On stably stratified homogeneous shear flows subjected to rotation

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Theoretical aspects of modeling stratified turbulent flows subjected to rotation are considered. The structural equilibrium behavior of second-moment closure (SMC) models is explored, guided by bifurcation analysis. It is shown that the ability of the models to predict a critical gradient Richardson number in the absence of system rotation $Ri_g^{cr} \approx 0.25$ is largely dependent on the model for the pressure-strain correlation tensor. It is also found that the most commonly used linear models are ill-posed when the combined effect of system rotation and stratification is imposed; the models do not exhibit a steady state solution.

1. Introduction

The combined effect of body forces associated with density stratification, system rotation, and streamline curvature is important in a wide variety of turbulent flows. These body forces exert profound effects on the vertical mixing in, for instance, geophysical boundary layers as well as in engineering applications such as turbomachinery flows. The importance is manifested through the direct effects on the turbulence fluctuations; the turbulence can be suppressed (stabilized) or enhanced (destabilized). For instance, the stabilizing effect associated with the flow over a convex surface leads to significantly reduced skin friction.

The most physically appealing approach to account for body force effects within the framework of RANS modeling is full second-moment closures (SMC). The main reason is that body force effects are accounted for in a systematic manner. The more frequently adopted eddy-viscosity type closures such as the standard $k-\varepsilon$ model are not particularly well suited for these situations. It is, therefore, unfortunate that the more elaborate SMC models are not very often employed in practice for complex fluid flow predictions mainly because of numerical stiffness problems.

Analysis of homogeneous shear flow provides theoretical insight on the stabilizing and destabilizing effects of imposed system rotation, flow curvature, or density stratification on the turbulent stresses. This is particularly valuable from a turbulence modeler's perspective since it also provides a mean for systematic approximations of SMC closures.

Equilibrium and bifurcation analysis have emerged as simple and powerful guides to closure model formulations. Analytical solutions of SMCs provide a methodology for a systematic derivation of simpler eddy-viscosity models that retain some of the predictive capabilities of the more elaborate SMC models, in particular, the ability to respond to imposed body forces. Although homogeneous shear flow is superficially simple, it has

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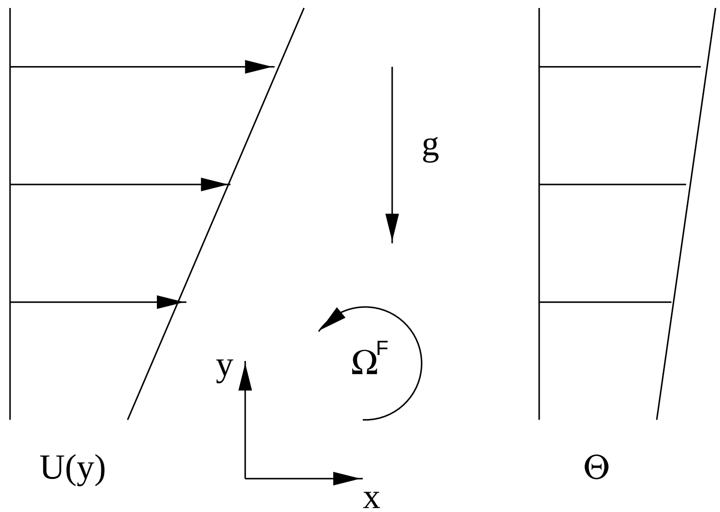

FIGURE 1. Schematic of flow configuration.

proven to be a useful reference point – including for models intended to compute complex flows far from equilibrium.

The present study examines the response of full SMC models in stratified homogeneous shear flow subjected to orthogonal mode rotation, see Fig. 1. The analysis is based on structural equilibrium, and the behavior of these models is elucidated by bifurcation analysis. In contrast to the modeling of passive scalars, density stratification mathematically couples the equations governing the Reynolds stresses and the turbulent heat fluxes; this significantly adds to the complexity of modeling stratified turbulence. The motivation of this study is two-fold: first, to provide some theoretical guidance to closure formulations, in particular for turbulent heat flux modeling, and second, to serve as guidance for improved eddy-viscosity formulations. Equilibrium and bifurcation analyses have successfully been used to incorporate rotational effects in eddy-viscosity models in Pettersson Reif, Durbin & Ooi (1999).

The theoretical analysis is based on the *general* linear models for the Reynolds stresses and turbulent heat fluxes. However, only numerical results obtained with the most widely used model, the IP model, is presented here, but the generality of the approach sets the stage for analyzing all existing linear (or quasi-linear) SMC models. Similar analyses of the Mellor-Yamada SMC have previously been performed, cf. e.g. Hassid & Galperin (1994), Baumert & Peters (2000) and Kantha, Rosati & Galperin (2000). Mellor-Yamada, which also is a subset of the general linear formulation, has received much attention in the geophysical fluid dynamics community, but it has rarely been used in engineering fluid flow predictions; the reason is mainly due to the highly simplified pressure-strain correlation model.

2. Equilibrium and bifurcation analysis

Structural equilibrium of the Reynolds stresses is defined by constant values of the anisotropy tensor $d_t b_{ij} = d_t(\overline{u_i u_j}/k) = 0$ and of the turbulent to mean flow time-scale ratio $d_t(\mathcal{S}k/\varepsilon) = 0$ where $\mathcal{S}$ is the mean shear dU/dy. True structural equilibrium requires also that $d_t(S_{ij}k/\varepsilon) = 0 = d_t(\Omega^A_{ij}k/\varepsilon)$ where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right) \tag{2.1}$$

and

$$\Omega^A_{ij} = \frac{1}{2}\left(\frac{\partial U_j}{\partial x_i} - \frac{\partial U_i}{\partial x_j}\right) + \epsilon_{ijk}\Omega^F_k \tag{2.2}$$

are the mean rate of strain and absolute mean vorticity tensors, respectively. Ω^F_k is the angular velocity of the reference frame about the x_k axis. It should be noted that k and ε themselves are not in general constants at equilibrium. However, the evolution of the kinematic Reynolds stresses $\overline{u_iu_j}$ is the same as k; this is the reason why $d_tb_{ij} = 0$. Similarly, the evolution of the turbulent and mean flow time scales are also identical but not necessarily $d_t(k/\varepsilon) = 0 = d_t\mathcal{S}$.

Equilibrium of the turbulent fluxes is defined by $d_tK_{ij} = 0$ and $d_t(\partial_i\Theta) = -\partial_iU_k\partial_k\Theta$ where $\mathbf{K}$ is the dispersion tensor and Θ is the mean temperature. The dispersion tensor relates the turbulent fluxes to the mean temperature field;

$$\overline{u_i\theta} = -K_{ij}\frac{k^2}{\varepsilon}\frac{\partial\Theta}{\partial x_j}. \tag{2.3}$$

For equilibrium to be attained, the mean temperature gradient is allowed to evolve as $d_t(\partial_j\Theta) = -\partial_jU_k\partial_k\Theta$.

2.1. *General linear models*

Under the assumption of homogeneity and local isotropy, the transport of the kinematic Reynolds stress tensor in a noninertial frame of reference is governed by

$$d_t\overline{u_iu_j} = \mathcal{P}_{ij} + \mathcal{R}_{ij} + \mathcal{G}_{ij} - \frac{2}{3}\varepsilon\delta_{ij} + \phi_{ij} \tag{2.4}$$

where

$$\begin{aligned}\mathcal{P}_{ij} &= -\left(\overline{u_iu_k}\partial_kU_j + \overline{u_ju_k}\partial_kU_i\right)\\ \mathcal{R}_{ij} &= -2\Omega^F_k\left(\epsilon_{jkl}\overline{u_iu_l}\right)\\ \mathcal{G}_{ij} &= -\beta\left(g_i\overline{u_j\theta} + g_j\overline{u_i\theta}\right).\end{aligned}$$

are the rate of production due to mean shear, system rotation, and buoyancy, respectively. The last term, which associated with buoyancy production, is responsible for the intercoupling between the turbulent stress and heat flux fields. The most general linear model for the pressure-strain correlation tensor ϕ_{ij} can be written as

$$\begin{aligned}\phi_{ij}/k = -C_1b_{ij}\varepsilon/k + 4/5S_{ij} + (C_2 + C_3)\left(b_{ik}S_{kj} + b_{jk}S_{ki} - \tfrac{2}{3}\delta_{ij}b_{mn}S_{nm}\right)\\ + (C_2 - C_3)\left(b_{ik}\Omega^A_{kj} + b_{jk}\Omega^A_{ki}\right) + C_4/k\left(\mathcal{G}_{ij} - \tfrac{1}{3}\delta_{ij}\mathcal{G}_{kk}\right).\end{aligned}$$

The corresponding equations for the turbulent kinetic energy and the dissipation rate are

$$\begin{aligned}d_tk &= \mathcal{P} - \varepsilon\\ d_t\varepsilon &= \tfrac{\varepsilon}{k}\left(C_{\varepsilon1}\mathcal{P} - C_{\varepsilon2}\varepsilon\right)\end{aligned}$$

where $\mathcal{P} = \frac{1}{2}(\mathcal{P}_{kk} + \mathcal{G}_{kk})$. The evolution of the Reynolds stress anisotropy tensor, $b_{ij} = \overline{u_i u_j}/k - 2/3\delta_{ij}$, and the turbulent to mean flow time-scale ratio can then be written as

$$\begin{aligned} d_t b_{ij} = & -\tfrac{8}{15} S_{ij} + b_{ij}\varepsilon/k\,(1 - C_1 - \mathcal{P}/\varepsilon) \\ & + (C_2 + C_3 - 1)\left(b_{ik}S_{kj} + b_{jk}S_{ki} - \tfrac{2}{3}\delta_{ij} b_{mn} S_{nm}\right) \\ & + (C_2 - C_3 - 1)\left(b_{ik}\Omega^A_{kj} + b_{jk}\Omega^A_{ki}\right) \\ & + (1 - C_4)\left(\mathcal{G}_{ij}/k - \tfrac{2}{3}\delta_{ij}\mathcal{G}/k\right) - \Omega^F_k\,(b_{il}\epsilon_{jkl} + b_{jl}\epsilon_{ikl})\,. \end{aligned}$$

and

$$d_\tau\left(\frac{\varepsilon}{\mathcal{S}k}\right) = \frac{\varepsilon}{\mathcal{S}k}\left((C_{\varepsilon 1} - 1)\frac{\mathcal{P}}{\varepsilon} - (C_{\varepsilon 2} - 1)\right) \tag{2.5}$$

where $\tau = \mathcal{S}t$.

Under homogeneous conditions, the transport of turbulent heat flux is governed by

$$d_t\overline{u_i\theta} = \mathcal{P}_{i\theta} + \mathcal{G}_{i\theta} + \phi_{i\theta} \tag{2.6}$$

where

$$\begin{aligned} \mathcal{P}_{i\theta} &= -\overline{u_i u_j}\partial_j\Theta - \overline{u_j\theta}\partial_j U_i \\ \mathcal{G}_{i\theta} &= -\beta g_i \overline{\theta^2} \end{aligned}$$

are the production terms. The corresponding general linear 'pressure-strain' model for turbulent fluxes can be written as

$$\begin{aligned} \phi_{ic} = -C_{1c}\varepsilon/k\overline{u_i\theta} + (C_{2c} + C_{3c})\,\overline{u_k\theta}S_{ki} + (C_{2c} - C_{3c})\,\overline{u_i\theta}\Omega^A_{ki} + C_{4c}\overline{u_i u_j}\partial_j\Theta \\ + C_{5c}\beta g_i\overline{\theta^2}. \end{aligned}$$

The last term in (2.7) depends on the *a priori* unknown temperature variance $\overline{\theta^2}$. A simple model equation for the temperature variance is adopted:

$$d_t\overline{\theta^2} = -2\overline{u_k\theta}\partial_k\Theta - C_{\mathcal{R}}\frac{\varepsilon}{k}\overline{\theta^2}. \tag{2.7}$$

The coefficient $C_{\mathcal{R}}$ denotes the ratio between the heat flux time-scale $\overline{\theta^2}/\varepsilon_\theta$ and the mechanical time-scale k/ε. $C_{\mathcal{R}}$ is for simplicity assumed to be constant; the alternative would be to solve a transport equation for the heat flux dissipation rate ε_θ.

The equation governing the evolution of the dispersion tensor $\mathbf{K}$, defined in (2.3), can then be written as

$$\begin{aligned} \partial_j\Theta d_t K_{ij} = & \\ \partial_j\Theta K_{ij}\varepsilon/k\,[\mathcal{P}/\varepsilon(C_{\varepsilon 1} - 2) + 2 - C_{\varepsilon 2}] & + K_{ij}\partial_k\Theta\left(S_{jk} + \Omega^A_{jk}\right) \\ & + (1 - C_{4c})\,\overline{u_i u_k}\varepsilon/k^2\partial_k\Theta \\ & - (1 - C_{2c} - C_{3c})\,K_{kj}S_{ki} - (1 - C_{2c} + C_{3c})\,K_{kj}\Omega^A_{ki} \\ & + (1 - C_{5c})\,\beta g_i\overline{\theta^2}\varepsilon/k^2 - C_{1c}\varepsilon/kK_{ij}\partial_j\Theta. \end{aligned}$$

The task is now to solve (2.5) – (2.5) and (2.7) – (2.8) for homogeneous shear flow subjected to orthogonal mode rotation, i.e. $\mathbf{U} = [U(y), 0, 0]$, $\mathbf{\Omega^F} = [0, 0, \Omega^F]$, $\mathbf{g} = [0, -g, 0]$ and $\mathbf{\Theta} = \Theta(y)$, see Fig. 1. The equilibrium solution of this set of equations depends on two parameters: $\Omega^F/\mathcal{S}$ and Ri_g. The gradient Richardson number is defined as $Ri_g = \beta g_2 \partial_y\Theta/\mathcal{S}^2$. Recall that $\mathcal{S} = dU/dy$.

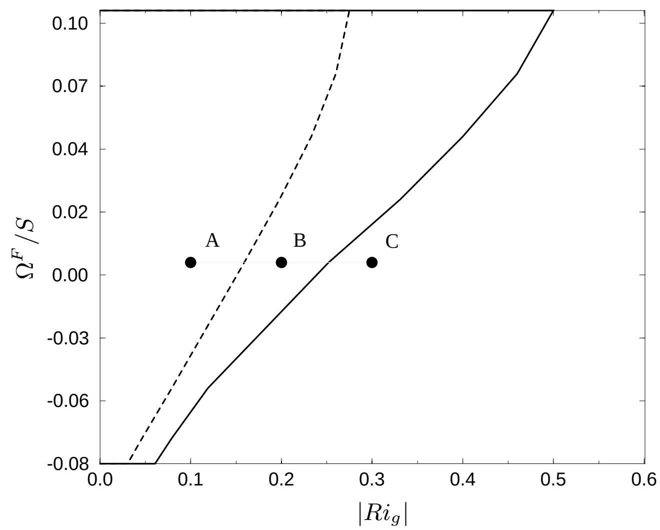

FIGURE 2. Bifurcation diagram for the IP model with $C_4 = 0.1$. Bifurcation line: ---- ; Stabilization line: —— .

The model coefficients for the IP model are

C_1	C_2	C_3	C_4	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$
1.8	3/5	0.0	0.5	1.5	1.8

whereas the following coefficients in the turbulent flux equation are used

C_{1c}	C_{2c}	C_{3c}	C_{4c}	$C_{\mathcal{R}}$
1.8	2.5	0.0	0.45	1.4

The value $C_4 = 0.5$ is the computationally optimized value suggested by Gibson & Launder (1978).

2.2. *Bifurcation of equilibria*

Equation (2.5) has two solutions for $d_t(\varepsilon/\mathcal{S}k) = 0$: (i) $\mathcal{P}/\varepsilon = (C_{\varepsilon 2} - 1)/(C_{\varepsilon 1} - 1)$ and (ii) $lim_{t\to\infty}\varepsilon/\mathcal{S}k = 0$. These solution branches, in the three-dimensional phase space $\varepsilon/\mathcal{S}k - Ri_g - \Omega^F/\mathcal{S}$, are referred to as the nontrivial and trivial branch, respectively. The solution on the nontrivial branch is thus $\varepsilon/\mathcal{S}k = Fcn(\Omega^F/\mathcal{S}, Ri_g)$ and $\mathcal{P}/\varepsilon = const$. It is associated with the exponential solution $k \propto e^{\sigma t}$ where $\sigma \propto \varepsilon/\mathcal{S}k$ depends on the model. On the trivial branch, $\varepsilon/\mathcal{S}k = 0$, and $\mathcal{P}/\varepsilon = Fcn(\Omega^F/\mathcal{S}, Ri_g)$ depends on the model. This branch is associated with an algebraic solution $k \propto t^{\lambda}$ where $\lambda \propto \mathcal{P}/\varepsilon - 1$. So as bifurcation occurs, i.e. when $\varepsilon/\mathcal{S}k = 0$, the exponential evolution of k is replaced by an algebraic. But more importantly, the argument of the algebraic solution becomes *negative* at certain values of $\Omega^F/\mathcal{S}$ and Ri_g, i.e $\mathcal{P}/\varepsilon < 1$. Hence, the evolution of k changes from an algebraic growth to an algebraic decay beyond the so-called point of stabilization (or neutral point): $\mathcal{P}/\varepsilon = 1$. The turbulence is thus *stabilized* by the imposed body force. It is this particular feature of SMC models that make them superior to traditionally $k - \varepsilon$ models. A more comprehensive description of bifurcation analysis can be found in Durbin & Pettersson Reif (1999).

The response of a particular closure model to an imposed body force can be neatly

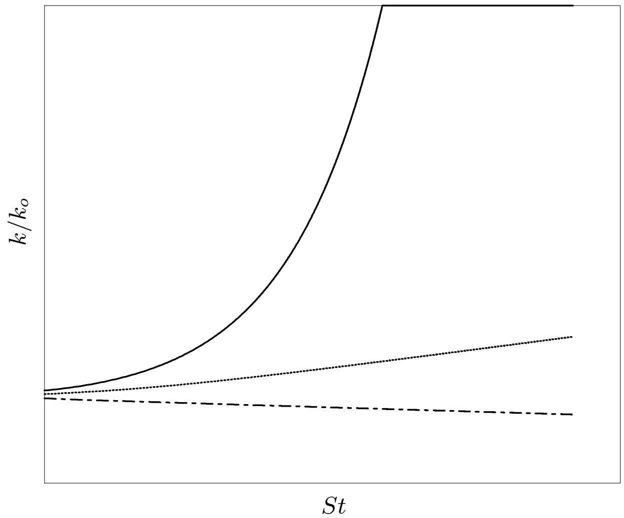

FIGURE 3. Evolution of turbulent kinetic energy in different regimes in the $\Omega^F/\mathcal{S}$ and Ri_g space, see Fig. 2. A: ——— ; B: ········ ; C: – – – – . The turbulent kinetic energy has been normalized with its initial value.

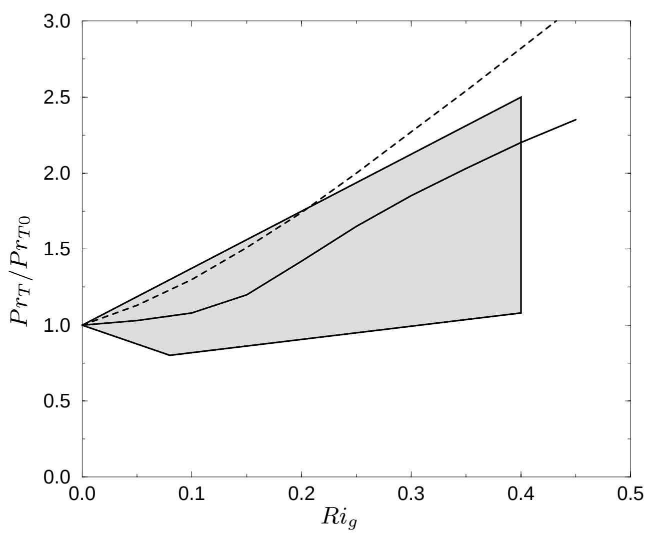

FIGURE 4. Turbulent Prandtl number as a function of gradient Richardson number ($\Omega^F/\mathcal{S} = 0$). The shaded region corresponds to the data reported by Rohr, Itsweire, Helland & Van Atta (1988). The lines are IP preditions; ——— : $C_4 = 0.1$; – – – – : $C_4 = 0.5$. The turbulent Prandtl number is normalized with its value at $Ri_g = 0$.

summarized in a bifurcation diagram, i.e. a plot that shows $\varepsilon/\mathcal{S}k$ versus Ri_g and $\Omega^F/\mathcal{S}$. Another issue related to the equilibrium solution of the model equations is whether it is stable or not. Stability in this context should not be confused with stabilization of turbulence; it relates to a characteristic of the solution to the governing model equations. Several solutions $\varepsilon/\mathcal{S}k$ might exist for the same pair Ri_g and $\Omega^F/\mathcal{S}$, but there should only be one physically realistic stable solution in order for the model formulation to make sense.

3. Results

There is experimental evidence that the critical mean gradient Richardson number $Ri_g^{cr} \approx 0.25$ ($\Omega^F/\mathcal{S} = 0$); fully turbulent conditions cease to exist as Ri_g is increased beyond Ri_g^{cr}. It has also been well established that homogeneous shear flow relaminarizes due to an imposed system rotation outside the range $-0.1 < \Omega^F/\mathcal{S} < 0.5$. In contrast to pure stratification, there exist *two* critical rotation numbers. SMC models tend to predict, in agreement with linear stability analysis, maximum growth rate of turbulent kinetic energy at $\Omega^F/\mathcal{S} \approx 0.25$ ($Ri_g = 0$); as the rotation rate is increased from $\Omega^F/\mathcal{S} = 0$ to $\Omega^F/\mathcal{S} = 0.25$, the turbulence is *destabilized*, i.e. the turbulent time scale k/ε decreases. In a stably stratified environment, the magnitude of the critical gradient Richardson number is, therefore, expected to increase for positive $\Omega^F/\mathcal{S}$ up to some point where it again is reduced. On the other hand, for negative and sufficiently high positive rotation rates, $|Ri_g^{cr}|$ is expected to be reduced.

Figure 2 shows the bifurcation and stabilization lines for the IP model. The predictions partly confirm the above conjecture; the magnitude of the critical Richardson number is increased as the rotation is increased. It should be noted that the constant $C_4 = 0.1$ has been used instead of the original value 0.5 suggested by Gibson & Launder (1978) in order to predict $Ri_g^{cr} \approx 0.25$ at $\Omega^F/\mathcal{S} = 0$. The critical gradient Richardson number corresponds to the stabilization line in the figure. If $C_4 = 0.5$ is used, the model predicts $Ri_g^{cr} \approx 0.8$, which is inconsistent with laboratory observations. The accompanying results in Fig. 3 show the evolution of turbulent kinetic energy in the different regions of the bifurcation diagram; it exhibits an algebraic or exponential behavior depending on the values of the imposed parameters.

The turbulent Prandtl number $Pr_T = Ri_g b_{12}/K_{22} = Ri_g/Ri_f$ depends on the predictions at a given Ri_g. Figure 4 displays the turbulent Prandtl number as a function of gradient Richardson number. Two different IP predictions are included in the figure to illustrate the dependence of the model constant C_4. Rohr, Itsweire, Helland & Van Atta (1988) argued that the Pr_T not is strongly dependent on Ri_g as long as $|Ri_g| < |Ri_g^{cr}|$. This behavior is reproduced by the IP model if the value $C_4 = 0.1$ is used.

4. Concluding remarks

It has been demonstrated that the model coefficient C_4 is crucial in terms of the ability to predict $Ri_g^{cr} \approx 0.25$ at zero rotation; the commonly used value $C_4 = 0.4$ should be replaced by $C_4 \approx 0.1$. The experimentally observed variation of the turbulent Prandtl number with the gradient Richardson number can then also be reproduced by the model.

It has only been possible to map the bifurcation diagram in the $\Omega^F/\mathcal{S} - Ri_g$ phase space within a limited range of rotation rates ($Ri_g \neq 0$), see Fig. 2. This behavior seems closely related to the particular set of model coefficients that is used. However, all commonly known linear and quasi-linear SMCs have been tested, and all of them seem to be 'ill-posed'. It is also interesting to note that under certain conditions the solution of an SMC takes the form of relaxed oscillations; no numerically stable steady state solution could be found for a given set of $\Omega^F/\mathcal{S} - Ri_g$.

It has been demonstrated that the stabilizing effect of stratification can be properly accounted for in a non-rotating frame of reference. The fact that all existing linear models seem ill-posed as system rotation is imposed is a bit worrying. Further work is definitely needed in order to resolve this problem, which probably also is of concern when computing complex flows relevant in engineering or geophysics. The added complexity of the model equations that results from an imposed density stratification makes this task a challenge.

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