau_{ij}^{ub} = C_4\Delta^2|\bar{\bar{\vec{j}}}|\bar{J}_{ij} + \overline{\bar{b}_i\bar{u}_j} - \overline{\bar{b}_j\bar{u}_i} - \left(\bar{\bar{b}}_i\bar{\bar{u}}_j - \bar{\bar{b}}_j\bar{\bar{u}}_i\right), \tag{2.9}$$

with C_3 and C_4 dynamically computed.

Finally, a last approach is inspired by works in RANS context (Yoshizawa 1990). Thus, Miki & Menon (2008) model τ_{ij}^{ub} as

$$\tau_{ij}^{ub} = -2\epsilon_{kij}E_k, \tag{2.10}$$

where ϵ_{kij} is the Levi-Civita symbol and E_k is the turbulence electromotive force defined as

$$E_k = \alpha\bar{b}_k - \beta\bar{j}_k + \gamma\bar{\omega}_k. \tag{2.11}$$

In this equation, α, β and γ are spatially dependent quantities defined with the kinetic and magnetic subgrid-scale energy, $k^{sgs} = \frac{1}{2}\left(\overline{u_iu_i} - \bar{u}_i\bar{u}_i\right)$ and $k^{sgs,b} = \frac{1}{2}\left(\overline{b_ib_i} - \bar{b}_i\bar{b}_i\right)$, and computing model coefficient locally and dynamically [see Miki & Menon (2008) for details]. Note that this model needs to solve two additional transport equations for k^{sgs} and $k^{sgs,b}$, with additional unknown terms to model.

Eight SGS models for τ_{ij}^{ub} presented above have been summarized in Table 1. Some of these models are very easy to apply (especially without dynamic coefficient computation) but some others need a non-negligible implement effort and have a non-negligible computational cost. The goal of this work is to be able to determine which model appears as the best way to model τ_{ij}^{ub}. Our analysis is mainly based on a priori tests using direct numerical simulation (DNS) data. Our a priori tests measure two types of performance. Using the distinction between structural and functional models proposed by Sagaut (2005), we measure the SGS model performance in terms of structural and functional performance.

Name	Equations	Symbol	Remark
Yoshizawa	(2.1) and (2.2)	○	2 constant coefficients
Theolbald	(2.1) and (2.3)	□	1 dynamic coefficient
Cross-helicity	(2.1) and (2.4)	◇	1 dynamic coefficient
Gradient	(2.6)	△	1 constant coefficient
Scale-similarity	(2.7)	▽	Explicit filtering
Mixed-Gradient	(2.8)	△(dotted line)	1 dynamic coefficient
Mixed-Scale-similarity	(2.9)	▽(dotted line)	1 dynamic coefficient
Miki	(2.10) and (2.11)	▷	2 additional transport equations

TABLE 1. Models for the unknown SGS term τ_{ij}^{ub}

2.2. *Structural performance and optimal estimator*

The structural modeling strategy consists of using the best approximation of the unknown SGS term by constructing it from the known structure of small-scales. For example, the gradient model, Eq.(2.6), is a structural model based on a Taylor series expansion of the filtering operation. This type of model is known to give a good approximation of the unknown term with a high correlation between the unknown term and the model in a priori test. From this definition, we define the structural performance of a model by its capacity to locally approximate the SGS term to be modeled.

To evaluate the structural performance, the optimal estimation theory is used. In the framework of optimal estimation theory (Deutsch 1965), the models are compared using the notion of an optimal estimator (Moreau *et al.* 2006). Based on this idea, if a quantity τ is modeled with a finite set of variables ϕ, an exact model cannot be guaranteed. If the exact solution τ is known, for example from DNS, the optimal estimator of τ in terms of the set of variables ϕ is given by the expectation of the quantity τ conditioned on the variables in the set, i.e. $\langle \tau | \phi \rangle$, where the angle brackets indicate statistical averaging over a suitable ensemble. A quadratic error can consequently be defined as the average of the square of the difference at each point between the conditional mean value given by the value of ϕ at this point and the exact value of the quantity,

$$\epsilon_{\min} = \langle (\tau - \langle \tau | \phi \rangle)^2 \rangle, \tag{2.12}$$

where $\epsilon_{\min}$ is the error. It should be noted that any model formulated using the variable set ϕ will introduce an error that is larger than or equal to this minimum error, with the best model formulation producing this minimum error. Consequently, this quadratic error $\epsilon_{\min}$ is referred to as the irreducible error. Only a change in the variable set may reduce the magnitude of this error. In contrast, the total quadratic error is given as

$$\epsilon_{\text{tot}} = \langle (\tau - f(\phi))^2 \rangle, \tag{2.13}$$

with $f(\phi)$ the proposed model for τ. The method allows comparison of different LES models by comparing their total errors. The method allow can also evaluate the improvement possibility of a given model without changing of the set of parameters. Indeed, if the total error of a given model is much higher than its irreducible part, improvement can be expected (by modification of the coefficient computation, for example). With this method the best set of quantities to model a sub-filter term can be obtained by comparing their irreducible error.
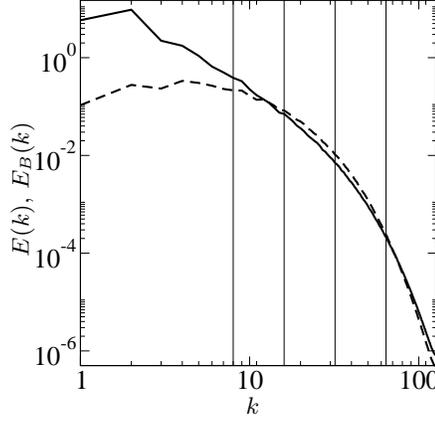
FIGURE 1. Kinetic, $E(k)$, and magnetic, $E_B(k)$, energy spectra: $E(k)$ ——— ; $E_B(k)$ ---- . The thin vertical lines show the location of some filters used in this work: $\Delta/\Delta x =$2, 4, 8 and 16.

2.3. *Functional performance*

The functional modeling strategy considers the action of the subgrid terms on the transported quantity (here, the magnetic field) and not the unknown term itself. It can introduce a dissipative term, for example, that has a similar effect but not necessarily the same structure. Because an adequate mechanism to dissipate magnetic energy from resolved to subgrid scales is essential, we define the functional performance as the model capacity that leads to good energy dissipation. The transport equation of the GS magnetic energy, $\bar{k}^b = \frac{1}{2}\bar{b}_i\bar{b}_i$, is

$$\frac{\partial \bar{k}^b}{\partial t} + \bar{u}_i \frac{\partial \bar{k}^b}{\partial x_i} = \frac{1}{2}D\frac{\partial^2 \bar{k}^b}{\partial x_i \partial x_i} - D\frac{\partial \bar{b}_i}{\partial x_j}\frac{\partial \bar{b}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\bar{b}_j \bar{b}_i \bar{u}_i\right) - \bar{u}_i \frac{\partial}{\partial x_j}\left(\bar{b}_j \bar{b}_i\right) - \frac{\partial}{\partial x_j}\left(\bar{b}_i \tau_{ij}^{ub}\right) + \tau_{ij}^{ub}\bar{J}_{ij}.$$

In this equation, the GS/SGS energy exchanges is the SGS dissipation, $\tau_{ij}^{ub}\bar{J}_{ij}$. Thus the functional performance is evaluated as the model capacity that best predicts this quantity in a statistical sense.

3. Results

3.1. *Numerical method*

As already explained, to clearly identify the performances of the various modeling strategy of τ_{ij}^{ub} without taking into account the modeling error of the other SGS unknown terms appearing in the filtered MHD equations (1.1)-(1.3), we performed a priori tests. These tests are conducted using direct numerical simulation (DNS) data from a forced homogeneous isotropic turbulence computation. A pseudo-spectral code with second-order explicit Runge-Kutta time-advancement is used. The viscous terms are treated exactly. The simulation domain is discretized using 256^3 grid points on a domain of length 2π. A classic 3/2 rule is used for de-aliasing the non-linear convection term, and a statistical stationarity is achieved using a forcing term (Alvelius 1999). The transport equation of the magnetic field is advanced simultaneously using an identical numerical scheme. First, a hydrodynamic (no magnetic field) case is performed; when the statistically stationary
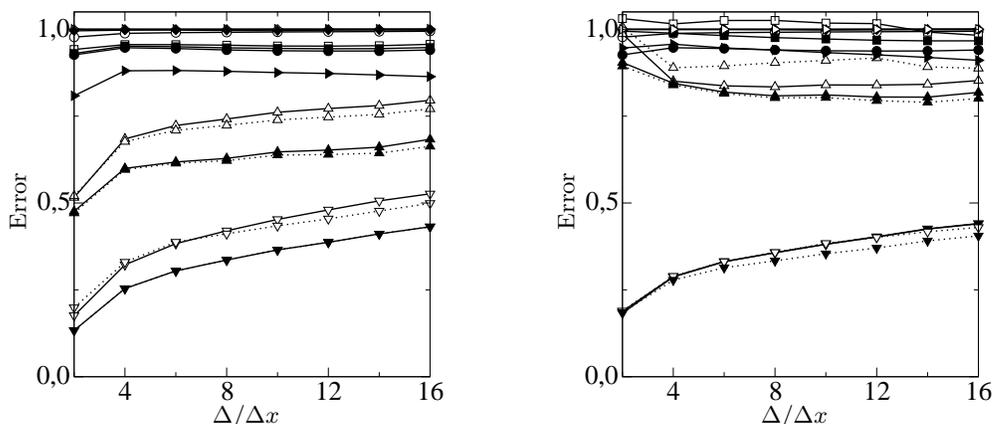
FIGURE 2. Structural performance of models. Evolution of the irreducible and total errors, Eq. (2.12) and (2.13), with the filter size for box (left) and spectral cutoff (right) filter. Open and solid symbols are for total and irreducible errors, respectively (see Table 1 for symbols correspondence).

state is obtained, the Reynolds number based on the Taylor microscale is around 100. The magnetic field is then initialized at small scales with a small amplitude. The magnetic Prandtl number is set to 0.5. Without external forcing, the magnetic energy grows leading quickly to a new statistical state. The a priori tests are performed when the flow is statistically stationary.

In the a priori tests, an explicit filter is used to replicate the behavior of the filter implicitly associated with the discretization in real LES. Two kinds of filter are used: a spectral cutoff filter to mimic spectral LES and a box filter to mimic LES using centered finite differences (Rogallo & Moin 1984). Several different filter sizes have been used such as $2 \leq \Delta/\Delta x \leq 16$, where Δ is the filter width and Δx is the grid spacing used in the DNS. The location in wavenumber space of the filters used are displayed in Fig. 1, which shows the kinetic and magnetic energy spectra.

3.2. *Models performances*

As already explained, to first evaluate the structural performance of the SGS models, the total error, Eq. (2.13), for each model is considered and compared with its irreducible error, Eq. (2.12). Figure 2 shows the evolution of the total and irreducible error with the filter size, for the different SGS models and for both box and spectral cutoff filter. In this figure, the error is normalized by the statistical variance of the SGS term. First conclusions can be addressed. As expected, SGS models based on a structural approach (models based on scale-similarity assumption or on Taylor series expansion) lead to the smallest errors. Note that for both filters the models based on the scale-similarity assumption have the smallest errors. In particular for the spectral cutoff filter the errors of the models based on a Taylor series expansion stay high. This is because the spectral cutoff filter leads to a divergent Taylor series owing to its non-localness (Sagaut 2005). All the other models have irreducible errors higher than the total error model of the models based on the structural approach, which shows that a structural improvement of these models can not be expected without adding new quantities in its set of parameters.
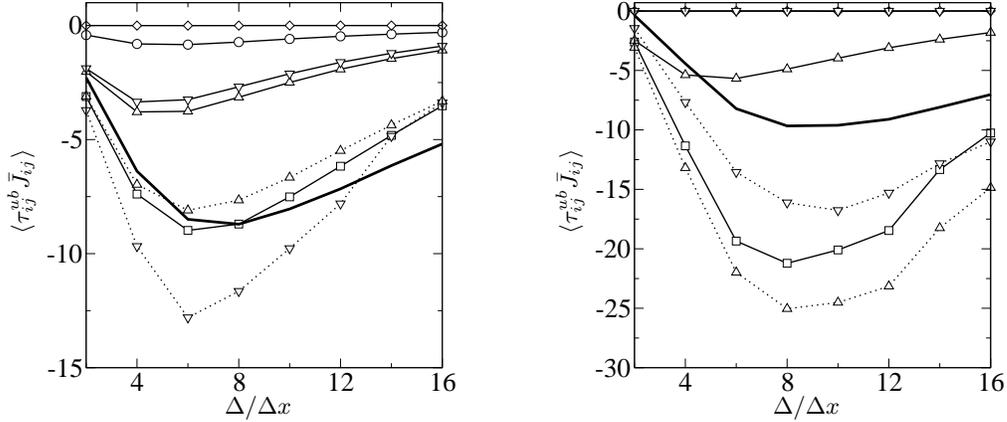
FIGURE 3. Functional performance of models. Evolution of the mean GS/SGS magnetic energy transfer with the filter size for box (left) and spectral cutoff (right) filter (see Table 1 for symbols correspondence). The thick line is the mean GS/SGS magnetic energy transfer from the filtered DNS.

This illustrates the improvement owing to the mixed approach in comparison with the models based only on an eddy diffusivity assumption.

As a second step, the functional performance is now studied from the evolution of the mean GS/SGS magnetic energy transfer, $\langle \tau_{ij}^{ub} \bar{J}_{ij} \rangle$, with the filter size. Figure 3 shows the results for box and spectral cutoff filters. First, it is shown that for the pure structural models, i.e. the gradient and scale-similarity models, there is insufficient transfer in comparison with the DNS results. This is a well-known problem for models based only on structural approach for hydrodynamic LES. Indeed, these models produce inadequate dissipation, which results in instable simulations. Conversely, Theobald's eddy-diffusivity based model predicts enough GS/SGS transfer and even an over-estimation for the spectral cutoff filter in comparison with the DNS results. Thus, this allows the mixed models to predict enough dissipation. Note that the other eddy-diffusivity based models do not lead to GS/SGS transfer to be a sufficient good candidates for building a mixed model. In particular, no GS/SGS transfer occurs in the cross-helicity model. Müller & Carati (2002) had already shown this property explaining that in LES performed with this SGS model, the lack of GS/SGS magnetic energy transfer is compensated by the transfer between kinetic and magnetic energy caused by the Lorentz force.

From this analysis, the best performing model appears to be the mixed-scale-similarity model. First, for structural performance, the analysis based on the total and irreducible errors shows that we can not expect any improvement in the other models to have the same performance of this model. Indeed the total error of this model is always smaller than the irreducible error, i.e. the smallest possible error, of the other models. For the functional performance, the eddy-diffusivity part of this model predicts enough dissipation for the simulation stability. It can be noted however that a possible improvement of this model would be to correct the over-estimation of the GS/SGS transfer.

3.3. *Dynamic procedure at the tensor divergence level*

In the results above, the mixed-scale-similarity model, Eq. (2.9), used a classic dynamic procedure to compute C_4. The classic dynamic procedure uses a second (test) filter,
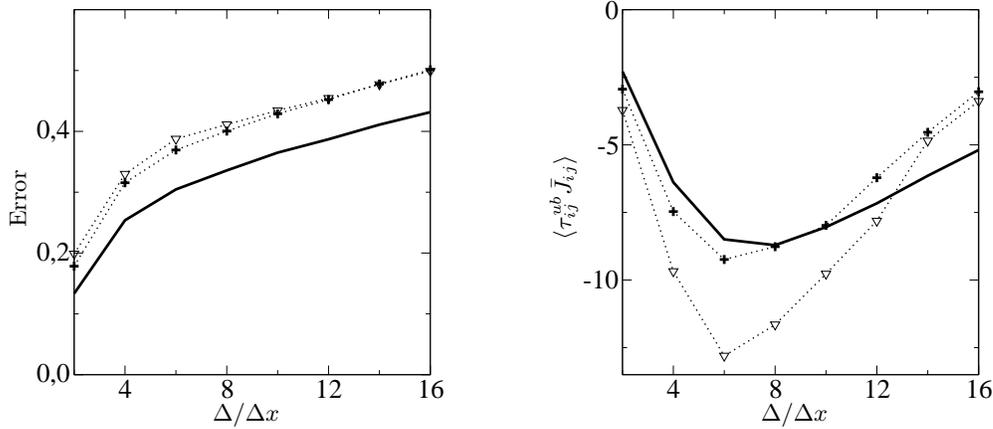
FIGURE 4. Comparison of model performance for the mixed-scale-similarity model with the classical (triangle) and with the divergence based (cross) dynamic procedure. Left: evolution of the irreducible (solid line) and total errors with the filter size. Right: Evolution of the mean GS/SGS magnetic energy transfer with the filter size and comparison with the mean GS/SGS magnetic energy transfer from the filtered DNS (solid line).

denoted $\hat{\ }$, of size $\hat{\Delta} = 2\Delta$. This procedure is based on the Germano identity (Germano *et al.* 1991),

$$L_{ij}^{ub} = T_{ij}^{ub} - \hat{\tau}_{ij}^{ub},$$

where $T_{ij}^{ub} = \widehat{\overline{b_i u_j}} - \hat{\bar{b}}_i \hat{\bar{u}}_j - \left(\widehat{\overline{u_i b_j}} - \hat{\bar{u}}_i \hat{\bar{b}}_j \right)$. Thus, $L_{ij}^{ub} = \widehat{\bar{b}_i \bar{u}_j} - \hat{\bar{b}}_i \hat{\bar{u}}_j - \left(\widehat{\bar{u}_i \bar{b}_j} - \hat{\bar{u}}_i \hat{\bar{b}}_j \right)$ can be computed directly from the resolved field. Assuming that, T_{ij}^{ub}, which is the subgrid tensor corresponding to the second filtering level, is also modeled with the mixed-scale-similarity model and with the same value of C_4. An equation for C_4 can then be written from a least squares averaging procedure (Lilly 1992). In fact, from the filtered transport equation of the magnetic field, Eq. (1.2), it is noted that only the vector given by the divergence of the tensor, $\partial \tau_{ij}^{ub} / \partial x_j$ has to be known and not the tensor, τ_{ij}^{ub}, itself. In this sense, Clark *et al.* (1979) explained the efficiency of the Smagorinsky model for the Navier-Stokes equations. The correlation between the Smagorinsky model and the SGS term is weak at the tensor level but higher at the vector level. Thus to improve the prediction of the GS/SGS magnetic energy transfer with the mixed-scale-similarity model, a dynamic computation of the C_4 coefficient is tested for $\partial \tau_{ij}^{ub} / \partial x_j$ instead of τ_{ij}^{ub}. Therefore, the starting point is to use the divergence of the Germano identity

$$\frac{\partial L_{ij}^{ub}}{\partial x_j} = \frac{\partial T_{ij}^{ub}}{\partial x_j} - \frac{\widehat{\partial \tau_{ij}^{ub}}}{\partial x_j},$$

and the same steps of the classical dynamic procedure are then used. Figure 4 shows the structural and functional performance of the mixed-scale-similarity model using a divergence based dynamic procedure. The results are compared with the classic mixed-scale-similarity model. It is important to note that the structural performance has not deteriorated. The total error of the model stays close to its irreducible error and the error is much smaller than the error of the other models (Fig. 2). Moreover, the functional performance is improved. The new dynamic procedure allows a better prediction of the GS/SGS magnetic energy transfer. The over-prediction observed with the classic dynamic
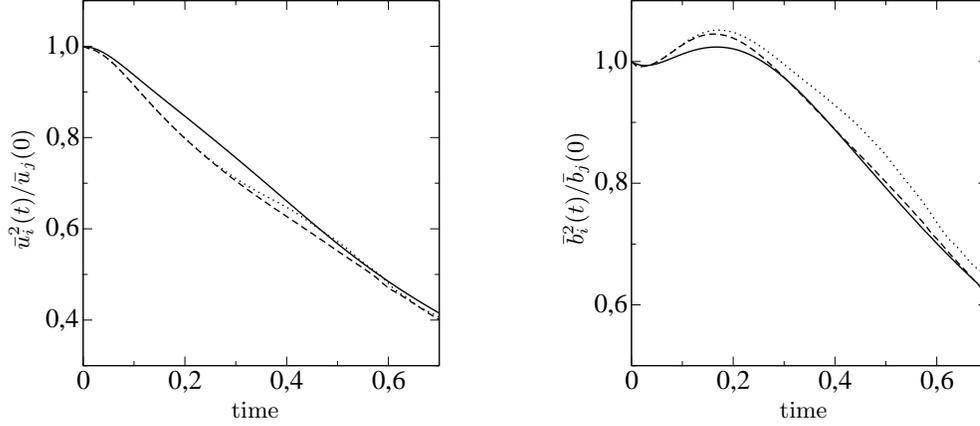
FIGURE 5. Time evolution of the filtered kinetic (left) and magnetic (right) energy. DNS (solid line), LES with mixed scale similarity model using a classical dynamic procedure (dashed line) and LES with mixed scale similarity model using a divergence based dynamic procedure (dotted line)

procedure disappears and the transfer predicted by the model is close to the transfer computed from the filtered DNS.

Direct and large-eddy simulations have been performed for decaying MHD turbulence. LES use mixed-scale-similarity models, both with classical and divergence-based dynamic procedures. For these LES, the computational domain is discretized using 64^3 grid points. The filtered Navier-Stokes equation is closed by a dynamic Smagorinsky model to evaluate $\tau_{ij}^u - \tau_{ij}^b$. The influence of the SGS Lorentz force is just taken into account in the dynamic coefficient [e.g. see Müller & Carati (2002)]. Figure 5 shows the decaying of the filtered kinetic (left) and filtered magnetic (right) energy for LES and filtered DNS. The results seem to show a better agreement with the divergence-based dynamic model, but it is difficult to be sure because of the coupling of the modeling error and an over-dissipation of the kinetic energy in the first stage of the simulations. In further work, the modeling improvement of τ_{ij}^u and τ_{ij}^b will be addressed again using a priori tests as starting point.

4. Conclusions

In this work, the modeling of the subgrid-scale (SGS) term appearing in the transport equation of the filtered magnetic field is addressed. From a priori tests, the performances of several SGS models have been evaluated. The measure of model performance is defined as structural and functional performances. The structural performance is defined as the model capacity to locally reproduce the unknown SGS term, whereas the functional performance is defined as the model capacity to reproduce the energetic action of the unknown term. The structural performance is thus evaluated by using the optimal estimation theory. This allows one to compare the models but also to evaluate the improvement possibility of a given model. The functional performance is evaluated by the comparison of the GS/SGS magnetic energy transfer given by the model with the expected GS/SGS magnetic energy transfer from the DNS data. Here in, the mixed model based on the scale-similarity model with a divergence-based dynamic procedure has the best performance. This work could be the starting point of a methodology to

improve SGS modeling in various configurations. In further work, the modeling of other SGS terms appearing in the equations of MHD flows will be addressed.

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