Time filtering in large eddy simulations

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An explicit time filter is applied to the Navier-Stokes equation prior to a space filter. The time filter is supposed to be smooth, and an exact expansion depending on the time derivatives of the velocity is derived for the associated stress tensor. On the contrary, the effect of the space filter is treated as usual and an eddy viscosity model is introduced in the LES equation. The total stress is thus represented using a new class of mixed models combining time and space derivatives of the LES field.

1. Introduction

Large eddy simulations are usually regarded as numerical experiments in which only the largest structures are computed explicitly while the effects of the small scales are modeled. The separation between large and small scales is traditionally assumed to be obtained by applying a spatial filter to the Navier-Stokes equation. Various types of filters have been introduced, like the Gaussian filter, the top-hat filter, or the Fourier cut-off. However, some of these filters, like the Gaussian and the top-hat filters, do not really reduce the number of degrees of freedom. They simply transform the turbulent field u_i into a new field v_i for which the evolution equation requires as much information to be solved as the Navier-Stokes equation. We will refer to these filters as "smooth filters." In order to reduce the number of degrees of freedom, Fourier cut-off type of filters are required. We will refer to these filters as "projective filters" since they project the turbulent field u_i onto a field $\overline{u}_i$ which can be captured on a coarser grid.

In a recent study, Carati, Winckelmans and Jeanmart (2000) have analyzed the advantages of using a combination of smooth and projective filters for defining the LES field. Their study gives some theoretical support to the use of mixed models (Vreman *et al.*, 1997, Leonard & Winckelmans, 1999, Winckelmans *et al.*, 2000) containing an eddy viscosity term representing the effects of the projective filters and a product of first order derivatives of the velocity characterizing the effects of the smooth filter. In the present study, we will investigate the possibility of replacing the smooth spatial filter by a smooth time filter.

2. Time filtering and the Navier-Stokes τ equation

2.1. *Derivation of the Navier-Stokes τ equation*

Let us consider that the Navier-Stokes equation is filtered using a time averaging:

$$v_i(t) = \int dt'\ G(\frac{t-t'}{\tau})\ u_i(t')\,. \tag{2.1}$$

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Here τ is the temporal width of the filter. In this preliminary study, it will be considered as a constant both in space and in time. This guarantees the commutation between the filtering and both the space and time derivatives. For any symmetric filter with a C^∞ Fourier transform (which is true for a very wide class of filters including the top-hat and the Gaussian filters), Carati, Winckelmans and Jeanmart have proved the following equality:

$$\int G(\frac{t-t'}{\tau})u_i(t')\,u_j(t')\,dt' = \sum_{r,s}\ c_{rs}\ \tau^{r+s}\ \partial_t^r v_i\ \partial_t^s v_j\,. \tag{2.2}$$

The coefficient c_{rs} is derived from a generating function based on the Fourier transform $\tilde{G}(\omega\tau)$ of $G(\frac{t-t'}{\tau})$. The simplest situation corresponds to the Gaussian filter, in which case the double series reduces to a single one:

$$\int G(\frac{t-t'}{\tau})u_i(t')\,u_j(t')\,dt' = \sum_{r=0}^{\infty}\ \frac{\tau^{2r}}{r!}\ \partial_t^r v_i\ \partial_t^r v_j\,. \tag{2.3}$$

The LES equation thus reads:

$$\partial_t v_i + \partial_j \sum_{r=0}^{\infty} \frac{\tau^{2r}}{r!}\frac{\partial^r v_i}{\partial t^r}\frac{\partial^r v_j}{\partial t^r} = -\partial_i p + \nu\nabla^2 v_i\,. \tag{2.4}$$

Keeping all of the terms in the series (2.2) amounts to assuming that the LES is infinitely accurate in time, which is not realistic. Hence, we have to truncate the series. In the case when only the lowest orders in τ are kept, the filtered equation reads:

$$\dot{v}_i = -\partial_i p - \partial_j\big(v_i\,v_j + \tau^2\,\dot{v}_i\,\dot{v}_j\big) + \nu\nabla^2 v_i\,, \tag{2.5}$$

where we use the same notation p in the equations for both u_i and v_i since it is only used to enforce continuity. Quite remarkably, this lowest order expansion is not restricted to the Gaussian filter and is valid for all symmetric filters with a C^∞ Fourier transform. We will refer to this generic equation as the Navier-Stokes tau (NS-τ) equation. Here $\dot{v}_i$ refers to the Eulerian time derivative: $\dot{v}_i = \partial_t v_i$. The only difference between the NS and the NS-τ equations comes from the term proportional to τ^2 that can be seen as a forcing:

$$f_i \equiv -\partial_j \dot{v}_i\,\dot{v}_j\,. \tag{2.6}$$

Strictly speaking, f_i has the dimension of a density of force divided by the square of a time since we did not include the factor τ^2 in the definition of f_i. This choice will allow us to keep track explicitly of the power of τ in each term. Of course, this notation hides the fact that the NS-τ equation is implicit in $\dot{v}_i$. Yet, it will be useful in simplifying the discussions on the balance equations.

It is important to remark that the NS-τ equation is not equivalent to the original Navier-Stokes equation since the higher order terms in the series (2.2) have been neglected. However, since the NS-τ is only intended to be used in LES, it is hoped that the higher order terms can be lumped into the subgrid-scale models when projective filters are applied to the NS-τ equation.

2.2. *Balance equations*

We have not been able to derive a conservation law for the inviscid NS-τ equation. Nevertheless, we have derived two balance equations that can be useful in code checking and in appreciating the effect of the τ term on the dynamics of v_i. First, we consider the evolution of $\langle v_i v_i/2\rangle$:

$$\frac{d}{dt}\frac{1}{2}\langle v_i v_i\rangle = -\tau^2\langle v_i \partial_j \dot{v}_i \dot{v}_j\rangle = \tau^2\langle v_i f_i\rangle\,. \tag{2.7}$$

Here, the brackets denote the volume average. Not surprisingly, since the additional τ term acts like a force, its contribution to the "energy" balance takes the usual $\langle v_i f_i\rangle$ form. The equation for the second order time derivative is readily derived from the NS-τ equation,

$$\ddot{v}_i = -\partial_i \dot{p} - \partial_j\big(\dot{v}_i\, v_j + v_i\, \dot{v}_j + \tau^2\, \ddot{v}_i\, \dot{v}_j + \tau^2\, \dot{v}_i\, \ddot{v}_j\big) + \nu\Delta\dot{v}_i\,. \tag{2.8}$$

From this, we can derive another balance equation (always in the limit of zero viscosity):

$$\frac{d}{dt}\frac{1}{2}\langle \dot{v}_i \dot{v}_i\rangle = -\langle \dot{v}_i \partial_j v_i \dot{v}_j\rangle - \tau^2\langle \dot{v}_i \partial_j \ddot{v}_i \dot{v}_j\rangle\,. \tag{2.9}$$

Using the identity $\langle v_i\partial_j \dot{v}_i\dot{v}_j\rangle + \langle \dot{v}_i\partial_j v_i\dot{v}_j\rangle = \langle \partial_j v_i\dot{v}_i\dot{v}_j\rangle = 0$, we obtain the following balance equation:

$$\frac{d}{dt}\frac{1}{2}\langle v_i v_i + \tau^2\dot{v}_i\dot{v}_i\rangle = -\tau^4\langle \dot{v}_i \partial_j \ddot{v}_i \dot{v}_j\rangle\,. \tag{2.10}$$

Finally, we remark that $\langle \dot{v}_i\partial_j \dot{v}_i\dot{v}_j\rangle = 0$. Hence, the time derivative of this quantity also vanishes, which implies, $\langle \dot{v}_i\partial_j \ddot{v}_i\dot{v}_j\rangle = -\langle \ddot{v}_i\partial_j \dot{v}_i\dot{v}_j\rangle = \langle \ddot{v}_i f_i\rangle$ (here we also have used the property $\langle \dot{v}_i\partial_j \dot{v}_i\ddot{v}_j\rangle = 0$). The balance equation (2.10) finally reads:

$$\frac{d}{dt}\frac{1}{2}\langle v_i v_i + \tau^2\dot{v}_i\dot{v}_i\rangle = \tau^4\langle \ddot{v}_i f_i\rangle\,. \tag{2.11}$$

In the limit of small τ, the right-hand side of this balance will be very small and $\langle v_i v_i + \tau^2\dot{v}_i\dot{v}_i\rangle$ should be almost conserved. This means that if the "energy" $\langle v_i v_i\rangle$ starts to grow, the average rate of change of v_i, $\langle \dot{v}_i\dot{v}_i\rangle$ should decrease, preventing an exponential growth of the energy. Although these arguments are obviously not fully rigorous, they at least support the idea that the NS-τ equation should not lead to major instabilities in the limit of small τ.

3. Spatial filtering of the NS-τ equation

Since the NS-τ equation implicitly assumes the use of a smooth time filter, its simulation is likely to require the same type of grid as the simulation of the NS equation. It is thus necessary to apply a projective filter on the NS-τ equation. The hope is that the modeling of the subgrid scale tensor will be easier for the NS-τ equation, which should contain less energy in the small scales, than for the original Navier-Stokes equation. The effect of the projective spatial filter, which will be denoted by an overbar, $\overline{v}_i$, should then be taken into account through a model. The total equation for $\overline{v}_i$ would then read:

$$\dot{\overline{v}}_i = -\partial_i\overline{p} - \partial_j\big(\overline{v}_i\,\overline{v}_j + \tau^2\,\dot{\overline{v}}_i\,\dot{\overline{v}}_j\big) + \nu\nabla^2\overline{v}_i - \partial_j\tau_{ij}\,, \tag{3.1}$$

where $\tau_{ij} = \overline{v_i\, v_j} - \overline{v}_i\,\overline{v}_j + \tau^2\left(\overline{\dot{v}_i\,\dot{v}_j} - \dot{\overline{v}}_i\,\dot{\overline{v}}_j\right)$. Let us consider that this projective filter removes all the information related to scales smaller than $\overline{\Delta}$. In this case, τ_{ij} could be modeled through an eddy viscosity term, for which we have used two types of scalings.

3.1. *Kolmogorov scaling for the eddy viscosity*

The Kolmogorov scaling for the eddy viscosity has been introduced in Wong & Lilly, (1994) and Carati *et al.* (1995):

$$\tau_{ij}^{M} \approx -2\, C_\epsilon\, \overline{\Delta}^{4/3}\, \overline{S}_{ij}\,, \tag{3.2}$$

where $\overline{S}_{ij} = (\partial_i \overline{v}_j + \partial_j \overline{v}_i)/2$. A dynamical approach (Germano, 1992, Germano *et al.*, 1991, Ghosal *et al.*, 1995) has been implemented by considering two levels of discretization. The second level of discretization corresponds to the elimination of scales smaller than $\widehat{\overline{\Delta}}$ and would be obtained either by the application of the operator $\widehat{\cdots}$ to Eq. (3.1):

$$\widehat{\dot{\overline{v}}}_i = -\partial_i \widehat{\overline{p}} - \partial_j \big(\widehat{\overline{v}}_i\, \widehat{\overline{v}}_j + \tau^2\, \widehat{\dot{\overline{v}}}_i\, \widehat{\dot{\overline{v}}}_j\big) + \nu \nabla^2 \widehat{\overline{v}}_i - \partial_j \widehat{\tau_{ij}} - \partial_j L_{ij}\,, \tag{3.3}$$

or directly by the application of the operator $\widehat{\overline{\cdots}}$ to Eq. (2.5):

$$\widehat{\overline{\dot{v}}}_i = -\partial_i \widehat{\overline{p}} - \partial_j \big(\widehat{\overline{v}}_i\, \widehat{\overline{v}}_j + \tau^2\, \widehat{\overline{\dot{v}}}_i\, \widehat{\overline{\dot{v}}}_j\big) + \nu \nabla^2 \widehat{\overline{v}}_i - \partial_j T_{ij}\,. \tag{3.4}$$

The comparison of these two versions of the same equation leads to a Germano type identity:

$$\widehat{\tau_{ij}} + L_{ij} = T_{ij}\,, \tag{3.5}$$

where

$$T_{ij} = \widehat{\overline{v_i\, v_j}} - \widehat{\overline{v}}_i\, \widehat{\overline{v}}_j + \tau^2 \left(\widehat{\overline{\dot{v}_i\, \dot{v}_j}} - \widehat{\overline{\dot{v}}}_i\, \widehat{\overline{\dot{v}}}_j\right)\,, \tag{3.6}$$

$$L_{ij} = \widehat{\overline{v}}_i\, \widehat{\overline{v}}_j - \widehat{\overline{v}_i\, \overline{v}_j} + \tau^2 \left(\widehat{\dot{\overline{v}}}_i\, \widehat{\dot{\overline{v}}}_j - \widehat{\dot{\overline{v}}_i\, \dot{\overline{v}}_j}\right)\,. \tag{3.7}$$

Assuming that C_ϵ is independent of the level of discretization, T_{ij} has to be modeled using the same Kolmogorov scaling $T_{ij}^{M} \approx -2\, C_\epsilon\, \widehat{\overline{\Delta}}^{4/3}\, \widehat{\overline{S}}_{ij}$. If we assume, moreover, that C_ϵ is constant in space, this parameter can be derived by minimizing the volume average of the square of the difference between the right-hand side and the left-hand side of:

$$-2\, C_\epsilon\, \overline{\Delta}^{4/3}\, \widehat{\overline{S}}_{ij} + L_{ij} \approx -2\, C_\epsilon\, \widehat{\overline{\Delta}}^{4/3}\, \widehat{\overline{S}}_{ij}\,, \tag{3.8}$$

which leads to:

$$C_\epsilon \approx \frac{1}{2\,(\widehat{\overline{\Delta}}^{4/3} - \overline{\Delta}^{4/3})}\, \frac{\langle L_{ij}\, \widehat{\overline{S}}_{ij}\rangle}{\langle \widehat{\overline{S}}_{ij}\, \widehat{\overline{S}}_{ij}\rangle}\,. \tag{3.9}$$

3.2. *Smagorinsky scaling for the eddy viscosity*

The Smagorinsky (1963) scaling for the eddy viscosity has also been used

$$\tau_{ij}^{M} \approx -2\, C\, \overline{\Delta}^2 \sqrt{2 \overline{S}_{kl} \overline{S}_{kl}}\, \overline{S}_{ij}\,. \tag{3.10}$$

In which case, the dynamic prediction for C reads:

$$C \approx \frac{1}{2\,(\widehat{\overline{\Delta}}^2 - \overline{\Delta}^2)}\, \frac{\langle L_{ij}\, M_{ij}\rangle}{\langle M_{ij}\, M_{ij}\rangle}\,, \tag{3.11}$$

where

$$M_{ij} = \sqrt{2 \widehat{\overline{S}}_{kl} \widehat{\overline{S}}_{kl}}\, \widehat{\overline{S}}_{ij} - \widehat{\sqrt{2 \overline{S}_{kl} \overline{S}_{kl}}\, \overline{S}_{ij}}\,. \tag{3.12}$$

4. Solving the spatially filtered NS-τ equation

Solving the NS-τ equation requires solving an implicit quadratic equation for $\dot{v}_i$. Different methods have been considered. First, an iterative scheme has been implemented for solving the unfiltered NS-τ equation:

$$\dot{v}_i^{n+1} = (1-\lambda)\dot{v}_i^n + \lambda\left(-\partial_i p - \partial_j\big(v_i\, v_j + \tau^2\, \dot{v}_i^n\, \dot{v}_j^n\big) + \nu\nabla^2 v_i\right). \tag{4.1}$$

Although this method converges, it usually requires many iterations, especially for $\tau/dt \gg 1$. However, when the spatially filtered NS-τ equation is considered, convergence properties appear to be much more favorable. For this reason, we adopted the simplest scheme, namely using the previous time step value of $\dot{v}_i$ in the the right-hand side of the spatially filtered NS-τ equation:

$$\dot{v}_i(t+dt) = -\partial_i p - \partial_j\big(v_i\, v_j + \tau^2\, \dot{v}_i(t)\, \dot{v}_j(t)\big) + \nu\nabla^2 v_i - \partial_j \tau_{ij}\,. \tag{4.2}$$

We also report an alternative technique that could be used for solving this implicit equation: by using the equation for the second order time derivative, one still has to solve an implicit relation (for $\ddot{v}_i$), but it is now linear in $\ddot{v}_i$. This motivates the formulation of the NS-τ equation in terms of two variables v_i and $w_i \equiv \dot{v}_i$. The filtered version of these equations is

$$\dot{\overline{v}}_i = \overline{w}_i\,, \tag{4.3}$$

$$\dot{\overline{w}}_i = \nu\nabla^2\overline{w}_i - \partial_i\mathcal{P} - \partial_j\big(\overline{w}_i\overline{v}_j + \overline{v}_i\overline{w}_j + \tau^2\dot{\overline{w}}_i\overline{w}_j + \tau^2\overline{w}_i\,\dot{\overline{w}}_j\big) - \partial_j\mathcal{T}_{ij}\,, \tag{4.4}$$

where $\mathcal{T}_{ij} = -2\,C_\epsilon\,\overline{\Delta}^{4/3}\mathcal{S}_{ij}$ and $\mathcal{S}_{ij} = (\partial_i\overline{w}_j + \partial_j\overline{w}_i)/2$. The parameter C_ϵ is the same as in (3.9). The only approximation here is that the time derivative of C_ϵ can be neglected. The pressure-type term $\partial_i\mathcal{P}$ is only used for enforcing incompressibility.

This alternative method also has the advantage of carrying all the information required to check the second balance equation derived in Section 2. However, we have not yet implemented this technique.

5. Numerical results

Various tests have been made for different values of τ/dt. It must be noted that this ratio is similar to the ratio between the filter width Δ and the mesh spacing dx in more traditional LES. It is usually considered that $\Delta/dx \geq 1$ is a minimal requirement for numerical accuracy and that much larger values of this ratio will waste computational resources. Hence, values of Δ/dx somewhat larger than 1 are usually considered. The same approach has been adopted here. We have found that $\tau/dt \approx 1$ produces results indistinguishable from $\tau = 0$ and that large values of τ/dt lead to numerical instability when the previous time step value of $\dot{\overline{v}}_i$ is used on the right-hand side of the NS-τ equation. We thus adopted a value, $\tau/dt = 4$, for which the effect of the τ term is observable while the convergence of the NS-τ equation is still ensured.

Tests with both the Kolmogorov and the Smagorinsky scalings for the eddy viscosity have been made. In both cases, the model parameter has been computed dynamically. Tests have been made for decaying turbulence, comparing a 512^3 DNS with a 64^3 LES. The temporal evolution of both the resolved energy and the resolved dissipations are presented in Figs. 1-4. It should be noted that the dynamic Smagorinsky model is known to produce good results for this case though it systematically overpredicts the total en-

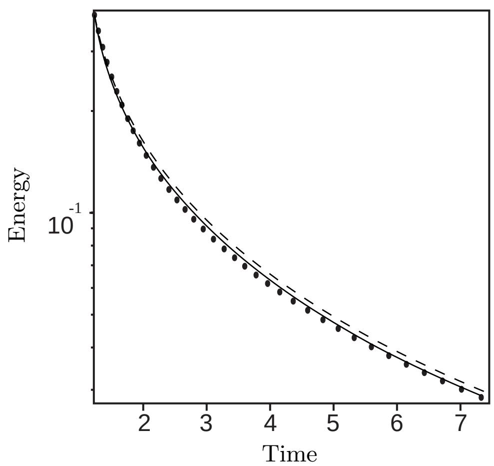

FIGURE 1. Decay of the resolved energy for truncated DNS (•), LES with the dynamic Smagorinsky model (– – – –) and LES with the dynamic Smagorinsky model and the τ term (———).

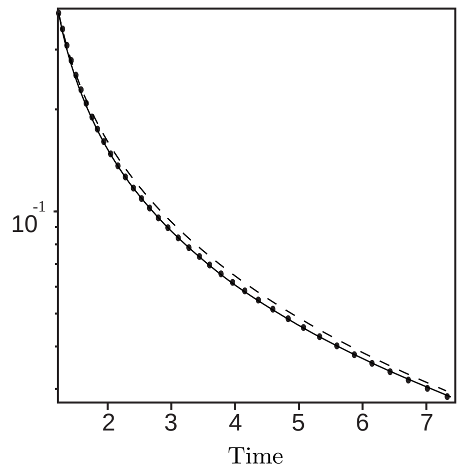

FIGURE 2. Decay of the resolved energy for truncated DNS (•), LES with the dynamic Kolmogorov model (– – – –) and LES with the dynamic Kolmogorov model and the τ term (———).

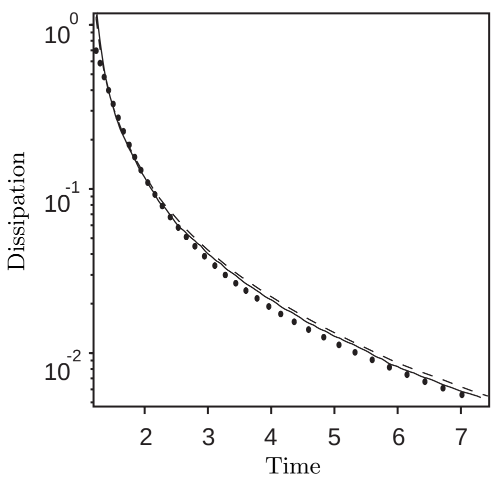

FIGURE 3. Decay of the resolved dissipation for truncated DNS (•), LES with the dynamic Smagorinsky model (– – – –) and LES with the dynamic Smagorinsky model and the τ term (———).

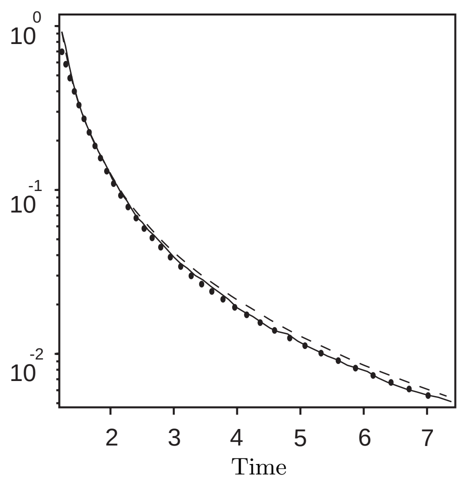

FIGURE 4. Decay of the resolved dissipation for truncated DNS (•), LES with the dynamic Kolmogorov model (– – – –) and LES with the dynamic Kolmogorov model and the τ term (———).

ergy. Also, without the τ term, the Smagorinsky and Kolmogorov scalings are known to produce almost indistinguishable results. The results presented in Figs. 1-4 are interesting in two respects. First, the effect of the time filtering, though quite limited, definitely improves the prediction for both the resolved energy and the resolved dissipation. Remarkably, the improvement is systematically better when the Kolmogorov scaling is used

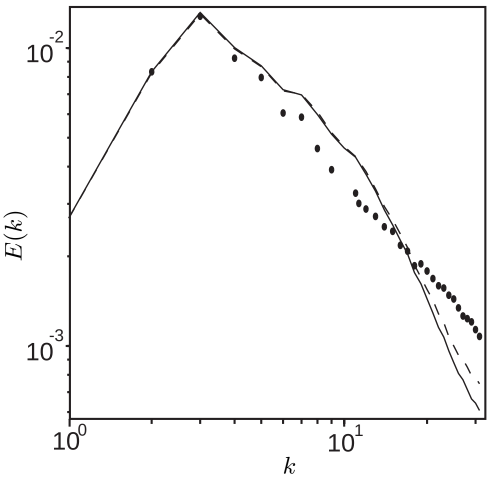

FIGURE 5. Energy spectrum at $t = 2.6$ in the units of Figs. 1-4 for truncated DNS (•), LES with the dynamic Smagorinsky model (– – – –) and LES with the dynamic Smagorinsky model and the τ term (———).

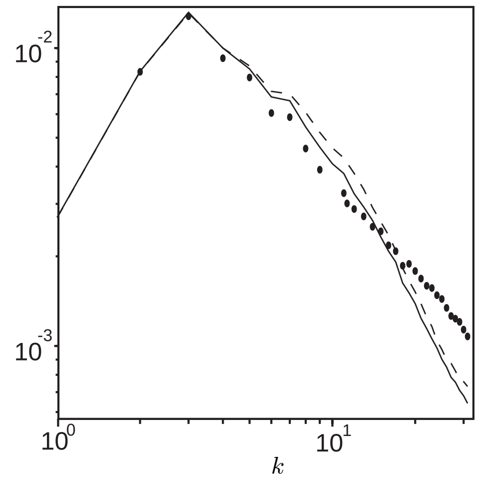

FIGURE 6. Energy spectrum at $t = 2.6$ in the units of Figs. 1-4 for truncated DNS (•), LES with the dynamic Kolmogorov model (– – – –) and LES with the dynamic Kolmogorov model and the τ term (———).

for the eddy viscosity. Actually, when this model is used for the NS-τ equation, the results almost perfectly fit the DNS results.

It should be noted, however, that the energy spectra do not show such a perfect agreement between DNS and LES. In all cases the energy in the low wavenumber range of the LES is overpredicted while the high wavenumber energy is underpredicted. Also, the difference between the Smagorinsky and Kolmogorov scalings is more clearly observed in the spectra. The addition of the τ term does not improve the performance of the dynamic Smagorinsky model in the energy-containing range, but the overprediction of the energy in the low wavenumber range is somewhat improved when the τ term is added to the dynamic Kolmogorov scaling.

6. Conclusion

The combination of a smooth time filter and a projective space filter has been proposed and tested for LES of decaying isotropic turbulence. The effect of the smooth time filtering has been taken into account through an exact expansion which does not require any modeling. Only the first term of this expansion, which is generic for a wide class of smooth filters, has been retained in our approach. The resulting equation, referred to as the Navier-Stokes τ equation, only differs from the original equation by a new forcing term depending on the first order time derivatives of the velocity. As a consequence, the Navier-Stokes τ equation is implicit in this time derivative. Three major points have been learned from this study:

(1) Using the previous time step value of the time derivative in the new term representing the effect of the time filter has been shown to be easily implementable and stable for values of τ/dt of the order of 4.

(2) The effect of the new term is rather limited but does tend to improve the predictions when compared to a pure eddy viscosity model.

(3) The mixed model including the τ term and a dynamic eddy viscosity term is more

sensitive to the type of scaling used for the eddy viscosity than are purely dynamic eddy viscosity models.

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