Budget of disturbance energy in gaseous reacting flows

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This paper presents an energy analysis of the disturbances that occur in gaseous combustion. It builds upon the previous work of Myers (1991) by including species and heat release terms, thus extending Myers' exact and linearized energy corollaries to combusting flows. These energy corollaries identify additional and significant energy density, flux, and source terms, thereby generalizing the recent results of Nicoud and Poinsot (2005) to include non-zero mean flow quantities, large amplitude disturbances, and varying specific heats. The associated stability criterion is therefore significantly different from the Rayleigh criterion in several ways. The closure of the exact equation is performed on an oscillating 2-D laminar flame. Results show that in this case the general equation can be substantially simplified by considering only entropy, heat release, and heat flux terms. The first one behaves as source term whereas the latter two dissipate the disturbance energy. Moreover, terms associated with the non-zero baseline flow are found to be important for the global closure of the balance even though the mean Mach number is small.

1. Introduction

Combustion stability has received sustained attention in both the academic and industrial communities, particularly over the last fifty years. During this time, the literature on this issue has grown enormously, and now spans numerous applications, including rockets (Flandro 1985; Culick 2001), afterburners (Bloxsidge *et al.* 1988), gas turbines (Dowling and Stow 2003; Poinsot and Veynante 2001), and industrial burners (Putnam 1971). The "Rayleigh criterion" is the most common argument for explaining combustion stability. While Rayleigh himself only first stated this criterion in prose form (Lord Rayleigh 1878), it is often written as

$$\int_{\Omega} \overline{p'\omega'_T} d\mathbf{x} > L, \tag{1.1}$$

where p', ω'_T, L, and Ω are the static pressure and heat release rate disturbances at a point in space, the losses from the combustor, and the combustor volume, respectively. $(\bar{\ })$ denotes the time average. This criterion states that the combustor is unstable when the relative phase of the pressure and heat release disturbances over the combustor volume are such that the integral is larger than the (at present unspecified) losses.

Despite the Rayleigh criterion having its origin well over a century ago, a recent paper by Nicoud and Poinsot (2005) suggests that it is still at the very least unclear under what conditions this criterion can be derived from the equations governing combusting

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fluid motion. The earlier works of Chu (1956, 1965), Bloxsidge *et al.* (1988), and Dowling (1997) attempted to show precisely this, using progressively more general definitions of acoustic or disturbance energies that have appeared over the last fifty or so years. These works illustrate that deriving the Rayleigh criterion from the governing equations first requires a valid conservation equation for the energy contained in the disturbances, whether these disturbances are considered acoustic or otherwise.

In combusting flows, any equation stating disturbance energy conservation must start from equations of motion that at least include non-zero mean flow quantities and entropy variation. To ignore either the mean flow or entropy variation causes conceptual problems, discussed later in this paper. It appears that only the energies defined by Morfey (1971) and Myers (1991) do this. Viscous dissipation and heat conduction, while included in these works, are not essential and are usually small. The derivation in Dowling (1997) of an acoustic energy conservation equation for combusting flows extended the approach of Morfey (1971) for non-combusting flows. In Morfey's analysis, any entropy disturbances are shifted into the source term. Myers (1991) allows entropy disturbances to remain in both the energy density and flux terms. Myers' equation was consistent with those developed earlier by Chu (1965) and Pierce (1981) for zero-mean flow.

Nicoud and Poinsot (2005) rederived the fluctuating energy equation of Chu (1965) and argued that the Rayleigh criterion is an incomplete description of the significant sources of fluctuating energy in combustion. In the limit of small disturbance amplitude, a source term proportional to $T_1\omega_{T1}$ was found where ω_{T1} and T_1 are the first terms in the heat release and static temperature asymptotic expansions. This term is analogous but significantly different to the Rayleigh term in Eq. (1.1). Entropy disturbances through the flame were also argued to be a significant source of disturbance energy, but Nicoud and Poinsot's formulation was conceptually problematic because they assumed zero-mean flow quantities. Bloxsidge *et al.* (1988) and Dowling (1997) also showed that terms other than the Rayleigh term existed for their differently defined acoustic energy equation, but both argued that these terms were small in practice.

This difference of opinion on such a fundamental and practically important problem needs to be resolved. This can be achieved by first deriving a general equation for disturbance energy, as this paper does, and then studying numerically the magnitudes of all the identified source terms. The basic equation should not be linear as it is often the case when dealing with acoustics. Indeed the temperature, entropy, and velocity disturbances in particular can be large within flames and non-linear effects are already known to be significant in the acoustic energy analysis of solid rocket combustion (Flandro 1985; Culick 2001). Since Myers (1991) presented both exact and linearized disturbance energy equations, comparison of the two would determine the applicability of the linearized equations on a given combustor if the generalization of Myers' approach to combusting flows can be handled appropriately.

This paper supports the questioning of the validity of the Rayleigh criterion (Nicoud and Poinsot 2005) in common combusting flows. It draws heavily on Myers' exact and linearized energy corollaries (Myers 1991), and extends Nicoud and Poinsot's results to non-zero mean Mach numbers, large amplitude disturbances, and varying specific heats. Preliminary testing of the proposed exact disturbance energy equation is then performed by post-processing numerical simulation of a 2-D laminar oscillating flame.

2. Formulation

2.1. *What is a "disturbance energy" ?*

The notion of a disturbance energy Ed is somewhat vague; we first present a set of properties that we believe this quantity should meet in order to be useful when analyzing combustion stability. Many of these properties are obtained when expanding E_d or any other quantity as $f(\mathbf{x},t) = f_0(\mathbf{x}) + \sum \epsilon^i f_i(\mathbf{x},t)$, where ϵ is a small parameter.

- **P0**: E_d should be zero when there are no fluctuations, that is $E_{d0} = 0$,
- **P1**: E_d should be quadratic in the primitive variable fluctuations, that is $E_{d1} = 0$, and reduce properly to the well-established energies derived earlier for small amplitude disturbances (Chu 1965; Myers 1991; Nicoud and Poinsot 2005) when proper assumptions are made. **P1** is also a pre-requisite for **P3**,
- **P2**: The leading order term of E_d, viz. E_{d2}, should only depend on the first-order term of the primitive variable fluctuations ρ_1, p_1, $\mathbf{u}_1$, etc.
- **P3**: E_{d2} should be definite positive so that it increases with the amplitude of the fluctuations. The disturbance energy itself should remain positive even for large amplitude fluctuations.

While **P0** is an obvious statement, **P1** is enforced for consistency with previous works. Noting E the sensible stagnation energy of the mixture, **P1** disqualifies $E - E_0$ as a disturbance energy since $(E - E_0)_1 = C_v T_1$ for a calorifically perfect mixture at rest. Property **P2** is required for practical use: if E_{d2} were depending on both p_1 and p_2, for instance, one would have to define and handle two different pressure fluctuations when computing/analysing E_{d2}. Eventually, let us assume a conservation equation of the form

$$\frac{\partial E_d}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{W} = D \tag{2.1}$$

for E_d, where $\mathbf{W}$ and D stand for the flux vector and source of the disturbance energy. If property **P3** is satisfied, a stability criterion can easily be derived by integrating Eq. (2.1) over the flow domain Ω bounded by the surface S:

$$\text{STABILITY} \qquad \Leftrightarrow \qquad \int_\Omega D d\mathbf{x} - \int_S \mathbf{W} \cdot \mathbf{n} dS < 0, \tag{2.2}$$

where $\mathbf{n}$ is the outward normal vector.

At this point, it is unclear whether a disturbance energy satisfying **P0-P3** exists and if it is unique. A potential candidate for which we can show that **P0-P2** and at least partly **P3** are satisfied is discussed in the rest of this paper.

2.2. *Basic equations*

The exact disturbance energy conservation equation is derived from statements of mass conservation, species mass conservation, momentum transport, energy conservation, and entropy transport for a mixture of n gaseous species,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{m} = 0, \tag{2.3}$$

$$\frac{\partial \rho Y_k}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{m} + \mathbf{q}_k) = \omega_k, \qquad \text{for} \qquad k = 1, 2, ..., n \tag{2.4}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\zeta} + \boldsymbol{\nabla} H - T\boldsymbol{\nabla} s = \boldsymbol{\psi} + \boldsymbol{\psi}^*, \tag{2.5}$$

$$\frac{\partial}{\partial t}(\rho H - p) + \boldsymbol{\nabla} \cdot (\mathbf{m}H) - \mathbf{m} \cdot \boldsymbol{\psi} = TQ, \tag{2.6}$$

$$\frac{\partial \rho s}{\partial t} + \boldsymbol{\nabla} \cdot (s\mathbf{m}) = Q + Q^*, \tag{2.7}$$

In Eqs. (2.3)–(2.7), ρ is the volumetric density, $\mathbf{u}$ is the velocity vector, $\mathbf{m} = \rho\mathbf{u}$ is the momentum, $\boldsymbol{\xi} = \boldsymbol{\nabla} \times \mathbf{u}$ is the vorticity and $\boldsymbol{\zeta} = \boldsymbol{\xi} \times \mathbf{u}$, $H = h_s + \mathbf{u} \cdot \mathbf{u}/2$ is the sensible stagnation enthalpy, T is the temperature, s is the sensible entropy, $\boldsymbol{\psi}$ is such that $\psi_{ij} = (1/\rho)(\partial\tau_{ij}/\partial x_i)$ where τ_{ij} is the ij^{th} component of the viscous stress tensor. Noting Y_k, ω_k, h_{sk}, g_{sk}, $\mathbf{V}_k$ and $\mathbf{q}_k = \rho Y_k \mathbf{V}_k$ the mass fraction, volumic rate of production, sensible enthalpy, sensible Gibbs free energy, diffusion velocity and diffusion mass flux of the k^{th} species respectively, one also defines $\boldsymbol{\psi}^* = \sum_{k=1}^{n} g_{sk} \boldsymbol{\nabla} Y_k$, $TQ^* = -\sum_{k=1}^{n} g_{sk} \left(\omega_k - \boldsymbol{\nabla} \cdot \mathbf{q}_k\right)$ and $TQ = -\boldsymbol{\nabla} \cdot \mathbf{q} + \phi + \omega_T$ where $\mathbf{q} = -\lambda \boldsymbol{\nabla} T + \rho \sum_{k=1}^{n} Y_k h_{sk} \mathbf{V}_k$ is the heat flux, ϕ is the viscous dissipation, and ω_T is the heat release per unit volume. Splitting each quantity () into time-averaged $(^-)$ and fluctuating components $()'$, Eqs. (2.3)–(2.7) can be time averaged to give:

$$\boldsymbol{\nabla} \cdot \bar{\mathbf{m}} = 0, \tag{2.8}$$

$$\boldsymbol{\nabla} \cdot (\bar{Y}_k \bar{\mathbf{m}}) + \boldsymbol{\nabla} \cdot \bar{\mathbf{q}}_k + \boldsymbol{\nabla} \cdot \overline{\mathbf{m}' Y_k'} = \bar{\omega}_k, \tag{2.9}$$

$$\bar{\boldsymbol{\zeta}} + \boldsymbol{\nabla} \bar{H} - \bar{T} \boldsymbol{\nabla} \bar{s} - \overline{T' \boldsymbol{\nabla} s'} = \bar{\boldsymbol{\psi}} + \bar{\boldsymbol{\psi}}^*, \tag{2.10}$$

$$\boldsymbol{\nabla} \cdot (\bar{\mathbf{m}} \bar{H}) + \boldsymbol{\nabla} \cdot \overline{\mathbf{m}' H'} - \bar{\mathbf{m}} \cdot \bar{\boldsymbol{\psi}} - \overline{\mathbf{m}' \cdot \boldsymbol{\psi}'} = \bar{T} \bar{Q} + \overline{T' Q'}, \tag{2.11}$$

$$\boldsymbol{\nabla} \cdot (\bar{\mathbf{m}} \bar{s}) + \boldsymbol{\nabla} \cdot \overline{\mathbf{m}' s'} = \bar{Q} + \bar{Q}^*. \tag{2.12}$$

Note that Eqs. (2.8)–(2.12) have been obtained by assuming that the time averages of time derivatives are zero, which implies averaging either over a very long period of time or over a finite number of periods of oscillation. For variables $\mathbf{u}$, p, $\mathbf{m}$, Y_k, $\mathbf{q}_k$, ω_k, $\boldsymbol{\zeta}$, H, T, s, $\boldsymbol{\psi}$, $\boldsymbol{\psi}^*$, Q, Q^*, e_s, g_{sk} and $\Omega_k = \omega_k - \boldsymbol{\nabla} \cdot \mathbf{q}_k - \boldsymbol{\nabla} \cdot (\mathbf{m} Y_k)$, $f(\mathbf{x}, t) = \bar{f}(\mathbf{x}) + \sum \epsilon^i f_i(\mathbf{x}, t)$ where $\bar{f}(\mathbf{x})$ defines the baseline flow. Note that f' is therefore not equivalent to f_1 as defined in Section 2.1. It is exactly $\sum_{i=1}^{+\infty} f_i(\mathbf{x}, t)\epsilon^i$ and thus equals ϵf_1 to order ϵ^2. Note also that Myers (1991) defined his disturbances around a laminar base flow, which has little meaning in many combusting cases, particularly those undergoing strong limit cycle oscillations. Nonetheless, if the disturbances are sufficiently small, the time averages of the products of the disturbances in Eqs. (2.8)–(2.12) are negligible and the laminar equations are recovered.

2.3. *Non-linear disturbance energy*

Myers' approach (1991) consists of subtracting from Eq. (2.6) an appropriate linear combination of Eqs. (2.3), (2.5), and (2.7) in order to obtain an *exact* conservation equation for a disturbance energy. The same approach has been followed in this study, including combustion and mixing effects by including the species mass conservation Eq. (2.3) into the linear combination that is removed from (2.6). A guide to determine a proper linear combination of these equations is to eventually obtain a disturbance energy that satisfies properties **P0-P3**. One shows that a proper choice is to multiply Eq. (2.3) by $(\bar{H} - \bar{T}\bar{s} - \sum_{k=1}^{n} \overline{g_{sk}}\bar{Y}_k)$, Eq. (2.7) by $\bar{T}$, Eq. (2.5) by $\bar{\mathbf{m}}$, and Eq. (2.4) by $\bar{g_{sk}}$ and then subtract all these from (2.6). Making use of Eqs. (2.8), (2.9), (2.11), and (2.12), one eventually obtains an equation of the form (2.1), where the disturbance energy, flux vector, and source terms are defined respectively as

$$E_d = \rho[H - \bar{H} - \bar{T}(s - \bar{s})] - \bar{\mathbf{m}} \cdot (\mathbf{u} - \bar{\mathbf{u}}) - (p - \bar{p}) - \sum_{k=1}^{n} \overline{g_{sk}}\rho(Y_k - \bar{Y}_k) - \overline{\rho' e'_s} \quad (2.13)$$

$$\begin{aligned} \mathbf{W} = {} & (\mathbf{m} - \bar{\mathbf{m}})[(H - \bar{H}) - \bar{T}(s - \bar{s})] + \bar{\mathbf{m}}(T - \bar{T})(s - \bar{s}) \\ & + \overline{\mathbf{m}' H'} - T\overline{\mathbf{m}' s'} \end{aligned} \quad (2.14)$$

and

$$D = D_\xi + D_s + D_Q + D_{Q^*} + D_\psi + D_{\psi^*} + D_{Y_k}, \quad (2.15)$$

where

$$\begin{aligned} D_\xi &= -(\mathbf{m} - \bar{\mathbf{m}}) \cdot (\boldsymbol{\zeta} - \bar{\boldsymbol{\zeta}}) - \overline{\mathbf{m}' \cdot \boldsymbol{\xi}'}, \\ D_s &= -(\mathbf{m} - \bar{\mathbf{m}}) \cdot (s - \bar{s})\boldsymbol{\nabla}\bar{T} + (s - \bar{s})\bar{\mathbf{m}} \cdot \boldsymbol{\nabla}(T - \bar{T}) \\ & \quad - \overline{\mathbf{m}' s'} \cdot \boldsymbol{\nabla} T - \mathbf{m} \cdot \overline{T' \boldsymbol{\nabla} s'} \\ D_Q &= (T - \bar{T})(Q - \bar{Q}) + \overline{T' Q'} \\ D_{Q^*} &= (T - \bar{T})(Q^* - \bar{Q}^*) + \overline{T' Q^{*\prime}} \\ D_{\boldsymbol{\psi}} &= (\mathbf{m} - \bar{\mathbf{m}}) \cdot (\boldsymbol{\psi} - \bar{\boldsymbol{\psi}}) + \overline{\mathbf{m}' \cdot \boldsymbol{\psi}'} \\ D_{\boldsymbol{\psi}^*} &= (\mathbf{m} - \bar{\mathbf{m}}) \cdot (\boldsymbol{\psi}^* - \bar{\boldsymbol{\psi}}^*) + \overline{\mathbf{m}' \cdot \boldsymbol{\psi}^{*\prime}} \\ D_{Y_k} &= \sum_{k=1}^{n} g'_{sk}\Omega'_k + \sum_{k=1}^{n} (g'_{sk} Y_k + \overline{g_{sk}} Y'_k)\, \boldsymbol{\nabla} \cdot \mathbf{m}' \\ & \quad + \sum_{k=1}^{n} \overline{g'_{sk}\Omega'_k} + \overline{g'_s \boldsymbol{\nabla} \cdot \mathbf{m}'}, \end{aligned}$$

where $\Omega_k = \omega_k - \boldsymbol{\nabla} \cdot \mathbf{q}_k - \boldsymbol{\nabla} \cdot (\mathbf{m}Y_k)$. Note that the correlation term in Eq. (2.13) as been introduced so that the leading order term of the disturbance energy does not contain a constant contribution and is positive. Also note that the disturbance energy contains the fluctuation of the turbulent kinetic energy, $E_k = \bar{\rho}\frac{1}{2}(u'_i u'_i - \overline{u'_i u'_i})$.

2.4. *Linearization*

It is obvious from Eq. (2.13) that E_d satisfies property **P0**. Disturbances of the form $()' = () - (\bar{\ }) = \sum_{i=1}^{\infty} \epsilon^i ()_i$ are then substituted into the exact Eq. (2.1), and only the lowest order terms in ϵ are retained. In keeping with the other studies of disturbance energy in isentropic and homentropic flows (Morfey 1971; Cantrell and Hart 1964; Pierce 1981), the remaining terms are of second order in the disturbances, meaning that E_d also satisfies **P1**. Retention of all second-order terms in the exact flux vector and source terms results in a rather complex disturbance energy equation, where much of the complexity is contained in viscous stress, dissipation, and heat conduction terms that can be argued to be negligible in most combusting flows. Ignoring such terms as well as the vorticity terms results in the following linearized disturbance energy equation,

$$\frac{\partial E_{d2}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{W}_2 = D_2, \tag{2.16}$$

where the disturbance energy density E_{d2}, flux vector $\mathbf{W}_2$, and source D_2 terms are

$$E_{d2} = \frac{p_1^2}{2\bar{\rho}\bar{c}^2} + \frac{1}{2}\bar{\rho}\mathbf{u}_1 \cdot \mathbf{u}_1 + \rho_1 \bar{\mathbf{u}} \cdot \mathbf{u}_1 + \frac{\bar{\rho}\bar{T}s_1^2}{2\bar{c_p}} + E_{Y2}, \tag{2.17}$$

$$\mathbf{W}_2 = (p_1 + \bar{\rho}\bar{\mathbf{u}} \cdot \mathbf{u}_1)\left(\mathbf{u}_1 + \frac{\rho_1}{\bar{\rho}}\bar{\mathbf{u}}\right) + \bar{\mathbf{m}}T_1 s_1 + \overline{\mathbf{m}'H'} - T\overline{\mathbf{m}'s'}, \tag{2.18}$$

and

$$\begin{aligned} D_2 = &-\overline{\mathbf{m}'s'} \cdot \boldsymbol{\nabla}\bar{T} - \bar{\mathbf{m}} \cdot \overline{T'\boldsymbol{\nabla}s'} \\ &- s_1\mathbf{m}_1 \cdot \boldsymbol{\nabla}\bar{T} + s_1\bar{\mathbf{m}} \cdot \boldsymbol{\nabla}T_1 \\ &+ \left(\frac{\omega_1 T_1}{\bar{T}} - \frac{\bar{\omega_T}T_1^2}{(\bar{T})^2}\right) + \overline{T'\omega_T'} \\ &+ T_1 Q_1^* + \overline{T'Q^{*'}} + \mathbf{m}_1 \cdot \boldsymbol{\psi}_1^* + \overline{\mathbf{m}' \cdot \boldsymbol{\psi}^{*'}} \\ &+ \sum_{k=1}^{n} g_{sk1}\Omega_{k1} + \sum_{k=1}^{n}\left(g_{sk1}\bar{Y_k} + \bar{g_{sk}}Y_{k1}\right)\boldsymbol{\nabla} \cdot \mathbf{m}_1 \\ &+ \sum_{k=1}^{n}\overline{g_{sk1}\Omega_{k1}} + \overline{g_s'\boldsymbol{\nabla} \cdot \mathbf{m}'}. \end{aligned} \tag{2.19}$$

The E_{Y2} term in Eq. (2.17) is the contribution from the $\rho\bar{g_{sk}}(Y_k - \bar{Y_k})$ terms in Eq. (2.13) and is equal to

$$\begin{aligned} E_{Y2} = &\frac{\bar{p}}{2}\sum_{k=1}^{n}\left[\left(1 + \frac{W}{W_k} - \frac{s_k W}{R}\right)\frac{g_{sk} - e_{sk}}{C_v T} + \frac{W}{W_k}\left(1 + \frac{1}{Y_k}\right)\right]Y_{k1}^2 \\ &+ \bar{p}\sum_{k=1}^{n}\sum_{j\neq k}\left[\left(1 + \frac{W}{W_k} - \frac{s_k W}{R}\right)\frac{g_{sj} - e_{sj}}{C_v T} + \frac{W}{W_j}\right]Y_{k1}Y_{j1} \\ &+ \frac{\rho}{C_v}\sum_{k=1}^{n}\left(g_{sk} - e_{sk}\right)s_1 Y_{k1} + \sum_{k=1}^{n}\left[(\gamma - 1)(g_{sk} - e_{sk}) + \frac{RT}{W_k}\right]\rho_1 Y_{k1}. \end{aligned} \tag{2.20}$$

From Eqs. (2.17)-(2.20), the proposed disturbance energy also satisfies property **P2** since its leading order term only depends on first-order quantities. Note also that Eq. (2.16) simplifies to other, existing acoustic energy conservation equations under the condition of homentropic flow and homogeneous mixture. The energy density E_{d2} and flux $\mathbf{W}_2$ terms then become those defined by Cantrell and Hart (1964) for acoustic propagation in a non-stationary medium. Under the zero Mach number flow assumption and calorific perfection, Eq. (2.16) reduces to the form given in Nicoud and Poinsot (2005). The last three lines in Eq. (2.19) as well as E_{Y2} are related to mixture inhomogeneities over space. These terms do not seem to have been reported elsewhere and require further investigation. Although not negligible, they are not necessary to obtain a reasonable closure of the disturbance energy budget (see Section 3.2).

2.5. *Definite positivity of E_{d2}*

It is not evidenced that E_{d2} as defined in Eq. (2.17) is a positive definite quadratic form (property **P3**). Even in the case of a homogeneous mixture where all the mass fraction fluctuations are zero, this property is not clearly established because of the $\rho_1 \bar{\mathbf{u}} \cdot \mathbf{u}_1$ term (Hanifi *et al.* 1996). We propose in the following a simple proof that E_{d2} is indeed definite positive in the case of a flow without mass fraction fluctuations ($E_{Y2} = 0$) at small enough mean Mach number. Making use of $\rho_1/\bar{\rho} = p_1/(\gamma\bar{p}) - s_1/C_p$, which is true only at chemical equilibrium, and then rewriting E_{d2} in the following matrix form:

$$E_{d2} = F^t A F, \tag{2.21}$$

where F is the reduced first order fluctuation vector $F^t = [p_1/(\bar{\rho}\bar{c}^2) \;\; ||\mathbf{u}_1||/\bar{c} \;\; s_1/C_p]$ and A is the following matrix:

$$A = \frac{\bar{\rho}\bar{c}^2}{2} \begin{bmatrix} 1 & M\mathbf{n}_0 \cdot \mathbf{n}_1 & 0 \\ M\mathbf{n}_0 \cdot \mathbf{n}_1 & 1 & -M\mathbf{n}_0 \cdot \mathbf{n}_1 \\ 0 & -M\mathbf{n}_0 \cdot \mathbf{n}_1 & 1/(\gamma - 1) \end{bmatrix} = \frac{\bar{\rho}\bar{c}^2}{2} B, \tag{2.22}$$

where $\mathbf{n}_0$ and $\mathbf{n}_1$ are $\bar{\mathbf{u}}/||\bar{\mathbf{u}}||$ and $\mathbf{u}_1/||\mathbf{u}_1||$ respectively. Note that B being symmetric, it admits three real eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$ and that the eigenvalues of A are then simply $\lambda_i = \bar{\rho}\bar{c}^2\mu_i/2$. Thus E_{d2} is positive definite as soon as $\lambda_3 > 0$, viz. $\mu_3 > 0$. From Eq. (2.22), the characteristic polynomial of B is proportional to

$$P(\mu) = (\gamma - 1)\mu^3 - (2\gamma - 1)\mu^2 + (\gamma + 1)\mu - (1 - \gamma M^2(\mathbf{n}_0 \cdot \mathbf{n}_1)^2)$$

and its roots are such that $\mu_1 + \mu_2 + \mu_3 = (2\gamma - 1)/(\gamma - 1) > 0$ and $\mu_1\mu_2\mu_3 = (1 - \gamma M^2(\mathbf{n}_0 \cdot \mathbf{n}_1)^2)/(\gamma - 1)$, which is strictly positive as soon as $M < 1/\sqrt{\gamma} \leq 1/(|\mathbf{n}_0 \cdot \mathbf{n}_1|\sqrt{\gamma})$. Under this latter restriction, it follows that the eigenvalues of A are either **(a)** all positive or **(b)** such that $\mu_1 > 0$ and $\mu_3 \leq \mu_2 < 0$. Since $\gamma > 1$, it is obvious that the two roots of the derivative of $P(\mu)$, viz. $P'(\mu) = 3(\gamma - 1)\mu^2 - 2(2\gamma - 1)\mu + \gamma + 1$, are both strictly positive so that at least two roots of $P(\mu)$ are positive. Thus only the **(a)** choice is acceptable and $\mu_3 > 0$ as long as $M < 1/\sqrt{\gamma}$, which is not a restrictive condition in many combusting flows. In other words, the linearized disturbance energy can be recast under the form:

$$E_{d2} = \lambda_1\theta_1^2 + \lambda_2\theta_2^2 + \lambda_3\theta_3^2,$$

where the λ_i's are all positive and the θ_i's are linear combinations of $\bar{\mathbf{u}} \cdot \mathbf{u}_1$, p_1, and s_1. Further work is required to investigate whether this **P3** property still holds in the case where mass fraction fluctuations are accounted for ($E_{Y2} \neq 0$).

3. Numerical Results

3.1. *Configuration*

The 2-D laminar oscillating flame configuration considered for testing the closure of the budget of the disturbance energy is adapted from an experiment described in Le Helley (1994). The burner consists of a ducted premixed propane-air flame that is stabilized thanks to a perforated plate with multiple holes. For certain operation modes this burner features small laminar Bunsen tip flames behind each hole and no turbulence effects are

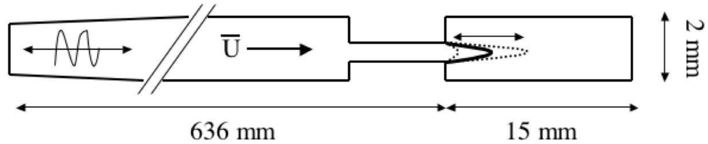

FIGURE 1. Schematic of the computational domain.

FIGURE 2. Heat release when the flame is the shortest (most left) and the longest (second on the left) during the cycle of oscillation. The above illustrates time averaged heat release (first on the right) and temperature (most right).

present. We limit our study to this regime. The 2-D planar computational mesh has been chosen such that chemistry and thermodynamic effects are resolved on the mesh (50,000 nodes). Chemistry is modeled using a single-step Arrhenius law involving five gaseous species. The pre-exponential factor, mass fraction exponent, and activation temperature are fitted to produce the proper flame speed in the lean regime. The main characteristics of the computational flow domain are depicted in Fig. 1 (not to scale). The velocity and temperature are imposed at the inlet while static pressure is prescribed at the outlet. In both cases partially reflecting characteristic boundary conditions (Kaufmann *et al.* 2002; Selle *et al.* 2004) have been used for numerical stability reasons. For the time average mass flow rate ($4.1\mathbf{x}10^{-3}kg.s^{-1}$) concerned and at stoichiometric equivalence ratio, the flame is self-excited at a frequency close to 820 Hz, which corresponds to the third longitudinal acoustic mode of the combustor including the air feeding line.

The extreme positions of the flame over the cycle are shown in Fig. 2, where the time averaged temperature and heat release fields in the flame are shown. Recall that the time averaged solution is the baseline flow and corresponds to no fluctuation and zero disturbance energy in the present analysis.

The numerical tool used in this section is the unstructured combustion code AVBP developed at CERFACS (AVBP 2006). AVBP solves the complete Navier-Stokes equations including chemistry in two and three spatial dimensions. The unstructured approach allows computing not only the combustor but also the whole air feeding line as well as the exhaust system. This code was selected because it solves the complete compressible Navier-Stokes equations under a form that is mathematically equivalent to Eqs. (2.3), (2.4), (2.5), and (2.6). Its ability to reproduce the unsteady behavior of the Le Helley's flame has been demonstrated elsewhere (Kaufmann *et al.* 2002).

3.2. *Energy budget*

The disturbance energy and all the sources and flux terms in Eq. (2.1) have been computed by post-processing 40 fields over two periods of the limit cycle of the flame depicted in Fig. 2. These quantities have subsequently been integrated over the computational domain to obtain

$$\mathcal{E}_d = \int_\Omega E_d d\Omega \qquad \mathcal{W} = \int_S \mathbf{W}\cdot\mathbf{n} dS \qquad \mathcal{D} = \int_\Omega D d\Omega.$$

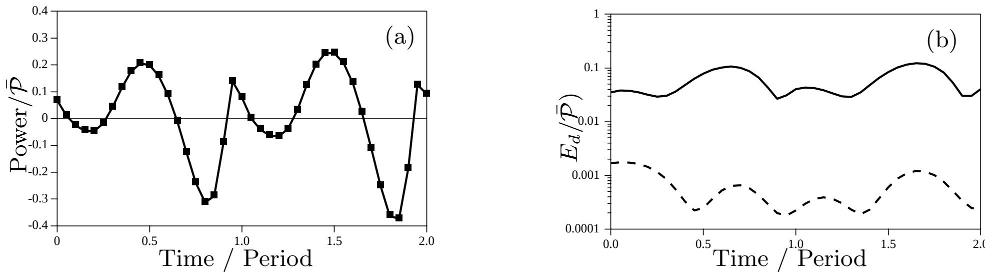

FIGURE 3. Time evolution of **(a)** the time derivative of the total disturbance energy dE_d/dt (■) and spatial terms $D - W$ (———) and **(b)** the disturbance energy E_d for $3.4\mathbf{x}10^{-3} < |\mathbf{m}| < 4.8\mathbf{x}10^{-3}$ kg/s. (present flame, ———) and $1.1\mathbf{x}10^{-3} < |\mathbf{m}| < 1.3\mathbf{x}10^{-3}$ kg/s (- - - -).

As shown in Fig. 3a, the global budget $d\mathcal{E}_d/dt = \mathcal{D} - \mathcal{W}$ closes nicely when all the terms are included. Note that the scaling is by the time averaged total heat release of the flame $\bar{\mathcal{P}} = \int_\Omega \bar{q} d\Omega$. The net power curve shows a near derivative discontinuity at time $t \approx 0.9T$. It corresponds to the shrinking of the flame front, which quickly creates disturbance energy. Note that the time sampling shown on Fig. 3 is not the one used for the assessment of the time derivative of E_d. Fig. 3b depicts the time evolution of the disturbance energy for the flame of Fig. 2 as well as for a calculation with smaller mean and oscillating mass flow rate ($1.1\mathbf{x}10^{-3} < |\mathbf{m}| < 1.3\mathbf{x}10^{-3}$ kg/s instead of $3.4\mathbf{x}10^{-3} < |\mathbf{m}| < 4.8\mathbf{x}10^{-3}$ kg/s). In both cases the disturbance energy remains positive (property **P4**), E_d being larger for larger oscillating mass flow rate.

Order of magnitude analysis suggests that the viscous stress, thermal diffusion, viscous dissipation ($D_{\boldsymbol{\psi}}$) terms are usually small in combustion. The vorticity term (D_ξ) should also be insignificant in most combusting flows although several orders of magnitude larger than the other small terms ($D_{\boldsymbol{\psi}}$). This is confirmed by the numerical results from the 2-D oscillating laminar flame, as shown in Fig. 4. Some terms are 1000 times (or more) smaller than $d\mathcal{E}/dt$ (see Fig. 4a) and do not contribute to the global budget. Note that besides the vorticity and viscous dissipation terms, the boundary terms ($\mathcal{W}$) belong to this category. This result seems to contradict the findings of Martin *et al.* (2004), where the boundary terms were balancing the first-order Rayleigh term $p'\omega'_T$. The difference comes from the fact that only the acoustic part of the fluctuating energy was considered in Martin *et al.* (2004), while E_d also contains the entropy fluctuations. Although the boundary terms are still of the same order as $p'\omega'_T$ (not shown), they are much smaller than the first-order term in the total disturbance energy balance, viz. $T'\omega'_T$. Indeed, the Rayleigh term is approximately $1.5\mathbf{x}10^{-4}$ $\bar{\mathcal{P}}$, while the $T'\omega'_T$ term is roughly $2\mathbf{x}10^{-1}$ $\bar{\mathcal{P}}$ (see Fig. 5).

Other terms in the energy balance are only a few percent of $d\mathcal{E}_d/dt$ and only contribute slightly to the global budget. As shown in Fig. 4b, these terms are the contributions of the D_{Q^*}, $D_{\boldsymbol{\psi}^*}$, and D_{Y_k} terms in Eq. (2.15) and are related to the mixture inhomogeneities and mass fraction fluctuations. Eventually, the first-order term in the energy balance are the contributions from D_Q and D_s (see Fig. 4c). Since $Q = (-\boldsymbol{\nabla} \cdot \mathbf{q} + \phi + \omega_T)/T$ is almost equal to $\omega_T/T - \boldsymbol{\nabla} \cdot \mathbf{q}/T$, the entropy, heat release, and heat flux terms are the most important terms in the energy balance. Figure 4d shows that keeping only these large terms leads to a reasonable closure of the disturbance energy equation.

Figure 5a,b shows the contributions of the different terms in the definition of D_Q and D_s (see Eq. (2.15)). In both cases, all the terms have approximately the same magnitude,

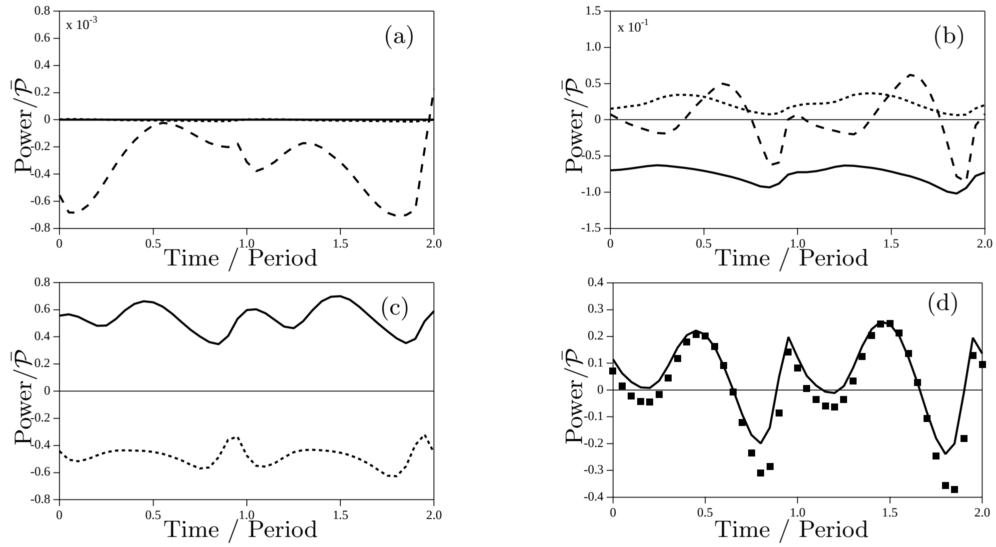

FIGURE 4. Time evolution of the volume integrals of **(a)** the boundary ($\mathcal{W}$, - - - -), D_ξ (———) and D_ψ (········) terms, **(b)** the D_{Q^*} (———), D_{ψ^*} (········) and D_{Y_k} (- - - -) terms, **(c)** the D_Q (········) and D_s (———) terms. Approximate budget **(d)** based on the heat release (D_Q) and entropy (D_s) terms only: exact time derivative of the total disturbance energy (■) and approximate spatial terms (———)

meaning that no further simplification can be made in the energy budget. Noticeably, the correlation terms are important and cannot be neglected in the disturbance energy balance. Recall that these correlation terms arise from the choice of the time averaged solution as the baseline flow. Although it is natural that the definition of the no-fluctuation state appears in the disturbance energy equation, it seems that the importance of these correlation terms has not been reported elsewhere. Figure 5c also shows that only the heat release and heat flux terms contribute in the $T'Q'$ term, the viscous dissipation based term being negligible. The heat release term can be further split in three terms by using the identity

$$+T'\left(\frac{\omega_T}{T}\right)' = T'\frac{\omega'_T}{T} + \bar{\omega_T}T'\left[\frac{1}{T} - \overline{\left(\frac{1}{T}\right)}\right] - T'\overline{\left(\omega'_T\left(\frac{1}{T}\right)'\right)},$$

where the three contributions on the RHS have similar amplitude as shown in Fig. 5d. Note however that the time average of $T'\overline{\omega'_T(1/T)'}$ is zero so that this term does not contribute to the (in)stability of the flow. The first term in this decomposition, viz. $T'\omega'_T/T$, is a *source* term. Note that assuming small amplitude fluctuations, this term becomes $T_1\omega_{T1}/\bar{T}$, the corrected Rayleigh term of Nicoud and Poinsot (2005). The second term in the decomposition of $T'(\omega_T/T)'$ is proportional to $\bar{\omega_T}$ and can be linearized as $-\bar{\omega_T}(T_1/\bar{T})^2$ for small amplitude fluctuations, indicating that this term tends to dissipate disturbance energy. Figure 5d indeed shows that the $\bar{\omega_T}$ term is a *sink* term in which amplitude is comparable to the corrected Rayleigh *source* term so that the heat flux term eventually contributes more than the heat release terms (Fig. 5c). Note that under the zero Mach number assumption used by Nicoud and Poinsot (2005), the time averaged heat release $\bar{\omega_T}$ is null and only the positive contribution of $T'(\omega_T/T)'$, viz. $T_1\omega_{T1}/\bar{T}$, is present. This is another output of the present analysis that non-zero Mach number terms

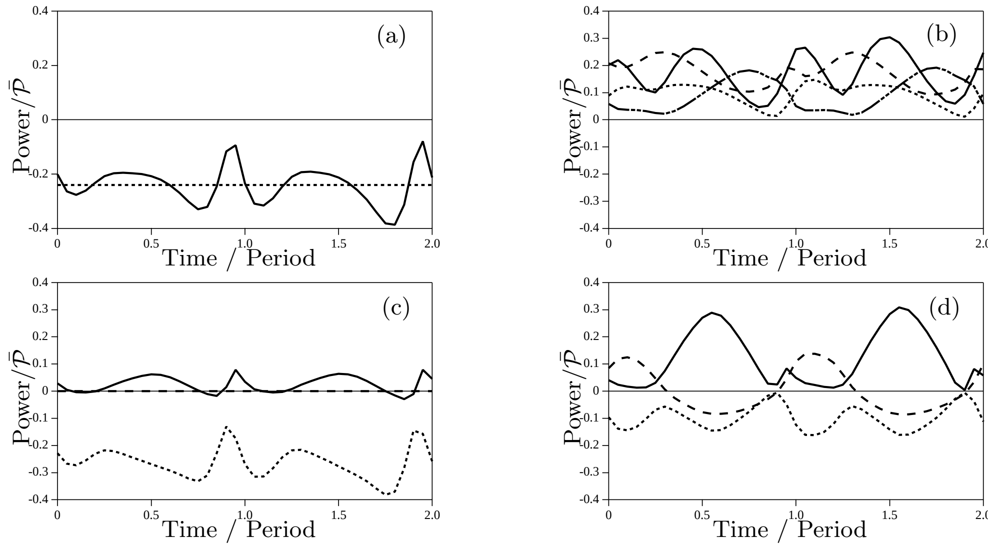

FIGURE 5. Time evolution of the volume integrals of **(a)** the terms in D_Q, viz. $T'Q'$ (———) and $\overline{T'Q'}$ (········) and **(b)** the terms in D_s, viz. $-\mathbf{m'} \cdot s'\boldsymbol{\nabla}\bar{T}$ (———), $s'\bar{\mathbf{m}} \cdot \boldsymbol{\nabla}T'$ (········), $-\overline{\mathbf{m}'s'} \cdot \boldsymbol{\nabla}T$ (– – – –) and $-\mathbf{m} \cdot \overline{T'\boldsymbol{\nabla}s'}$ (—·–), **(c)** the terms in $T'Q'$, viz. $T'(\omega_T/T)'$ (———), $-T'(\boldsymbol{\nabla}\cdot\mathbf{q}/T)'$ (········) and $T'(\phi/T)'$ (– – – –), **(d)** the terms in $T'(\omega_T/T)'$, viz. $T'\omega_T'/T$ (———), $\bar{\omega_T}T'(1/T - \overline{1/T})$ (········) and $-T'\overline{\omega_T'(1/T)'}$ (– – – –)

can have a significant contribution in the energy balance, even though the mean Mach number is very small (of order $3\text{x}10^{-2}$ for the flame considered).

The production terms for the disturbance energy equation are related to entropy as shown in Fig. 5b. In the low Mach number limit, $\boldsymbol{\nabla}\bar{p} = 0$ and $\boldsymbol{\nabla}\bar{T} \propto \boldsymbol{\nabla}\bar{s}$ so that the $-\mathbf{m'} \cdot s'\boldsymbol{\nabla}\bar{T}$ term is proportional to the classical production term $-\mathbf{m'} \cdot s'\boldsymbol{\nabla}\bar{s}$ for scalars. Assuming that acoustic fluctuations are negligible in the reaction zone, one obtains that $s' \propto T'$ so that the $s'\bar{\mathbf{m}} \cdot \boldsymbol{\nabla}T'$ term in D_s is proportional to $\bar{\mathbf{m}} \cdot \boldsymbol{\nabla}(s'^2)$, which is most likely positive since $\bar{\mathbf{m}}$ is from the fresh to the burnt gas and the entropy fluctuations are generated in the flame region. This is indeed also confirmed by Fig. 5b. The remaining entropy terms are based on time averaged correlations and require further investigation.

4. Conclusions

The exact transport equation derived for the disturbance energy from the basic governing equations for a combusting gaseous mixture can be used for generating the most general stability criterion if E_{d2} is indeed positive definite. Although the positive definiteness of E_{d2} has only been shown analytically in cases where mass fraction fluctuations can be neglected, the numerical results obtained for a 2-D laminar flame suggest that it might hold also in the case of large amplitude fluctuations and variable mass fractions. Moreover, the numerical results suggest that the time evolution of the global fluctuating energy is mostly governed by the heat release, heat flux, and entropy source terms, the contribution arising from the mixture changes over space and time being the largest of the negligible terms. Previous classical energy forms for homentropic flows are recovered as special cases of the fluctuating energy defined in this study. It is also shown that the terms proportional to the mean Mach number can be significant even if the baseline flow speed is very small ($M = 3\mathbf{x}10^{-2}$ in the present study).

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REFERENCES

AVBP 2006 Avbp code : www.cerfacs.fr/cfd/avbp_code.php and www.cerfacs.fr/cfd/cfdpublications.html.

Bloxsidge, G., Dowling, A., Hooper, N. and Langhorne, P. 1988 Active control of reheat buzz. *AIAA J.* **26**, 783–790.

Cantrell, R. and Hart, R. 1964 Interaction between sound and flow in acoustic cavities: mass, momentum and energy considerations. *J. Acous. Soc. America* **36**, 697–701.

Chu, B. 1965 On the energy transfer to small disturbances in fluid flow (part i). *Acta Mechanica* pp. 215–234, cTR04.

Chu, B. T. 1956 Stability of systems containing a heat source – the Rayleigh criterion. *NACA Research memorandum* **NACA RM 56D27**.

Chu, B. T. and Kovasnay, L. S. G. 1958 Non–linear interactions in a viscous heat–conducting compressible gas. *J. Fluid Mech.* **3**, 494–514.

Culick, F. E. C. 2001 Dynamics of combustion systems: fundamentals, acoustics and control. In *Active control of engine dynamics*, pp. 89–206. NATO / Von Karman Institute.

Dowling, A. 1997 Nonlinear self-exited oscillations of a ducted flame. *J. Fluid Mech.* **346**, 271–290.

Dowling, A. and Stow, S. 2003 Acoustic analysis of gas turbine combustors. *J. Prop. Power* **5** (19), 751–764.

Flandro, G. A. 1985 Energy balance analysis of nonlinear combustion instability. *J. Prop. Power* **3** (1), 210–221.

Hanifi, A., Schmid, P. J. and Henningson, D. S. 1996 Transient growth in compressible boundary layer fow. *Phys. Fluids* **8**, 51–65.

Kaufmann, A., Nicoud, F. and Poinsot, T. 2002 Flow forcing techniques for numerical simulation of combustion instabilities. *Combust. Flame* **131**, 371–385.

Le Helley, P. 1994 Etude théorique et expérimentale des instabilités de combustion et de leur contrôle dans un brûleur laminaire prémélangé. PhD thesis, UPR 288, CNRS et Ecole Centrale.

Lord Rayleigh 1878 The explanation of certain acoustical phenomena. *Nature* **18**, 319–321.

Martin, C., Benoit, L., Nicoud, F. and Poinsot, T. 2004 Analysis of acoustic energy and modes in a swirled combustor. In *Proc of the Summer Program*, pp. 377–394. Stanford: Center for Turbulence Research, NASA Ames/Stanford Univ.

Morfey, C. L. 1971 Acoustic energy in non–uniform flows. *J. Sound Vib.* **14** (2), 159–170.

Myers, M. 1991 Transport of energy by disturbances in arbitrary steady flows. *J. Fluid Mech.* **226**, 383–400.

Nicoud, F. and Poinsot, T. 2005 Thermoacoustic instabilities: Should the Rayleigh criterion be extended to include entropy changes? *Combust. Flame* **142**, 153–159.

Pierce 1981 *Acoustics: an introduction to its physical principles and applications*. New York: McGraw Hill.

Poinsot, T. and Veynante, D. 2001 *Theoretical and numerical combustion*. R.T. Edwards.

Putnam, A. 1971 *Combustion driven oscillations in industry*. American Elsevier.

Selle, L., Nicoud, F. and Poinsot, T. 2004 The actual impedance of non-reflecting boundary conditions: implications for the computation of resonators. *AIAA J.* **42**, 958–964.