

Magnetic turbulence in compressible fluids

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We present high resolution numerical simulations of compressible magnetohydrodynamic (MHD) turbulence. We concentrate on studies of spectra and anisotropy of velocity and density. We describe a technique of separating different magnetohydrodynamic (MHD) modes (slow, fast and Alfvén) and apply it to our simulations. For the mildly supersonic case, velocity and density show anisotropy. However, for the highly supersonic case, we observe steepening of the slow mode velocity spectrum and isotropization of density. Our present studies show that the spectrum of density gets substantially *flatter* than the Kolmogorov one.

1. Introduction

Most astrophysical systems, e.g. accretion disks, stellar winds, the interstellar medium (ISM) and intercluster medium are turbulent with an embedded magnetic field that influences almost all of their properties. This turbulence which spans from km to many kpc (see discussion in Armstrong, Rickett, & Spangler 1995) holds the key to many astrophysical processes (e.g., transport of mass and angular momentum, star formation, fragmentation of molecular clouds, heat and cosmic ray transport, magnetic reconnection). Statistics of turbulence is also essential for the cosmic microwave background (CMB) radiation foreground studies (Lazarian & Prunet 2002). In this brief, using high resolution simulations, we discuss statistics of 3D MHD turbulence and present new results on density fluctuations.

Why do we expect astrophysical fluids to be turbulent? A fluid of viscosity ν becomes turbulent when the rate of viscous dissipation, which is $\sim \nu/L^2$ at the energy injection scale L , is much smaller than the energy transfer rate $\sim V_L/L$, where V_L is the velocity dispersion at the scale L . The ratio of the two rates is the Reynolds number $Re = V_L L/\nu$. In general, when Re is larger than 10 – 100 the system becomes turbulent. Chaotic structures develop gradually as Re increases, and those with $Re \sim 10^3$ are appreciably less chaotic than those with $Re \sim 10^8$. Observed features such as star forming clouds are very chaotic for $Re > 10^8$. This makes it difficult to simulate realistic turbulence. The currently available 3D simulations containing 512 grid cells along each side can have Re up to $\sim O(10^3)$ and are limited by their grid sizes. Therefore, it is essential to find “*scaling laws*” in order to extrapolate numerical calculations ($Re \sim O(10^3)$) to real astrophysical fluids ($Re > 10^8$). We show below that even with its limited resolution, numerics is a great tool for *testing* scaling laws.

Kolmogorov theory provides a scaling law for *incompressible non-magnetized* hydrodynamic turbulence. This law provides a statistical relation between the relative velocity v_l of fluid elements and their separation l , namely, $v_l \sim l^{1/3}$. An equivalent description is to express spectrum $E(k)$ as a function of wave number k ($\sim 1/l$). The two descriptions are related by $kE(k) \sim v_l^2$. The famous Kolmogorov spectrum is $E(k) \sim k^{-5/3}$. The

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applications of Kolmogorov theory range from engineering research to meteorology (see Monin & Yaglom 1975) but its astrophysical applications are poorly justified and the application of the Kolmogorov theory can lead to erroneous conclusions (see reviews by Lazarian *et al.* 2003; Lazarian & Yan 2003)

Let us consider *incompressible* MHD turbulence first. There have long been an understanding that the MHD turbulence is anisotropic (e.g. Shebalin *et al.* 1983). Substantial progress has been achieved recently by Goldreich & Sridhar (1995; hereafter GS95), who made an ingenious prediction regarding relative motions parallel and perpendicular to magnetic field \mathbf{B} for incompressible MHD turbulence. An important observation that leads to understanding of the GS95 scaling is that magnetic field cannot prevent mixing motions of magnetic field lines if the motions are perpendicular to the magnetic field. Those motions will cause, however, waves that will propagate along magnetic field lines. If that is the case, the time scale of the wave-like motions along the field, i.e. $\sim l_{\parallel}/V_A$, (l_{\parallel} is the characteristic size of the perturbation along the magnetic field and $V_A = B/\sqrt{4\pi\rho}$ is the local Alfvén speed) will be equal to the hydrodynamic time-scale, l_{\perp}/v_l , where l_{\perp} is the characteristic size of the perturbation perpendicular to the magnetic field. The mixing motions are hydrodynamic-like[†]. They obey Kolmogorov scaling, $v_l \propto l_{\perp}^{1/3}$, because incompressible turbulence is assumed. Combining the two relations above we can get the GS95 anisotropy, $l_{\parallel} \propto l_{\perp}^{2/3}$ (or $k_{\parallel} \propto k_{\perp}^{2/3}$ in terms of wave-numbers). If we interpret l_{\parallel} as the eddy size in the direction of the local magnetic field and l_{\perp} as that in the perpendicular directions, the relation implies that smaller eddies are more elongated.

GS95 predictions have been confirmed numerically (Cho & Vishniac 2000; Maron & Goldreich 2001; Cho, Lazarian & Vishniac 2002a, hereafter CLV02a; see also CLV03); they are in good agreement with observed and inferred astrophysical spectra (see CLV03). However, the GS95 model considered incompressible MHD, but the media in molecular clouds is *highly compressible*. Does any part of GS95 model survives? Literature on the properties of compressible MHD is very rich (see reviews by Pouquet 1999; Cho & Lazarian 2004 and references therein). Higdon (1984) theoretically studied density fluctuations in the interstellar MHD turbulence. Matthaeus & Brown (1988) studied nearly incompressible MHD at low Mach number and Zank & Matthaeus (1993) extended it. In an important paper Matthaeus *et al.* (1996) numerically explored anisotropy of compressible MHD turbulence. However, those papers do not provide universal scalings of the GS95 type. After the GS95 model, Lithwick & Goldreich studied scaling relations for high- β plasmas and Cho & Lazarian (2002; hereafter CL02) for low- β plasmas.

The complexity of the compressible magnetized turbulence with magnetic field made some researchers believe that the phenomenon is too complex to expect any universal scalings for molecular cloud research. High coupling of compressible and incompressible motions is often quoted to justify this point of view.

Below we shall provide arguments that are suggestive that the fundamentals of compressible MHD can be understood and successfully applied to astrophysical fluids. In many astrophysical fluids the regular (or *uniform*) magnetic field is comparable with the fluctuating one. Therefore for most part of our discussion, we shall discuss results obtained for $\delta V \sim \delta B/\sqrt{4\pi\rho} \sim B_0/\sqrt{4\pi\rho}$, where δB is the r.m.s. strength of the random magnetic field.

Our work done during the summer program at CTR that we report here was an attempt to get a better insight into the physics of MHD turbulence using high resolution

[†] Recent simulations (Cho *et al.* 2003) suggest that perpendicular mixing is indeed efficient for mean magnetic fields of up to the equipartition value.

simulations (i.e. simulations within a 512^3 box). In what follows we discuss our numerical approach (sect. 2), our results obtained for the velocity and magnetic field statistics (sect. 3) and the density statistics (sect. 4). We concentrate on studies of spectra and turbulence anisotropy.

2. Numerical approach

We use a third-order accurate hybrid essentially non-oscillatory (ENO) scheme (see CL02) to solve the ideal isothermal MHD equations in a periodic box:

$$\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{v}) = 0, \quad (2.1)$$

$$\partial\mathbf{v}/\partial t + \mathbf{v} \cdot \nabla\mathbf{v} + \rho^{-1}\nabla(a^2\rho) - (\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi\rho = \mathbf{f}, \quad (2.2)$$

$$\partial\mathbf{B}/\partial t - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \quad (2.3)$$

with $\nabla \cdot \mathbf{B} = 0$ and an isothermal equation of state. Here \mathbf{f} is a random large-scale driving force, ρ is density, \mathbf{v} is the velocity, and \mathbf{B} is magnetic field. The rms velocity δV is maintained to be approximately unity, so that \mathbf{v} can be viewed as the velocity measured in units of the r.m.s. velocity of the system and $\mathbf{B}/\sqrt{4\pi\rho}$ as the Alfvén velocity in the same units. The time t is in units of the large eddy turnover time ($\sim L/\delta V$) and the length in units of L , the scale of the energy injection. The magnetic field consists of the uniform background field and a fluctuating field: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$.

We drive turbulence solenoidally in Fourier space and use 512^3 points, $V_A = B_0/\sqrt{4\pi\rho} = 1$, and $\rho_0 = 1$. The average rms velocity in statistically stationary state, δV , is around 1.

For our calculations we assume that $B_0/\sqrt{4\pi\rho} \sim \delta B/\sqrt{4\pi\rho} \sim \delta V$. In this case, the sound speed is the controlling parameter and basically two regimes can exist: supersonic and subsonic. Note that supersonic means low-beta and subsonic means high-beta. When supersonic, we consider mildly supersonic (or, mildly low- β) and highly supersonic (or, very low- β)[†].

2.1. Separation of MHD modes

Three types of waves exist (Alfvén, slow and fast) in compressible magnetized plasma. The slow, fast, and Alfvén bases that denote the direction of displacement vectors for each mode are given by

$$\hat{\xi}_s \propto (-1 + \alpha - \sqrt{D})k_{\parallel}\hat{\mathbf{k}}_{\parallel} + (1 + \alpha - \sqrt{D})k_{\perp}\hat{\mathbf{k}}_{\perp}, \quad (2.4)$$

$$\hat{\xi}_f \propto (-1 + \alpha + \sqrt{D})k_{\parallel}\hat{\mathbf{k}}_{\parallel} + (1 + \alpha + \sqrt{D})k_{\perp}\hat{\mathbf{k}}_{\perp}, \quad (2.5)$$

$$\hat{\xi}_A = -\hat{\varphi} = \hat{\mathbf{k}}_{\perp} \times \hat{\mathbf{k}}_{\parallel}, \quad (2.6)$$

where $D = (1 + \alpha)^2 - 4\alpha \cos\theta$, $\alpha = a^2/V_A^2 = \beta(\gamma/2)$, θ is the angle between \mathbf{k} and \mathbf{B}_0 , and $\hat{\varphi}$ is the azimuthal basis in the spherical polar coordinate system. These are equivalent to the expression in CL02:

$$\hat{\xi}_s \propto k_{\parallel}\hat{\mathbf{k}}_{\parallel} + \frac{1 - \sqrt{D} - \beta/2}{1 + \sqrt{D} + \beta/2} \left[\frac{k_{\parallel}}{k_{\perp}} \right]^2 k_{\perp}\hat{\mathbf{k}}_{\perp}, \quad (2.7)$$

[†] The terms ‘‘mildly’’ and ‘‘highly’’ are somewhat arbitrary terms. We consider these two supersonic cases to cover a broad range of parameter space. Note that Boldyrev, Nordlund, & Padoan (2002) recently provided a Mach number dependence study of the compressible MHD turbulence statistics where only two regimes are manifest: essentially incompressible and essentially compressible shock-dominated (with smooth transition at some M_s of order unity).

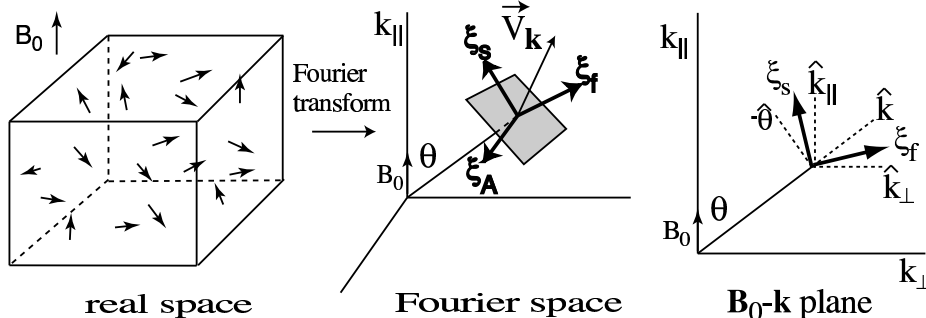


FIGURE 1. Separation method. We separate Alfvén, slow, and fast modes in Fourier space by projecting the velocity Fourier component \mathbf{v}_k onto bases ξ_A , ξ_s , and ξ_f , respectively. Note that $\xi_A = -\hat{\varphi}$. Slow basis ξ_s and fast basis ξ_f lie in the plane defined by \mathbf{B}_0 and \mathbf{k} . Slow basis ξ_s lies between $-\hat{\theta}$ and $\hat{k}_{||}$. Fast basis ξ_f lies between \hat{k} and \hat{k}_{\perp} . From Cho & Lazarian (2003).

$$\hat{\xi}_f \propto \frac{1 - \sqrt{D} + \beta/2}{1 + \sqrt{D} - \beta/2} \left[\frac{k_{\perp}}{k_{||}} \right]^2 k_{||} \hat{k}_{||} + k_{\perp} \hat{k}_{\perp}. \quad (2.8)$$

(Note that $\gamma = 1$ for isothermal case.)

Slow and fast velocity components can be obtained by projecting velocity Fourier component \mathbf{v}_k onto $\hat{\xi}_s$ and $\hat{\xi}_f$, respectively. See Cho & Lazarian (2003; hereinafter CL03) for discussion regarding how to separate slow and fast magnetic modes. We obtain energy spectra using this projection method. When we calculate the structure functions (e.g. for Alfvén modes) in real space, we first obtain the Fourier components using the projection and, then, we obtain the real space values by performing Fourier transform.

We tested our technique of separation in CL03 for a case when the separation is possible in real space and got essentially identical results with our *statistical* technique. Therefore we believe that our separation procedure works reliably.

3. Velocity and magnetic field spectra

We show in Fig. 2 new results from high resolution simulation of highly supersonic magnetically dominated media. The sonic Mach number, M_s , is ~ 10 . Fig. 2 shows that most of the scaling relations we previously found in the low resolution simulations are still valid in the high resolution simulation. Especially anisotropy of Alfvén, slow, and fast modes is almost identical to the one in the previous studies. However, the power spectra for slow modes do not show the Kolmogorov slope. The slope is close to -2 , which is suggestive of shock formation. At this moment, it is not clear whether or not the -2 slope is the true slope. In other words, the observed -2 slope might be due to the limited numerical resolution. Runs with higher numerical resolution should give the definite answer.

Formation of shocks is expected as within slow modes for sufficiently high Mach numbers the turbulence gets superAlfvénic. However, it is interesting that the turbulence anisotropy is definitely affected by magnetic field.

Alfvén modes follow the GS95 scaling as in the incompressible MHD turbulence. Alfvén perturbations cascade to small scales over just one wave period, while the other non-linear interactions require more time. Therefore we expect that the non-linear interactions with other types of waves should affect Alfvénic cascade only marginally. Moreover, as the

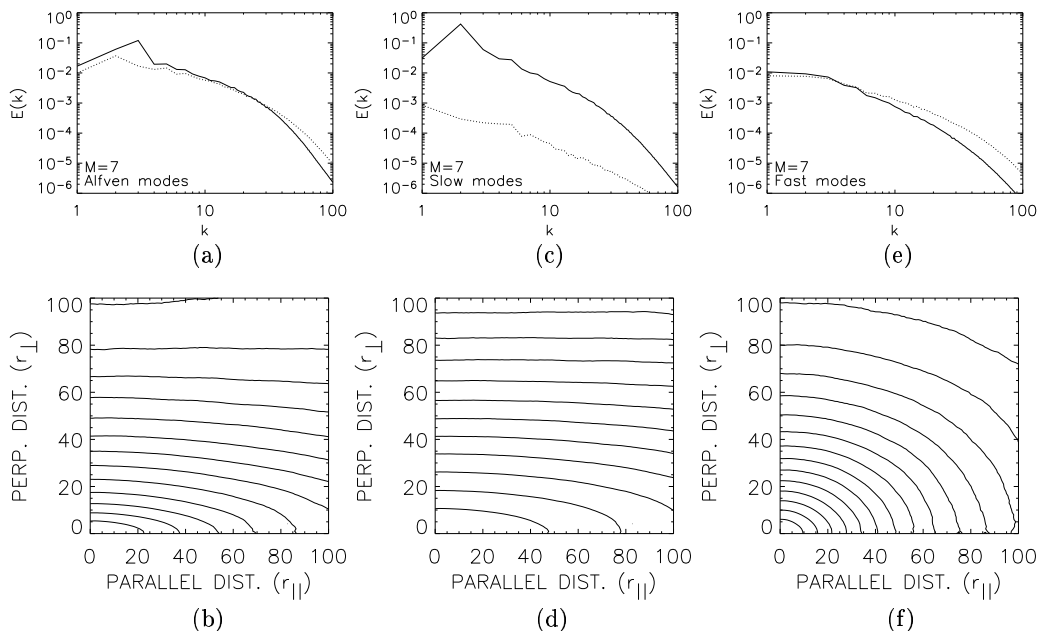


FIGURE 2. $M_s \sim 10$, $M_A \sim 0.5$, $\beta \sim 0.01$, and 512^3 grid points. Solid lines are for velocity spectra and dotted for magnetic field. (a) Spectra of Alfvén modes follow a Kolmogorov-like power law. (b) Eddy shapes (contours of same second-order structure function, SF_2) for velocity of Alfvén modes shows anisotropy similar to the GS95. ($r_{\parallel} \propto r_{\perp}^{2/3}$ or $k_{\parallel} \propto k_{\perp}^{2/3}$). The structure functions are measured in directions perpendicular or parallel to the local mean magnetic field in real space. We obtain real-space velocity and magnetic fields by inverse Fourier transform of the projected fields. (c) Spectra of slow modes are a bit steeper than the Kolmogorov-like power law. They closer to k^{-2} . (d) Slow mode velocity shows anisotropy similar to the GS95. We obtain contours of equal SF_2 directly in real space without going through the projection method, assuming slow mode velocity is nearly parallel to local mean magnetic field in low β plasmas. (e) Spectra of fast modes show a possible departure from the Kolmogorov spectrum or the IK spectrum. (f) The magnetic SF_2 of fast modes shows isotropy.

Alfvén waves are incompressible, the properties of the corresponding cascade do not depend on the sonic Mach number.

4. Density statistics

Density at low Mach numbers follow the GS95 scaling when the driving is incompressible (CL03). However, CL03 showed that this scaling substantially changes for high Mach numbers. Our high resolution results in Fig. 3 confirm the CL03 finding that at high Mach numbers density fluctuations get isotropic. Moreover, our present studies show that the spectrum of density gets substantially *flatter* than the GS95 one (see Fig. 4). Note, that a model of random shocks would produce a spectrum *steeper* than the GS95 one. A possible origin of the flat spectrum is the superAlfvénic perturbations created by fast modes within density perturbations originated from slow modes. This particular regime is clearly identified in a review by CLV03 (see Fig. 9 therein). It may also be related to the regime of superAlfvénic turbulence which is discussed in Norlund & Podoan (2003). However, alternative explanations of the shallow density fluctuations exist and

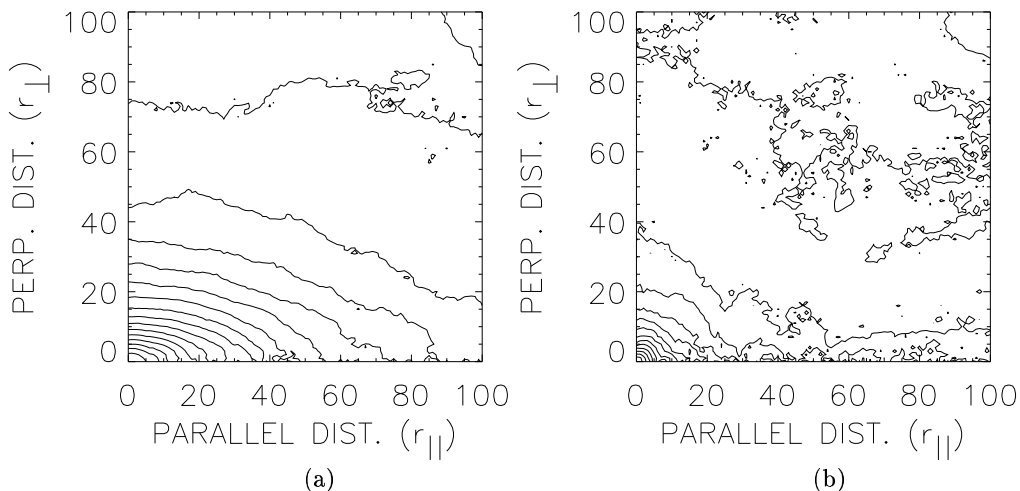


FIGURE 3. (a) Density structures for $M_s \sim 3$. (b) Density structures for $M_s \sim 10$. Density gets isotropic as the Mach number, M_s , increases.

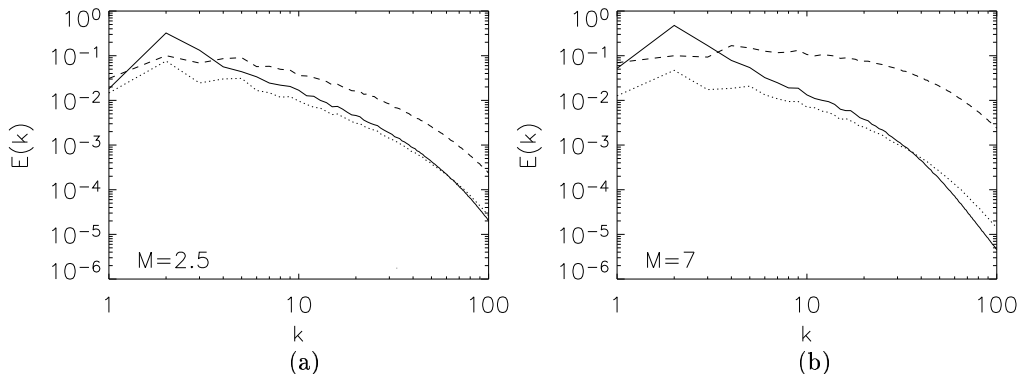


FIGURE 4. (a) Spectra for $M_s \sim 3$ at $t=3.7$. (b) Spectra for $M_s \sim 10$ at $t=6.0$. Dashed lines=density; solid=velocity; dotted=magnetic field. Density spectrum gets flatter as the Mach number, M_s , increases.

our ongoing work should clarify which process is actually responsible for the unusual density scaling that we observe.

The flat spectrum of density perturbations in high Mach number turbulence is definitely a very interesting phenomenon. It does affect the properties of interstellar medium, including the origin of the small scale clumps that are ubiquitous in the diffuse cold gas. This flat density spectrum should affect the star formation processes. A study in Boldyrev *et al.* (2002) indeed testifies that the density spectrum is flat in molecular clouds. However, a more systematic study is necessary.

5. Discussion

In the paper above we have studied scaling relations for Alfvén, fast and slow modes. First question that comes to mind is how good is to speak about different modes in strong MHD turbulence. It is well known, for instance, that for strong perturbations various linear modes are coupled. Well, our work shows that the coupling is indeed a

very important phenomenon at the large scales where the *total* Mach number M_{tot} , which normalizes the velocity perturbations by the sum of Alfvénic and sound velocities, is of the order of unity. At smaller scales when M_{tot} becomes less than unity the perturbations that we identify through our separation of mode procedure develop without much interaction between fast and Alfvén modes. This can be understood if we take into account that Alfvén modes cascade just over one eddy turnover time. Therefore there is not enough time for the non-linear interaction between fast and Alfvén modes to take place. At the same time, the shearing of slow modes by Alfvénic turbulence does take place over a turnover time. Therefore the slow modes mimic Alfvén mode anisotropy. A new effect that we start observing at high resolution is steepening of the slow mode spectrum. This is indicative that while shearing the Alfvén modes create shocks within the gas associated with slow modes. This is not so surprising as the Alfvén velocity is smaller within the slow modes in low beta medium.

We have not studied here a case of turbulence that is globally superAlfvénic. This is motivated by our belief that unless magnetic field is dynamically important the turbulence will behave like ordinary hydro turbulence. At the scale when the Alfvén velocity gets of the order of the perturbation velocity, we should encounter turbulence that is studied here.

Why would we care about those scalings? How wrong is it to use Kolmogorov scalings instead? Dynamics, chemistry and physics of molecular clouds (see Falgarone 1999) presents a complex of problems for which the exact scalings may be required to a different degree. If we talk about dynamics of interstellar dust or propagation of cosmic rays, one *must* account for the actual scalings and couplings of different modes (see reviews by Lazarian & Yan 2003, Lazarian *et al.* 2003). There are other problems, e.g. turbulent heat transport where the exact scaling of modes seems to be less important (Cho *et al.* 2003).

6. Conclusion

In the brief, we have studied statistics of compressible MHD turbulence with high resolution numerical simulation for highly ($M_s \sim 10$) supersonic low- β case. We provided the decomposition of turbulence into Alfvén, slow and fast modes. We have found that GS95 scaling is valid for *Alfvén modes*:

$$\text{Alfvén: } E^A(k) \propto k^{-5/3}, \quad k_{\parallel} \propto k_{\perp}^{2/3}.$$

Slow modes also follows the GS95 anisotropy for highly supersonic low- β case:

$$\text{Slow: } k_{\parallel} \propto k_{\perp}^{2/3}.$$

The velocity spectrum of slow modes tends to be steeper, which may be related to the formation of shocks.

Density shows anisotropy for the mildly supersonic case. But, for the highly supersonic case, density shows more or less isotropic structures. We suspect that shock formation is responsible for the isotropization of density. Our present studies show that the spectrum of density gets substantially *flatter* than the GS95 one (see Fig. 4).

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