Large-eddy simulations with explicit equations for subgrid-scale quantities

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Various alternative formulations of the LES equations have been explored in which additional evolution equations for variables such as the acceleration, the subgrid-scale stress tensor, or the subgrid-scale force are explicitly carried. Statistics of the velocity field obtained from the equation for the acceleration are shown to depend strongly on the initial conditions. This feature, which is independent of LES modeling issues, seems to prove that the velocity-acceleration formulation of the Navier-Stokes is not useful for numerical simulation. Equations for the subgrid-scale quantities appear to be much more stable. However, models required by this formulation of the LES problem still require additional study.

1. Introduction

The Navier-Stokes equation is usually written in terms of the velocity. However, it is legitimate to re-explore the turbulence problem by considering new formulations which are obtained by a change of variables. For instance, the equation for the vorticity can be considered as such an alternative formulation. The change of variables is in this case linear, and the two large-eddy simulation (LES) problems, obtained by projecting these two equations on a coarse grid, are more or less equivalent (Winckelmans *et al*, 1996). If, however, a nonlinear change of variables is considered, then the new LES problem would differ from the classical formulation of subgrid-scale modeling. The present project is devoted to the investigation of new LES formulations in which equations for certain nonlinear functions of the velocity are carried explicitly.

Our first attempt in that direction consisted in writing an explicit equation for the entire right-hand side of the velocity evolution equation, i.e., for the acceleration. In that case, the DNS formulation of the problem itself seems to be ill-conditioned since the turbulence statistics obtained from the acceleration equation appear to depend rather strongly on the initial conditions. For that reason, the related LES formulation has not been explored further.

Our second attempt in deriving a new formulation of the LES problem was to write explicit equations for the unknown subgrid-scale quantities. Equations for the Reynolds stress tensor in RANS have long been explored (Rodi, 1979), but in the LES context equations for the subgrid-scale stress have almost never been considered. One possible reason is that in LES it is a common practice to push the resolution to the limit of the available computational capacities. Consequently, carrying additional equations for the unknown subgrid-scale quantities has not been considered as a viable alternative to their modelling. Also, lack of knowledge about the phenomenology associated with the

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dynamics of the subgrid-scale quantities has probably discouraged LES practitioners from exploring these equations. One exception is the equation for the trace of the subgrid-scale stress tensor, often referred to as the turbulent kinetic energy, which has been used together with the resolved velocity equation. Our objective is to generalize and systematize this approach.

2. LES based on an equation for the acceleration

In a first attempt to derive an alternative formulation of the LES problem, we have considered the incompressible Navier-Stokes equation written in terms of the acceleration variable:

$$\partial_t u_i = a_i \tag{2.1a}$$

$$a_i = -\partial_j u_i u_j - \partial_i p + \nu \nabla^2 u_i \tag{2.1b}$$

$$\nabla^2 p = -\partial_i \partial_j u_i u_j \tag{2.1c}$$

Obviously equations (2.1) are closed and there is no need for an additional evolution equation for $\vec{a}$. Nevertheless, such an evolution equation can be derived easily, and the resulting equivalent formulation of the incompressible Navier-Stokes reads:

$$\partial_t u_i = a_i \tag{2.2a}$$

$$\partial_t a_i = -\partial_j \left(u_i a_j + u_j a_i\right) + \nu \nabla^2 a_i - \partial_i p_a \tag{2.2b}$$

Here the 'pseudo pressure' p_a has been introduced to enforce incompressibility of the vector $\vec{a}$. Like the pressure in the classical formulation of the incompressible Navier-Stokes equations, it is obtained by solving a Poisson equation. Since equation (2.2a) is linear, imposing incompressibility on both $\vec{a}$ and the initial condition $\vec{u}(t_0)$ guarantees that the velocity field remains divergence free. The set of equations (2.2) is thus equivalent to the original set of equations (2.1) provided a_i is properly initialized as in (2.1b). For DNS, solving (2.2) would be more complicated that solving (2.1), and there is no reason to consider this formulation. However, filtering the two sets of equations leads to different modeling problems. Filtering the set (2.1) yields the classical LES formulation. Filtering equations (2.2) yields:

$$\partial_t \overline{u}_i = \overline{a}_i \tag{2.3a}$$

$$\partial_t \overline{a}_i = -\partial_j \left(\overline{\overline{a}_i \overline{u}_j} + \overline{\overline{a}_j \overline{u}_i}\right) + \nu \nabla^2 \overline{a}_i - \partial_i \overline{p}_a - \partial_j \overline{\phi}^a_{ij} \tag{2.3b}$$

in which a model is required for $\overline{\phi}^a_{ij} \equiv \overline{u_i a_j + a_i u_j} - \overline{\overline{u}_i \overline{a}_j} - \overline{\overline{u}_j \overline{a}_i}$. Expressing the unclosed subgrid-scale quantities in terms of $\overline{u}_i$ and $\overline{a}_i$ would constitute an entirely new approach to subgrid scale modelling. Moreover, the evolution equation for $\overline{a}_i$ carries more information on the nonlinear dynamics than the classical equation for $\overline{u}_i$.

Coding the equations (2.2) is rather straightforward. We have checked that the results produced from a standard DNS code for (2.1) and from a code for the $u_i - a_i$ formulation (2.2) yield indistinguishable results when the acceleration vector is properly initialized. Unfortunately the equations (2.2) produce results that, even in the statistical sense, depend strongly on the initial conditions. If $a_i(t_0)$ is not set exactly from (2.1b), then global statistical quantities like the total energy and the total dissipation diverge from the DNS results.

The sensitivity to the initial conditions is so high that in the example presented here, the viscous decay of energy is not at all reproduced when the phase of the initial accel-

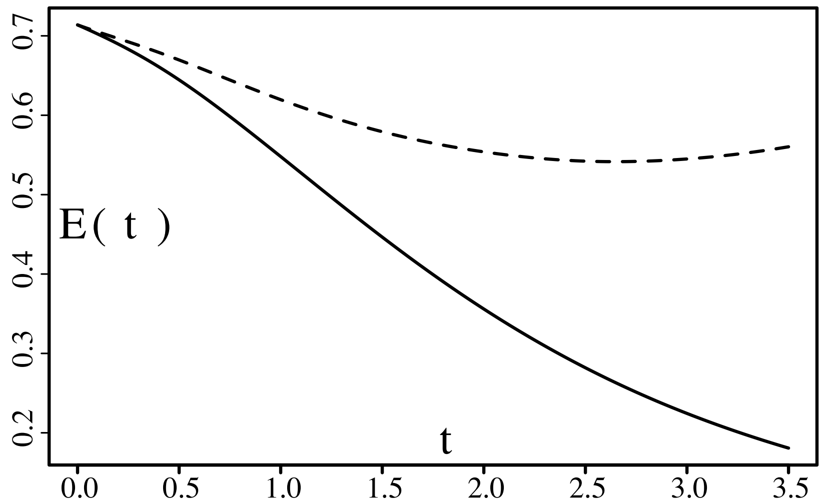

FIGURE 1. Total energy dissipation predicted by the equations (2.2) with the exact initial acceleration field (solid line) and with a field for which the phases have been slightly modified by multiplying each vector in Fourier space $a_i(\mathbf{k})$ by a factor $e^{i2\pi s}$, where $s \in [-0.1, 0.1]$ is a random variable.

eration are slightly modified. In the late stage of the computation, an increase of energy is even observed. We thus believe that the DNS velocity-acceleration formulation (2.2) is not reliable and, consequently, there is no point in further exploring the associated LES problem (2.3).

3. Equations for subgrid-scale quantities

The purpose of this section is to derive explicit evolution equations for subgrid-scale quantities. Two difficulties can immediately be anticipated. First, considering for instance the subgrid-scale stress tensor, $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$, it appears that this quantity is *not* strictly speaking a resolved variable since the product $\overline{u}_i \overline{u}_j$ cannot in general be fully captured on the same grid as the LES velocity $\overline{u}_i$. Carrying simultaneous equations for $\overline{u}_i$ and τ_{ij} thus requires two different grids, which of course is not desirable. This problem can be solved by rewriting the LES equation as follows:

$$\partial_t \overline{u}_i = -\partial_j \overline{\overline{u}_i \overline{u}_j} - \partial_i \overline{p} + \nu \nabla^2 \overline{u}_i - \partial_j \overline{\tau}_{ij}, \tag{3.1}$$

where now $\overline{\tau}_{ij} = \overline{u_i u_j} - \overline{\overline{u}_i \overline{u}_j}$. This equation implicitly assumes that the operator $\overline{\cdots}$ defining the LES field is a projection, $\overline{\cdots} = \overline{\overline{\cdots}}$, in order to ensure the Galilean invariance of the LES equations independently of the modelling approach used for representing $\overline{\tau}_{ij}$. Another obvious advantage of the formalism (3.1) is that all the terms in this equation, including the convective nonlinearity, appear as projected quantities (Carati, Winckelmans, & Jeanmart, 2001). We also note that by definition $\overline{\tau}_{ij} = \overline{\overline{\tau}_{ij}}$.

The second difficulty comes from the definition of the subgrid-scale quantities. It is obvious that the time derivative of a subgrid-scale quantity is a subgrid-scale quantity as well. It might thus be argued that the evolution of $\overline{\tau}_{ij}$ is entirely driven by subgrid-scale effects that require modelling and consequently the evolution equation for $\overline{\tau}_{ij}$ might be seen as entirely arbitrary. This difficulty will be partially worked around by deriving the evolution equation directly from the definition of $\overline{\tau}_{ij}$. However, there is still some arbitrariness in the expression of resolved and unresolved terms in this equation. Part of this arbitrariness will be removed by imposing that the evolution of any subgrid-scale quantity characteristic of a turbulent flow should be driven by an advection-dispersion equation.

3.1. *Equation for the subgrid-scale stress tensor*

The equation for $\overline{\tau}_{ij}$ is directly derived from its definition and from both the Navier-Stokes and the LES equations (3.1):

$$\partial_t\overline{\tau}_{ij} = \overline{\overline{u}_i\partial_l\overline{\tau}_{jl}} + \overline{\overline{u}_j\partial_l\overline{\tau}_{il}} + \nu\nabla^2\overline{\tau}_{ij} - \partial_l\overline{\phi}^c_{ijl} - \overline{\Phi}^c_{ij} - \overline{\Phi}^p_{ij} - \overline{\Phi}^\nu_{ij} \tag{3.2}$$

Four unclosed terms appear in this equation. They can be related to the convective nonlinearities $\overline{\phi}^c_{ijl}$ and $\overline{\Phi}^c_{ij}$, to pressure terms $\overline{\Phi}^p_{ij}$, and to viscous terms $\overline{\Phi}^\nu_{ij}$ and are fully symmetric second and third rank tensors:

$$\overline{\phi}^c_{ijl} = \overline{u_iu_ju_l} - \overline{\overline{u}_i\overline{u}_j\overline{u}_l} \tag{3.3}$$

$$\overline{\Phi}^p_{ij} = \overline{u_i\partial_jp} + \overline{u_j\partial_ip} - \overline{\overline{u}_i\partial_j\overline{p}} - \overline{\overline{u}_j\partial_i\overline{p}} \tag{3.4}$$

$$\overline{\Phi}^c_{ij} = -\overline{\overline{u}_i\partial_l(\overline{\overline{u}_l\overline{u}_j} - \overline{u}_l\overline{u}_j)} - \overline{\overline{u}_j\partial_l(\overline{\overline{u}_l\overline{u}_i} - \overline{u}_l\overline{u}_i)} \tag{3.5}$$

$$\overline{\Phi}^\nu_{ij} = 2\nu\left(\overline{\partial_lu_i\partial_lu_j} - \overline{\partial_l\overline{u}_i\partial_l\overline{u}_j}\right) \tag{3.6}$$

The dynamic equation (3.2) for $\overline{\tau}_{ij}$ does not contain an explicit advection term, though one is presumably hidden in $\overline{\phi}^c_{ijl}$. Hence, a reasonable expansion for this fully symmetric tensor that introduces the desirable advection term would be:

$$\overline{\phi}^c_{ijl} = \overline{\overline{u}_i\overline{\tau}_{jl}} + \overline{\overline{u}_j\overline{\tau}_{il}} + \overline{\overline{u}_l\overline{\tau}_{ij}} + \overline{\Phi}^c_{ijl} \tag{3.7}$$

where $\overline{\Phi}^c_{ijl}$ is a residual tensor that remains to be modelled. This expansion yields the following dynamic equation for $\overline{\tau}_{ij}$:

$$\partial_t\overline{\tau}_{ij} = -\overline{\overline{u}_l\partial_l\overline{\tau}_{ij}} + \nu\nabla^2\overline{\tau}_{ij} - \overline{\overline{S}^u_{jl}\overline{\tau}_{il}} - \overline{\overline{S}^u_{il}\overline{\tau}_{jl}} - \partial_l\overline{\Phi}^c_{ijl} - \overline{\Phi}^p_{ij} - \overline{\Phi}^c_{ij} - \overline{\Phi}^\nu_{ij} \tag{3.8}$$

where $\overline{S}^u_{ij} \equiv (\partial_i\overline{u}_j + \partial_j\overline{u}_i)/2$. The trace of this equation yields the evolution equation (Speziale, 1991,Ghosal *et al*, 1995) for the subgrid-scale energy $\overline{q} = \overline{\tau}_{ii}/2$:

$$\partial_t\overline{q} = -\overline{\overline{u}_l\partial_l\overline{q}} + \nu\nabla^2\overline{q} - \overline{\overline{S}^u_{ij}\overline{\tau}_{ij}} - \partial_l\overline{\Phi}^q_l - \overline{\epsilon}\,. \tag{3.9}$$

The production term $\overline{\overline{S}^u_{ij}\overline{\tau}_{ij}}$ is independent of $\overline{q}$ since $\overline{S}^u_{ij}$ is traceless for incompressible flows. The turbulent flux $\overline{\Phi}^q_l$ originates from $\overline{\Phi}^c_{iil}$ and from $\overline{\Phi}^p_{ii}$ which can be easily written as a divergence term. The status of $\overline{\Phi}^c_{ii}$ is less obvious. However, using the property of projectors $\langle\overline{a}\overline{b}\rangle = \langle a\overline{b}\rangle$, where $\langle\cdots\rangle$ is the volume average, the following identity can be derived: $\langle\overline{\Phi}^c_{ii}\rangle = 0$. Thus it is not inconsistent to take $\overline{\Phi}^c_{ii}$ to be a divergence term which then contributes to the subgrid-scale energy flux $\overline{\Phi}^q_l$. Finally, the subgrid-scale dissipation

$$\overline{\epsilon} = \overline{\Phi}^\nu_{ii}/2 \tag{3.10}$$

is not a flux term. It represents the dissipation of subgrid-scale energy due to viscous effects and has the property of remaining finite in the limit of very small viscosity. We will not consider further the equation for $\overline{\tau}_{ij}$ but will focus in the next section on the evolution equation for the subgrid-scale force.

3.2. *Equation for the subgrid-scale force*

The subgrid-scale stress tensor $\overline{\tau}_{ij}$ represents more information than required to close the equation for the resolved velocity $\overline{u}_i$. Only the divergence-free part of $\partial_j\overline{\tau}_{ij}$, which will be denoted $\overline{h}_i$, is actually needed for incompressible flows. The dynamics of $\overline{h}_i$ can be derived immediately from the equation for $\overline{\tau}_{ij}$ and by noting that the right-hand side of the evolution equation for $\overline{h}_i$ must be the divergence of a symmetric tensor. The complete set of LES equations then reads:

$$\partial_t\overline{u}_i = -\partial_j\overline{\overline{u}_i\overline{u}_j} - \partial_i\overline{p}_{\overline{u}} + \nu\nabla^2\overline{u}_i + \overline{h}_i + \overline{f}_i \tag{3.11}$$

$$\partial_t\overline{h}_i = -\partial_j(\overline{\overline{u}_j\overline{h}_i} + \overline{\overline{u}_i\overline{h}_j}) + \nu\nabla^2\overline{h}_i - \partial_j\overline{\psi}_{ij} - \partial_i\overline{p}_h \tag{3.12}$$

where a pseudo pressure $\overline{p}_h$ has been introduced to ensure that the subgrid-scale force remains divergence free. An external forcing term f_i has also been added to the Navier-Stokes equation for completeness and for keeping track of the possible effect of such a forcing on the balance equations derived in the following section. The LES pressure $\overline{p}_{\overline{u}}$ is *not* $\overline{p}$ because the incompressibility of the subgrid-scale force is enforced separately. It is thus a function of the LES velocity only:

$$\nabla^2\overline{p}_{\overline{u}} = -\partial_i\partial_j\overline{\overline{u}_i\overline{u}_j} \tag{3.13}$$

3.3. *Balance equations*

Balance equations for second order quantities based on $\overline{u}_i$ and $\overline{h}_i$ can be used to better understand the effects of the different terms entering the evolution equation for $\overline{h}_i$. The resolved kinetic energy balance is straightforwardly given by

$$\partial_t\langle\overline{u}_i\overline{u}_i/2\rangle = \langle\overline{u}_i\overline{f}_i\rangle + \langle\overline{u}_i\overline{h}_i\rangle - 2\nu\langle\overline{S}^u_{ij}\overline{S}^u_{ij}\rangle \tag{3.14}$$

There is nothing new in this relation. As expected the energy input rate due to the subgrid-scale force is given by the average value of the "cross-helicity" $\overline{u}_i\overline{h}_i$. In the absence of external forcing and of viscous effects, the subgrid-scale force is, as expected, the only source of variation of the resolved kinetic energy. Remarkably, in the same conditions, the evolution of the average cross-helicity depends only on the subgrid-scale force intensity $\overline{h}_i\overline{h}_i$ and on the unknown subgrid-scale tensors $\overline{\psi}_{ij}$:

$$\partial_t\langle\overline{u}_i\overline{h}_i\rangle = \langle\overline{h}_i\overline{h}_i\rangle + \langle\overline{S}^u_{ij}\overline{\psi}_{ij}\rangle + \langle\overline{f}_i\overline{h}_i\rangle - 4\nu\langle\overline{S}^u_{ij}\overline{S}^h_{ij}\rangle \tag{3.15}$$

where $\overline{S}^h_{ij} \equiv (\partial_i\overline{h}_j + \partial_j\overline{h}_i)/2$. In particular, the contributions of the nonlinear convective terms from the $\overline{u}_i$ and $\overline{h}_i$ equations cancel exactly. We can also write the balance equation for $\overline{h}_i\overline{h}_i$:

$$\partial_t\langle\overline{h}_i\overline{h}_i/2\rangle = -\langle\overline{h}_i\overline{h}_j\overline{S}^u_{ij}\rangle - 2\nu\langle\overline{S}^h_{ij}\overline{S}^h_{ij}\rangle + \langle\overline{S}^h_{ij}\overline{\psi}_{ij}\rangle \tag{3.16}$$

3.4. *The role of the new subgrid-scale tensor*

The unknown subgrid-scale tensor $\overline{\psi}_{ij}$ has contributions from both viscous and nonlinear effects. There is no reason here to separate these two contributions since they both appear with the same form (the divergence of a symmetric tensor) in the equation for $\overline{h}_i$. This generalized subgrid-scale stress tensor should be responsible for both the turbulent flux of $\overline{h}_i$ and the creation of $\overline{h}_i$ due to nonlinear interactions between modes from the resolved velocity field. A model for $\overline{\psi}_{ij}$ should thus have the property of remaining finite for finite $\overline{u}_i$ even when $\overline{h}_i = 0$ in order to be able to represent the creation of $\overline{h}_i$ due to the resolved velocity. However, this model should also remain finite for finite $\overline{h}_i$ when

$\overline{u}_i = 0$ in order to be able to represent the turbulent flux of $\overline{h}_i$ generated by subgrid-scale motions. Since only the projections of $\overline{\psi}_{ij}$ along the tensors $\overline{S}^u_{ij}$ and $\overline{S}^h_{ij}$ affect the global balance equations, it is tempting to model this tensor as:

$$\overline{\psi}_{ij} \approx -2\kappa_h \overline{S}^h_{ij} - 2\kappa_u \overline{S}^u_{ij} \tag{3.17}$$

Assuming that the Kolmogorov theory applies to the subgrid-scale tensor $\overline{\psi}_{ij}$, the coefficients κ_h and κ_u can be written as functions of the dissipation rate $\overline{\epsilon}$ and the projector characteristic width $\overline{\Delta}$ (or the wavenumber in Fourier space) only. Dimensional analysis then yields

$$\kappa_h = C_h \overline{\epsilon}^{1/3} \overline{\Delta}^{4/3} \tag{3.18}$$

$$\kappa_u = C_u \overline{\epsilon}^{2/3} \overline{\Delta}^{2/3} \tag{3.19}$$

where C_h and C_u are dimensionless parameters that remain to be determined.

4. Dynamic procedure for subgrid-scale quantities

The dynamic procedure (Germano *et al*, 1991; Germano, 1992), based on the introduction of a test operator $\widehat{\cdots}$, can be generalized for the pair of equations (3.11-3.12). The evolution of variables at the test level is:

$$\partial_t \widehat{\overline{u}}_i = -\partial_j \widehat{\overline{\widehat{\overline{u}}_i \widehat{\overline{u}}_j}} - \partial_i \widehat{\overline{p}}_{\widehat{\overline{u}}} + \nu \nabla^2 \widehat{\overline{u}}_i + \widehat{\overline{H}}_i + \widehat{\overline{f}}_i \tag{4.1}$$

$$\partial_t \widehat{\overline{H}}_i = -\partial_j \left(\widehat{\overline{\widehat{\overline{u}}_j \widehat{\overline{H}}_i}} + \widehat{\overline{\widehat{\overline{u}}_i \widehat{\overline{H}}_j}}\right) + \nu \nabla^2 \widehat{\overline{H}}_i - \partial_j \widehat{\overline{\Psi}}_{ij} - \partial_i \widehat{\overline{p}}_H \tag{4.2}$$

At this stage, it is convenient to introduce the following notations for the divergence-free nonlinear terms:

$$\overline{n}_i = -\partial_j \overline{\overline{u}_i \overline{u}_j} - \partial_i \overline{p}_{\overline{u}}\,, \tag{4.3}$$

$$\widehat{\overline{N}}_i = -\partial_j \overline{\widehat{\overline{u}}_i \widehat{\overline{u}}_j} - \partial_i \widehat{\overline{p}}_{\widehat{\overline{u}}}\,. \tag{4.4}$$

Because of relation (3.13), the vector $\overline{n}_i$ is known in terms of the LES velocity. Similarly, $\widehat{\overline{N}}_i$ is a function of $\widehat{\overline{u}}_i$ only. With these definitions, we can derive a generalized Germano identity between the subgrid scale force at the LES and test levels

$$\widehat{\overline{H}}_i = \widehat{\overline{h}}_i + \widehat{\overline{l}}_i \tag{4.5}$$

where $\widehat{\overline{l}}_i = \widehat{\overline{n}}_i - \widehat{\overline{N}}_i$. Because $\widehat{\overline{H}}_i \neq \widehat{\overline{h}}_i$, the comparison of $\widehat{\overline{\psi}}_{ij}$ with $\widehat{\overline{\Psi}}_{ij}$ cannot be done directly but requires an evolution equation for $\widehat{\overline{l}}_i$:

$$\begin{aligned} \partial_t \widehat{\overline{l}}_i = {} & -\partial_j \left(\widehat{\overline{\overline{u}_i \overline{n}_j}} + \widehat{\overline{\overline{u}_j \overline{n}_i}} - \widehat{\overline{\widehat{\overline{u}}_i \widehat{\overline{N}}_j}} - \widehat{\overline{\widehat{\overline{u}}_i \widehat{\overline{N}}_j}} \right) \\ & -\partial_j \left(\widehat{\overline{\overline{u}_i \overline{h}_j}} + \widehat{\overline{\overline{u}_j \overline{h}_i}} - \widehat{\overline{\widehat{\overline{u}}_i \widehat{\overline{H}}_j}} - \widehat{\overline{\widehat{\overline{u}}_i \widehat{\overline{H}}_j}} \right) - \partial_i \widehat{\overline{p}}_l + \nu \nabla^2 \widehat{\overline{l}}_i - \partial_j L^\nu_{ij} \end{aligned} \tag{4.6}$$

The comparison of equations (4.2-4.6) with the test-level version of equation (3.12) yields the following identity:

$$\widehat{\overline{\Psi}}_{ij} = \widehat{\overline{\psi}}_{ij} + \widehat{\overline{L}}^h_{ij} + \widehat{\overline{L}}^u_{ij} + \widehat{\overline{L}}^\nu_{ij} \tag{4.7}$$

where

$$\widehat{\overline{L}}_{ij}^{\nu} = -\nu(\widehat{\overline{\partial_l \overline{u}_i \partial_l \overline{h}_j}} - \overline{\partial_l \widehat{\overline{u}}_i \partial_l \widehat{\overline{h}}_j}) \tag{4.8}$$

$$\widehat{\overline{L}}_{ij}^{h} = 2\left(\widehat{\overline{\overline{u}_i \overline{h}_j}} + \widehat{\overline{\overline{u}_j \overline{h}_i}} - \overline{\widehat{\overline{u}}_i \widehat{\overline{h}}_j} - \overline{\widehat{\overline{u}}_j \widehat{\overline{h}}_i}\right) \tag{4.9}$$

$$\widehat{\overline{L}}_{ij}^{u} = \widehat{\overline{\overline{u}_i \overline{n}_j}} + \widehat{\overline{\overline{u}_j \overline{n}_i}} + \overline{\widehat{\overline{u}}_i \widehat{\overline{N}}_j} + \overline{\widehat{\overline{u}}_j \widehat{\overline{N}}_i} - 2\overline{\widehat{\overline{u}}_i \widehat{\overline{n}}_j} - 2\overline{\widehat{\overline{u}}_j \widehat{\overline{n}}_i} \tag{4.10}$$

In order to simplify the implementation of the dynamic procedure in this preliminary study, we introduce two approximations. First, we note that, in the limit of very small molecular viscosity, $\widehat{\overline{L}}_{ij}^{\nu}$ vanishes. For this reason, we neglect this term in the present approach, assuming a high Reynolds number flow. Second, the scaling of the coefficients κ_h and κ_u depends on the dissipation rate $\overline{\epsilon}$. It will be assumed that $\overline{\epsilon}$ measured at the grid level is the same as the dissipation rate $\widehat{\overline{\epsilon}}$ measured at the test level. Again, this approximation is compatible with the large Reynolds number limit. It is thus reasonable to rewrite the coefficients κ_h and κ_u as follows:

$$\kappa_h = C_h^{\epsilon}\ \overline{\Delta}^{4/3} \tag{4.11}$$

$$\kappa_u = C_u^{\epsilon}\ \overline{\Delta}^{2/3} \tag{4.12}$$

where the coefficients $C_h^{\epsilon} = C_h \overline{\epsilon}^{1/3}$ and $C_u^{\epsilon} = C_u \overline{\epsilon}^{2/3}$ should now be the same at the grid and test levels and can be determined using the dynamic procedure. This yields the following expressions for the unknown coefficients:

$$\kappa_h = \frac{1}{2(1-\alpha^{4/3})}\ \frac{\langle \widehat{\overline{L}}_{ij}^{tot} \widehat{\overline{S}}_{ij}^{h}\rangle\ \langle \widehat{\overline{S}}_{ij}^{h} \widehat{\overline{S}}_{ij}^{u}\rangle + \langle \widehat{\overline{L}}_{ij}^{tot} \widehat{\overline{S}}_{ij}^{u}\rangle\ \langle \widehat{\overline{S}}_{ij}^{h} \widehat{\overline{S}}_{ij}^{h}\rangle}{\langle \widehat{\overline{S}}_{ij}^{h} \widehat{\overline{S}}_{ij}^{h}\rangle\ \langle \widehat{\overline{S}}_{ij}^{u} \widehat{\overline{S}}_{ij}^{u}\rangle - \langle \widehat{\overline{S}}_{ij}^{u} \widehat{\overline{S}}_{ij}^{h}\rangle\ \langle \widehat{\overline{S}}_{ij}^{u} \widehat{\overline{S}}_{ij}^{h}\rangle} \tag{4.13}$$

$$\kappa_u = \frac{1}{2(1-\alpha^{2/3})}\ \frac{\langle \widehat{\overline{L}}_{ij}^{tot} \widehat{\overline{S}}_{ij}^{u}\rangle\ \langle \widehat{\overline{S}}_{ij}^{h} \widehat{\overline{S}}_{ij}^{u}\rangle + \langle \widehat{\overline{L}}_{ij}^{tot} \widehat{\overline{S}}_{ij}^{h}\rangle\ \langle \widehat{\overline{S}}_{ij}^{u} \widehat{\overline{S}}_{ij}^{u}\rangle}{\langle \widehat{\overline{S}}_{ij}^{h} \widehat{\overline{S}}_{ij}^{h}\rangle\ \langle \widehat{\overline{S}}_{ij}^{u} \widehat{\overline{S}}_{ij}^{u}\rangle - \langle \widehat{\overline{S}}_{ij}^{u} \widehat{\overline{S}}_{ij}^{h}\rangle\ \langle \widehat{\overline{S}}_{ij}^{u} \widehat{\overline{S}}_{ij}^{h}\rangle} \tag{4.14}$$

where $\alpha \equiv \overline{\Delta}/\widehat{\overline{\Delta}}$, $\widehat{\overline{\Delta}}$ is the characteristic length of the test operator, and $\widehat{\overline{L}}_{ij}^{tot} \equiv \widehat{\overline{L}}_{ij}^{u} + \widehat{\overline{L}}_{ij}^{h}$.

These models have been implemented in a spectral code for isotropic decaying turbulence. Preliminary tests have not yet produced successful results. The equation for the $\overline{h}_i$ variable appears to be unstable without a model for $\overline{\psi}_{ij}$. However, the models described here, despite the fact that they definitely improve the behavior of the equation, do not prevent instabilities after an initial decaying stage. At least two reasons could explain these difficulties. Firstly, the approximation of neglecting $\widehat{\overline{L}}_{ij}^{\nu}$ might be too crude for the fairly modest Reynolds number considered in the present tests (the reference data base corresponds to a 256^3 DNS). Second, the very simple model we have considered so far might be inadequate to model the unclosed term.

5. Conclusion

We have considered new formulations of the LES problem based on nonlinear changes of variables in the Navier-Stokes equation. The formulation of carrying an equation for the acceleration vector appears to be ill-conditioned and produces turbulence statistics that depend strongly on the initial conditions.

We have also considered a new formulation of the LES problem based on explicit equations for the unknown subgrid-scale quantities. In particular, the equation for the subgrid-scale force has been studied in detail and implemented in a spectral code. A dynamic procedure has been proposed for the related LES equations. The results obtained in preliminary runs are inconclusive as to whether this approach might eventually be successful.

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