

# Optimal and robust control of transition

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Optimal and robust control theories are used to determine feedback control rules that effectively stabilize a linearly unstable flow in a plane channel. Wall transpiration (unsteady blowing/suction) with zero net mass flux is used as the control. Control algorithms are considered that depend both on full flowfield information and on estimates of that flowfield based on wall skin-friction measurements only. The development of these control algorithms accounts for modeling errors and measurement noise in a rigorous fashion; these disturbances are considered in both a structured (Gaussian) and unstructured (“worst case”) sense. The performance of these algorithms is analyzed in terms of the eigenmodes of the resulting controlled systems, and the sensitivity of individual eigenmodes to both control and observation is quantified.

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## 1. Introduction

The behavior of infinitesimal perturbations in simple laminar flows is an important and well-understood problem. As the Reynolds number is increased, laminar flows often become unstable and transition to turbulence occurs. The effects of the turbulence produced in such flows are very significant and often undesirable, resulting in increased drag and heat transfer at the flow boundaries. Thus, a natural engineering problem is to study methods of flow control such that transition to turbulence can be delayed or eliminated.

Transition often occurs at a Reynolds number well below that required for linear instability of the laminar flow. Orszag & Patera (1983) demonstrate that finite amplitude two-dimensional perturbations can highly destabilize infinitesimal three-dimensional perturbations in the flow. Butler & Farrell (1992) show that the non-orthogonality of the eigenmodes of subcritical flows implies that perturbations of a particular initial structure will experience large amplification of energy before their eventual decay, and suggest that such amplification can sometimes lead to flow perturbations large enough for nonlinear instability to be triggered. Such nonlinear instabilities can lead to transition well below the critical Reynolds number at which linear instability occurs. Results such as these have renewed interest in the control of the small (linear) perturbations, as the mitigation of linear perturbations also lessens the potency of these nonlinear “bypass” mechanisms.

A firm theoretical basis for the control of small perturbations in viscous shear flows is only beginning to emerge. An important step in this direction is provided

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by Joslin *et al.* (1995) and Joshi, Speyer, & Kim (1996), who analyze this problem in a closed-loop framework, in which the dynamics of the flow system together with the controller are examined. Joslin *et al.* (1995) apply optimal control theory to a problem related to the one presented here; in their approach, the control is determined through an adjoint formulation requiring full flowfield information. Joshi, Speyer, & Kim (1996) consider essentially the same problem analyzed in this paper, and show that a simple constant gain feedback with an integral compensator may be used in a single-input/single-output (SISO) sense to stabilize the flow; a single output (the appropriate Fourier component of the streamwise drag) is multiplied by some scalar  $K$  and summed with a reference signal to determine the corresponding component of the control velocity. This proportional approach is a special case of a class of proportional-integral-derivative (PID) controllers, which combine terms which are proportional, integrals, and derivatives of a scalar output of a system.

The present work extends these analyses to rigorously account for state disturbances and measurement noise. A two-step control approach is used. First, a state estimate is developed from a (potentially inaccurate) model of the flow equations, with corrections to this state estimate provided by (noisy) flow measurements fed back through an output injection matrix  $L$ . This state estimate is then multiplied by a feedback matrix  $K$  to determine the control. Potentially, this approach can yield better results than a PID controller. In comparison to the PID approach, the present approach has many more parameters in the control law (specifically, the elements of the matrices  $K$  and  $L$ ), which are rigorously optimized for a clearly defined objective. In this manner, multiple-input/multiple-output (MIMO) systems are handled naturally and the controller is coupled with an estimator which models the dynamics of the system itself.

Many problems in fluid mechanics, especially those involving turbulence, are dominated by nonlinear behavior. In such problems, the linear analysis performed in this paper is not valid. However, optimal control approaches, which use full state information, may still be formulated (Abergel & Temam 1990) and performed (Moin & Bewley 1995) with impressive results. In order to make such schemes practical, one must understand how to account for disturbances in a rigorous fashion and how to estimate accurately the necessary components of the state (for instance, the location and strength of the near-wall coherent structures) based on limited flow measurements. The current paper makes these concepts clear in a fluid-mechanical sense, albeit for a linear problem, and thus provides a step in this development.

The controllers and estimators used in this work are determined by  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  approaches. These techniques have recently been cast in a compact form by Doyle *et al.* (1989), and are well suited to the current problem, in which the issue of interest is the ability of a closed-loop system to reject disturbances to a laminar flow when only a few noisy measurements of the flow are available.

In §2, we derive the governing equations for the present flow stability problem and cast these equations in a standard notation. In §3, the control problem is analyzed in terms of the controllability and observability of each individual eigenmode of the system developed in §2. In §4, the control approach developed in Doyle *et al.* (1989)

is summarized and applied to the present system. In this control approach, two Riccati equations describe a family of  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  controllers which take into account structured (Gaussian) and unstructured (“worst case”) disturbances. Results of these approaches are presented in §5, and §6 presents some concluding remarks.

## 2. Governing equations

This chapter derives the equations governing the perturbations to a laminar channel flow and casts them in a form to which standard control techniques may be applied. This familiar discussion is presented to precisely define the control problem under consideration.

### 2.1 Continuous form of flow equations

Consider a steady plane channel flow with maximum velocity  $U_0$  and channel half-width  $\delta$ . Non-dimensionalizing all velocities by  $U_0$  and lengths by  $\delta$ , the mean velocity profile in the streamwise direction ( $x$ ) may be written  $U(y) = 1 - y^2$  on the domain  $y \in [-1, 1]$ . The equations governing small, incompressible, three-dimensional perturbations ( $v, \omega$ ) are

$$\Delta \dot{v} = \{-i k_x U \Delta + i k_x U'' + \Delta(\Delta/Re)\} v \quad (1a)$$

$$\dot{\omega} = \{-i k_z U'\} v + \{-i k_x U + \Delta/Re\} \omega, \quad (1b)$$

where  $k_x$  is the streamwise wavenumber,  $k_z$  is the spanwise wavenumber,  $\Delta \equiv \partial^2/\partial y^2 - k_x^2 - k_z^2$  is the Laplacian,  $Re \equiv U_0 \delta/\nu$  is the Reynolds number,  $v$  is the Fourier component of the wall-normal velocity, and  $\omega$  is the Fourier component of the wall-normal vorticity. Equation (1a) is the (fourth order) Orr-Sommerfeld equation for the wall-normal velocity modes, and (1b) is the (second order) equation for the wall-normal vorticity modes. Note the one-way coupling between these two equations. Also note that, from any solution ( $v, \omega$ ), the streamwise velocity  $u$  and spanwise velocity  $w$  may be extracted by manipulation of the continuity equation and the definition of wall-normal vorticity into the form

$$u = \frac{i}{k_x^2 + k_z^2} \left( k_x \frac{\partial v}{\partial y} - k_z \omega \right) \quad (2a)$$

$$w = \frac{-i}{k_x^2 + k_z^2} \left( k_z \frac{\partial v}{\partial y} - k_x \omega \right). \quad (2b)$$

Control will be applied at the wall as a boundary condition on the wall-normal component of velocity  $v$ . The boundary conditions on  $u$  and  $w$  are no-slip ( $u = w = 0$ ), which implies that  $\omega = 0$  and (by continuity)  $\partial v/\partial y = 0$  on the wall.

In this development, it is assumed that an array of sensors, which can measure streamwise and spanwise skin friction, and actuators, which provide wall-normal blowing and suction with zero net mass flux, are mounted on the walls of a laminar channel flow. It is also assumed that a sufficient number of sensors and actuators are installed such that individual Fourier components of wall skin friction and wall transpiration may be approximated, and the analysis is carried through for a particular Fourier mode.

2.2 Discrete form of flow equations

The continuous problem described above is discretized on a grid of  $N + 1$  Chebyshev-Gauss-Lobatto points such that

$$y_l = \cos(\pi l/N) \quad \text{for } 0 \leq l \leq N.$$

An  $(N + 1) \times (N + 1)$  matrix  $\mathcal{D}$  may be expressed (Canuto *et al.* 1988, eqn. 2.4.31) such that the derivative of  $\omega$  with respect to  $y$  on the discrete set of  $N + 1$  points is given by

$$\omega' = \mathcal{D}\omega \quad \text{and} \quad \omega'' = \mathcal{D}\omega',$$

where the prime ( $'$ ) now indicates the (partial) derivative of the discrete quantity with respect to  $y$ . The homogeneous Neumann boundary condition on  $v$  is accomplished by modifying the first derivative matrix such that

$$\tilde{\mathcal{D}}_{lm} = \begin{cases} 0 & l = 0, N \\ \mathcal{D}_{lm} & 1 \leq l \leq N - 1. \end{cases}$$

Differentiation of  $v$  with respect to  $y$  is then given by

$$v' = \tilde{\mathcal{D}}v, \quad v'' = \mathcal{D}v', \quad v''' = \mathcal{D}v'', \quad \text{and} \quad v'''' = \mathcal{D}v''''.$$

With these derivative matrices, it is straightforward to write (1) in matrix form. This is accomplished by first expressing the matrix form of (1) on all  $N + 1$  collocation points such that

$$\dot{v} = \mathcal{L}v \tag{3a}$$

$$\dot{\omega} = \mathcal{C}v + \mathcal{S}\omega, \tag{3b}$$

where  $\mathcal{L}$ ,  $\mathcal{C}$ , and  $\mathcal{S}$  are  $(N + 1) \times (N + 1)$ . (Note that, for  $k_x^2 + k_z^2 \neq 0$ , the matrix form of the LHS of (1a) is invertible, so the form (3a) is easily determined.) The Dirichlet boundary conditions are explicitly prescribed as separate “forcing” terms. To accomplish this, decompose  $\mathcal{L}$ ,  $\mathcal{C}$ , and  $\mathcal{S}$  according to

$$\mathcal{L} = \begin{pmatrix} * & * & * \\ b_{11} & A_{11} & b_{12} \\ * & * & * \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} * & * & * \\ b_{21} & A_{21} & b_{22} \\ * & * & * \end{pmatrix} \quad \mathcal{S} = \begin{pmatrix} * & * & * \\ * & A_{22} & * \\ * & * & * \end{pmatrix}$$

where  $A_{11}$ ,  $A_{21}$ , and  $A_{22}$  are  $(N - 1) \times (N - 1)$  and  $b_{11}$ ,  $b_{12}$ ,  $b_{21}$ , and  $b_{22}$  are  $(N - 1) \times 1$ . Noting that  $\omega_0 = \omega_N = 0$  by the no-slip condition, and defining

$$x \equiv \begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \\ \omega_1 \\ \vdots \\ \omega_{N-1} \end{pmatrix} \quad A \equiv \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \quad B \equiv \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad u \equiv \begin{pmatrix} v_0 \\ v_N \end{pmatrix},$$

where  $x$  is  $2(N - 1) \times 1$ ,  $A$  is  $2(N - 1) \times 2(N - 1)$ ,  $B$  is  $2(N - 1) \times 2$ , and  $u$  is  $2 \times 1$ , we may express (3) in the standard form

$$\dot{x} = Ax + Bu. \quad (4)$$

The vector  $x$  is referred to as the “state”, and the vector  $u$  is referred to as the “control”.

### 2.3 Wall measurements

We will consider control algorithms using both full flowfield information and wall information only. For the latter case, we will assume that measurements made at the wall provide information proportional to the streamwise and spanwise skin friction

$$\begin{aligned} y_{m1} &= \left. \frac{\partial u}{\partial y} \right|_{\text{upper wall}} & y_{m2} &= \left. \frac{\partial u}{\partial y} \right|_{\text{lower wall}} \\ y_{m3} &= \left. \frac{\partial w}{\partial y} \right|_{\text{upper wall}} & y_{m4} &= \left. \frac{\partial w}{\partial y} \right|_{\text{lower wall}}. \end{aligned}$$

Equations (2a) and (2b) allow us to express these measurements as linear combinations of  $v$  and  $\omega$ . Defining  $a \equiv i k_x / (k_x^2 + k_z^2)$  and  $b \equiv -i k_z / (k_x^2 + k_z^2)$  and using the derivative matrices, the measurements are expressed as

$$\begin{aligned} y_{m1} &= \left( a \tilde{\mathcal{D}}^2 v + b \mathcal{D} \omega \right)_{\text{upper wall}} & y_{m2} &= \left( a \tilde{\mathcal{D}}^2 v + b \mathcal{D} \omega \right)_{\text{lower wall}} \\ y_{m3} &= \left( b \tilde{\mathcal{D}}^2 v + a \mathcal{D} \omega \right)_{\text{upper wall}} & y_{m4} &= \left( b \tilde{\mathcal{D}}^2 v + a \mathcal{D} \omega \right)_{\text{lower wall}}. \end{aligned}$$

Now decompose  $\tilde{\mathcal{D}}^2$  and  $\mathcal{D}$  according to

$$\tilde{\mathcal{D}}^2 = \begin{pmatrix} d_1 & c_1 & d_3 \\ * & * & * \\ d_2 & c_2 & d_4 \end{pmatrix} \quad \mathcal{D} = \begin{pmatrix} * & c_3 & * \\ * & * & * \\ * & c_4 & * \end{pmatrix},$$

where  $c_1, c_2, c_3$ , and  $c_4$  are  $1 \times (N - 1)$  and  $d_1, d_2, d_3$ , and  $d_4$  are  $1 \times 1$ . Finally, defining

$$y_m \equiv \begin{pmatrix} y_{m1} \\ y_{m2} \\ y_{m3} \\ y_{m4} \end{pmatrix} \quad C \equiv \begin{pmatrix} a c_1 & b c_3 \\ a c_2 & b c_4 \\ b c_1 & a c_3 \\ b c_2 & a c_4 \end{pmatrix} \quad D \equiv \begin{pmatrix} a d_1 & b d_3 \\ a d_2 & b d_4 \\ b d_1 & a d_3 \\ b d_2 & a d_4 \end{pmatrix},$$

where  $y_m$  is  $4 \times 1$ ,  $C$  is  $4 \times 2(N - 1)$ , and  $D$  is  $4 \times 2$ , allows us to express  $y_m$  in the standard form of a linear combination of the state  $x$  and the control  $u$

$$y_m = Cx + Du. \quad (5)$$

The vector  $y_m$  is referred to as the “measurement”.

### 3. Analysis of control problem

In §2, it was shown that the equations governing small perturbations in a laminar channel flow may be expressed in the standard form

$$\dot{x} = Ax + Bu \quad (6a)$$

$$y_m = Cx + Du, \quad (6b)$$

where all variables are complex and the system matrix  $A$  is dense and non-self-adjoint. We now discuss the eigenmodes of  $A$  and identify which of these modes may be modified by the control  $u$  and which may be detected by the measurements  $y_m$ .

It has been shown (Orszag 1971) that, for  $Re \leq 5772$ , the uncontrolled problem itself is stable and, for  $Re > 5772$ , weak instability is seen (though most of the eigenvalues remain stable), with the greatest instability near  $k_x = 1.0$  and  $k_z = 0.0$ . We seek a method to determine the control  $u$  which stabilizes the system in a manner which is robust to system uncertainties. To simplify our discussion, we will restrict our attention in the remainder of this work to the particular case  $Re = 10,000$ ,  $k_x = 1.0$ , and  $k_z = 0.0$ . Joshi, Speyer, & Kim (1996) explore the  $(Re, k_x, k_z)$  parameter space further.

For  $k_z = 0$  (two-dimensional perturbations),  $C = 0$  in (3), entirely decoupling the  $\omega$  eigenmodes from both the  $v$  eigenmodes and from the control  $u = (v_0, v_N)^T$ . In the language of control theory, the  $\omega$  eigenmodes are thus “uncontrollable” by the control  $u$ . (However, it is also seen that the  $\omega$  eigenmodes are stable, so these modes will, so to speak, “take care of themselves”.) Thus, for the remainder of this paper, we will restrict our attention to the  $v$  eigenmodes according to system (6) with

$$x = \begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \end{pmatrix} \quad A = \begin{pmatrix} A_{11} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \end{pmatrix} \quad u = \begin{pmatrix} v_0 \\ v_N \end{pmatrix},$$

where  $x$  is  $(N-1) \times 1$ ,  $A$  is  $(N-1) \times (N-1)$ ,  $B$  is  $(N-1) \times 2$ , and  $u$  is  $2 \times 1$ , and

$$y_m = \begin{pmatrix} y_{m1} \\ y_{m2} \\ y_{m3} \\ y_{m4} \end{pmatrix} \quad C = \begin{pmatrix} a c_1 \\ a c_2 \\ b c_1 \\ b c_2 \end{pmatrix} \quad D = \begin{pmatrix} a d_1 & b d_3 \\ a d_2 & b d_4 \\ b d_1 & a d_3 \\ b d_2 & a d_4 \end{pmatrix},$$

where  $y_m$  is  $4 \times 1$ ,  $C$  is  $4 \times (N-1)$ , and  $D$  is  $4 \times 2$ . (All the constituent matrices, vectors, and flow measurements are described in the previous section.)

#### 3.1 System analysis

We now address whether or not all of the current system's  $N-1$  eigenmodes may be controlled by the  $m=2$  control variables, and whether or not all of these eigenmodes may be observed with the  $p=4$  measurements. To accomplish this, it

is standard practice to consider two matrices which characterize the controllability and observability of the system as a whole (Lewis 1995). These are the system controllability Gramian  $L_c$  of  $(A, B)$  and the system observability Gramian  $L_o$  of  $(C, A)$ , which may be found by solution of

$$\begin{aligned} A L_c + L_c A^* + B B^* &= 0 \\ A^* L_o + L_o A + C^* C &= 0. \end{aligned}$$

Note that stable numerical techniques to solve equations of this form, referred to as Lyapunov equations, are well developed.

If  $L_c$  is (nearly) singular, there is at least one eigenmode of the system which is (nearly) unaffected by any choice of control  $u$ , and the system is called “uncontrollable”. If all uncontrollable eigenmodes are stable, and a controller may be constructed such that the dynamics of the system may be made stable by the application of control, the system is called “stabilizable”.

Similarly, if  $L_o$  is (nearly) singular, there is at least one eigenmode of the system which is (nearly) indiscernible by the measurements  $y_m$ , and the system is called “unobservable”. If all unobservable eigenmodes are stable, and an estimator may be constructed such that the dynamics of the error of the estimate may be made stable by appropriate forcing of the estimator equation, the system is called “detectable”.

For the present system, the smallest eigenvalue of both  $L_c$  and  $L_o$  are computed to be near machine zero, indicating that the present system as derived above is both uncontrollable and unobservable. Gramian analysis can not identify *which* of the eigenmodes are uncontrollable or unobservable, however, so it is impossible to predict from this analysis alone whether or not the system is stabilizable and detectable. For this reason, we now develop a method to determine which of the eigenmodes of a system may be affected by the control  $u$  and, similarly, which eigenmodes may be discerned by the measurements  $y_m$ .

### 3.2 Individual eigenmode analysis

We will now make use of the modal canonical form of the system (6) to quantify the sensitivity of each eigenmode of  $A$  to both control and observation (Kailath 1980). In order to clarify the derivation, we shall examine each eigenmode of the system separately. Define the eigenvalues  $\lambda_i$  and the right and left eigenvectors,  $\xi_i$  and  $\eta_i$ , of  $A$  such that

$$\begin{aligned} \text{right eigenvectors :} & \quad A \xi_i = \lambda_i \xi_i \\ \text{left eigenvectors :} & \quad \eta_i^* A = \lambda_i \eta_i^*, \end{aligned}$$

where the eigenvectors are normalized such that  $\|\xi_i\| = \|\eta_i\| = 1$  for all  $i$ . Assume  $A$  has distinct eigenvalues (this may be verified for the present system described above). Then any  $x$  may be decomposed as a linear combination of the (independent but not orthogonal) right eigenvectors such that

$$x = \sum_i \alpha_i \xi_i. \tag{7a}$$

Differentiating with respect to time,

$$\dot{x} = \sum_i \dot{\alpha}_i \xi_i. \quad (7b)$$

Also, note that left and right eigenvectors corresponding to different eigenvalues are orthogonal, but those corresponding to the same eigenvalues are not

$$(\eta_j, \xi_i) = 0 \quad j \neq i \quad (8a)$$

$$(\eta_j, \xi_i) \neq 0 \quad j = i. \quad (8b)$$

### 3.2.1 Definition of modal control sensitivity

By (6a) and (7), we have

$$\begin{aligned} \sum_i \dot{\alpha}_i \xi_i &= A \sum_i \alpha_i \xi_i + B u \\ &= \sum_i \alpha_i \lambda_i \xi_i + B u. \end{aligned}$$

Taking the inner product with  $\eta_j$  and noting (8a) yields

$$(\eta_j, \dot{\alpha}_j \xi_j) = (\eta_j, \alpha_j \lambda_j \xi_j) + (\eta_j, B u).$$

Noting (8b), this yields

$$\dot{\alpha}_j = \lambda_j \alpha_j + \frac{(B^* \eta_j)^* u}{\eta_j^* \xi_j}.$$

If the vector  $B^* \eta_j = 0$ , then  $\dot{\alpha}_j = \lambda_j \alpha_j$  for any  $u$ . In terms of equation (7a), the component of  $x$  parallel to  $\xi_j$  is not affected by the control  $u$ , and the eigenmode is said to be “uncontrollable”. Further, the norm of the coefficient of  $u$

$$f_j = \frac{|\eta_j^* B B^* \eta_j|^{1/2}}{|\eta_j^* \xi_j|}, \quad (9)$$

which we shall call the control sensitivity of mode  $j$ , is a quantitative measure of the sensitivity of the eigenmode  $j$  to the control  $u$ . Note the dependence of this expression on the matrix  $B B^*$ , which is the same term which drives the Lyapunov equation for controllability Gramian  $L_c$ .

### 3.2.2 Definition of modal observation sensitivity

By (6b) and (7) and assuming, for the moment, that  $u = 0$ , we have

$$y_m = \sum_j \alpha_j C \xi_j.$$

If the vector  $C \xi_j = 0$ , then  $y_m$  will not be a function of  $\alpha_j$ . In terms of equation (7a), the component of  $x$  parallel to  $\xi_j$  does not contribute to the measurements  $y_m$ , and the eigenmode is said to be “unobservable”. Further, the norm of  $C \xi_j$

$$g_j = |\xi_j^* C^* C \xi_j|^{1/2}, \quad (10)$$

which we shall call the observation sensitivity of mode  $j$ , is a quantitative measure of the sensitivity of the measurement  $y_m$  to eigenmode  $j$ . Note the dependence of this expression on the matrix  $C^* C$ , which is the same term which drives the Lyapunov equation for observability Gramian  $L_o$ .



$j$	$\lambda_j$	$f_j$	$g_j$
1	$0.00373967 - 0.23752649 i$	0.266545	102.61
3	$-0.03516728 - 0.96463092 i$	0.000215	72.85
4	$-0.03518658 - 0.96464251 i$	0.000005	1.45
5	$-0.05089873 - 0.27720434 i$	0.026606	347.98
6	$-0.06320150 - 0.93631654 i$	0.000513	81.39
7	$-0.06325157 - 0.93635178 i$	0.000021	2.90
8	$-0.09122274 - 0.90798305 i$	0.000931	83.36
9	$-0.09131286 - 0.90805633 i$	0.000056	4.32
10	$-0.11923285 - 0.87962729 i$	0.001587	77.67
11	$-0.11937073 - 0.87975570 i$	0.000124	5.37
12	$-0.12450198 - 0.34910682 i$	0.171859	69.50
13	$-0.13822653 - 0.41635102 i$	0.037660	252.09
14	$-0.14723393 - 0.85124584 i$	0.002833	63.31
15	$-0.14742560 - 0.85144938 i$	0.000268	5.59
16	$-0.17522868 - 0.82283504 i$	0.005581	44.14
$\vdots$	$\vdots$	$\vdots$	$\vdots$
38	$-0.32519719 - 0.63610486 i$	5.659801	0.78
39	$-0.34373267 - 0.67764346 i$	4.685315	0.64
$\vdots$	$\vdots$	$\vdots$	$\vdots$
53	$-0.66286552 - 0.67027520 i$	0.259581	11.58
$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 1. Least stable eigenmodes of  $A$  (no control) and the associated control and observation sensitivities. Note that all eigenvalues agree precisely with those reported by Orszag (1971). Calculation used Chebyshev collocation technique with  $N = 140$  in quad precision (128 bits per real number). The second eigenmode, which is not shown here, is spurious (see text). Note that the only unstable mode ( $j = 1$ ) for the present system is both sensitive to the control  $u$  and easily detected by the measurements  $y_m$ .

### 3.3 Sensitivity of eigenmodes of $A$ to control and observation

The least stable eigenvalues of  $A$  and their corresponding control and observation sensitivities  $f_j$  and  $g_j$  are tabulated in Table 1. Note that the fourth eigenmode is five orders of magnitude less sensitive than the first eigenmode to modifications in the control. In general, those modes in the upper branch of Fig. 1a (large  $|\Im(\lambda)|$ ) are much less sensitive to control than those in the lower branch (small  $|\Im(\lambda)|$ ). Near the intersection of the two branches ( $\Re(\lambda) \approx -0.3$ ), the control sensitivity is maximum, with this sensitivity decreasing slowly to the left of this intersection ( $\Re(\lambda) < -0.3$ ). It can be predicted that the eigenmodes corresponding to the largest  $f_j$  may be affected most upon application of some feedback control  $u$ .

Note that the flow measurements are two orders of magnitude less sensitive to the fourth eigenmode as they are to the first eigenmode. It can be predicted that the state estimates of the eigenmodes corresponding to the largest  $g_j$  will be most

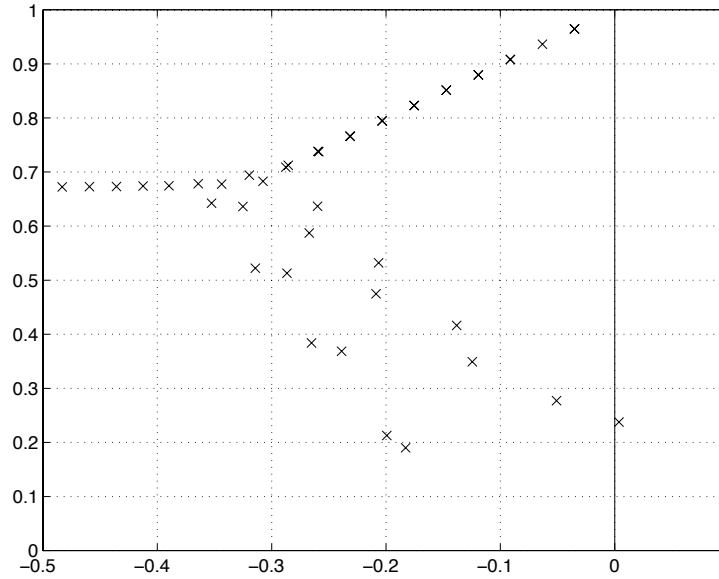


FIGURE 1a. Least stable eigenvalues:  $|\Im(\lambda_j)|$  versus  $\Re(\lambda_j)$ .

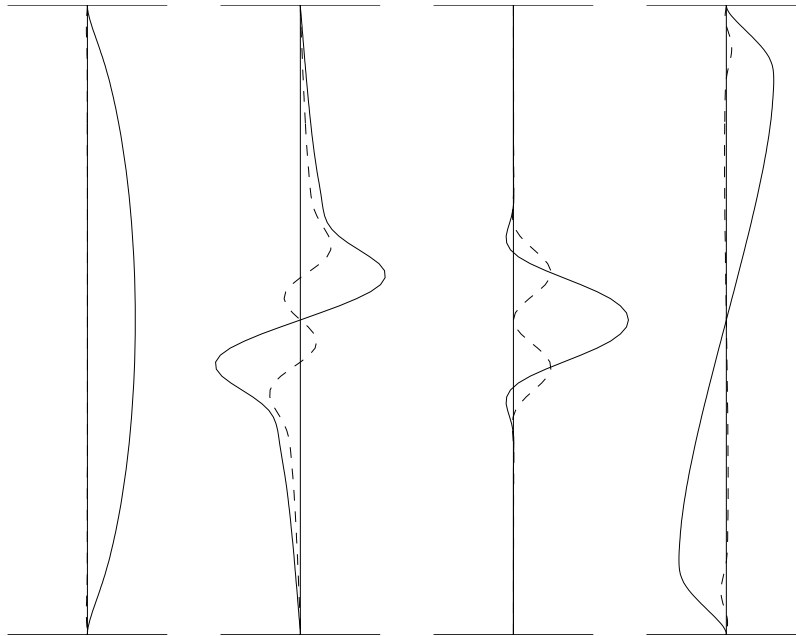


FIGURE 1b. Eigenvectors corresponding to (left to right):  $j = 1$  (unstable, lower branch),  $j = 3$  (stable, upper branch),  $j = 4$  (stable, upper branch), and  $j = 5$  (stable, lower branch), plotted as a function of  $y$  from the lower wall (bottom) to the upper wall (top). Real component of eigenvector is shown solid and imaginary component dashed. Corresponding eigenvalues are reported in Table 1.

accurate when estimating the state based on noisy measurements.

An important observation from Fig. 1b is that eigenvalues in the upper branch of Fig. 1a have corresponding eigenvectors with variations primarily in the center of the channel, and are thus less controllable via wall transpiration and less observable

via wall measurements than eigenvalues in the lower branch. This observation is quantified by reduced values of  $f_j$  and  $g_j$  for these modes in Table 1.

The second eigenvalue computed, at  $\lambda_2 = -0.0235 + 1.520i$  is spurious. Spurious eigenmodes may be easily identified two ways: i) the eigenvalue moves significantly when  $N$  is modified slightly, though the eigenvalues reported in Table 1 remain converged, and ii) when plotted, spurious modes are dominated by large oscillations from grid point to grid point across the entire domain, though converged eigenmodes are well resolved. Spurious eigenmodes are expected using this approach and may be disregarded.

#### 4. Summary of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theories

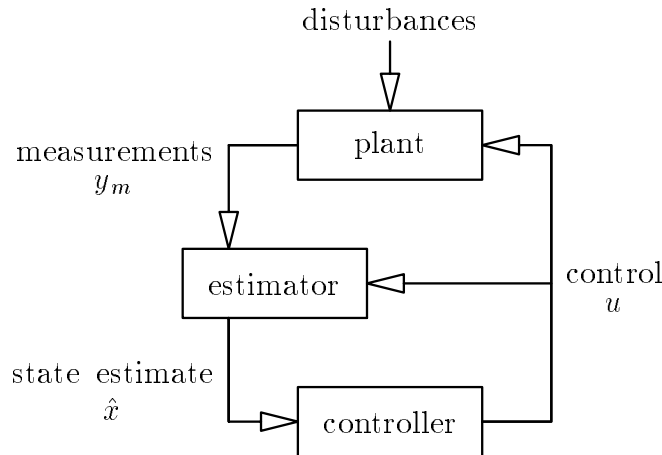
In §2, the equations governing the stability of a laminar channel flow were derived and cast in the form

$$\dot{x} = Ax + Bu \quad (11a)$$

$$y_m = Cx + Du, \quad (11b)$$

where the constituent matrices  $A$ ,  $B$ ,  $C$ , and  $D$  were summarized and discussed in §3. We now seek a simple method to determine a control  $u$  based on the measurements  $y_m$  to force the state  $x$  towards zero in a manner which rigorously accounts for state disturbances, to be added on the RHS of (11a), and measurement noise, to be added on the RHS of (11b).

The flow of information in this problem is illustrated schematically in the following block diagram.



The plant, which is forced by external disturbances, has an internal state  $x$  which cannot be observed. Instead, a few noisy measurements  $y_m$  are made, and with these measurements an estimate of the state  $\hat{x}$  is determined. This state estimate is then fed back to through the controller to determine the control  $u$  to apply back on the plant in order to regulate  $x$  to zero.

To be more precise, we will consider feedback of the measurements  $y_m$  such that a state estimate  $\hat{x}$  is first determined by the system model

$$\dot{\hat{x}} = A\hat{x} + Bu - \hat{u} \quad (12a)$$

$$\hat{y}_m = C\hat{x} + Du, \quad (12b)$$

$$\hat{u} = \mathcal{L}(y_m - \hat{y}_m), \quad (12c)$$

then this state estimate is used to produce the control

$$u = \mathcal{K}(\hat{x}). \quad (13)$$

Equation (11), with added disturbance terms on the RHS, is referred to as the “plant”, (12) is referred to as the “estimator”, and (13) is referred to as the “controller”. The problem at hand is to compute linear time-invariant (LTI) functions  $\mathcal{L}$  and  $\mathcal{K}$  such that i) the “output injection” term  $\hat{u}$  forces the state estimate  $\hat{x}$  in the estimator (12) towards the state  $x$  in the plant (11), and ii) the control  $u$  computed by the controller (13) forces the state  $x$  towards zero in the plant (11).

We will now demonstrate how to apply  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  control theories to determine  $\mathcal{L}$  and  $\mathcal{K}$ . (Note that we will redefine several variables used in §2 to derive the Orr-Sommerfeld equation. Considered in the context of this chapter, this should present no confusion.) With this presentation, one set of control equations, involving the solution of two Riccati equations, describes a family of  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  control algorithms. The reader is referred to Doyle *et al.* (1989), Dailey *et al.* (1990), and Zhou, Doyle, & Glover (1996) for derivation and further discussion of the control theories summarized here.

#### 4.1 $\mathcal{H}_2$ control theory

##### 4.1.1 Optimal control (LQR)

The first step in considering the system (11) is to consider the problem with no disturbances and measurements which identically determine full information about the state, so that  $\hat{x} = x$  (*i.e.* no estimation of the state is necessary). These assumptions are quite an idealization and can rarely be accomplished in practice, but this exercise is an important step to determine the best possible system performance. It is for this reason that the controller in this limit is referred to as optimal. Under these assumptions about the system, the objective of the optimal controller, of the form in (13), is to regulate (*i.e.* return to zero) some measure of the flow perturbation  $x$  from an arbitrary initial condition as quickly as possible without using excessive amounts of control forcing. Mathematically, a cost function for this problem may thus be expressed as

$$\mathcal{J}_{LQR} \equiv \int_0^\infty (||x||^2 + \ell^2 u^*u) dt. \quad (14)$$

The term involving  $||x||^2$  is a measure of the state disturbance  $x$  integrated over the time period over which the initial perturbation decays, which is taken as  $t \in [0, \infty)$ . The term involving  $u^*u$  is an expression of the magnitude of the control. These two terms are weighted together with a scalar  $\ell^2$ , which represents the price of the control. This quantity is small if the control is “cheap” (which generally results in

larger control magnitudes), and large if applying the control is “expensive”. As the state equation is linear, the cost quadratic, and the control objective regulation, this controller is also referred to as a linear quadratic regulator (LQR).

The mathematical statement of the present control problem, then, is the minimization of  $\mathcal{J}_{LQR}$ . This results in regulation of  $x$  without excessive use of control effort. Note that minimization of  $\mathcal{J}_{LQR}$  is equivalent to minimization of the integral of  $z^*z$ , where

$$z \equiv \begin{pmatrix} Q^{1/2} x/\ell \\ u \end{pmatrix}.$$

and where  $Q$  is a diagonal matrix with diagonal entries  $Q_{jj} = \pi/N$ , as required by the appropriate definition of the inner product (Canuto *et al.* 1988). In order to arrive at a form which is easily generalized in later sections, define

$$B_2 \equiv B \quad C_1 \equiv \begin{pmatrix} Q^{1/2}/\ell \\ 0 \end{pmatrix} \quad D_{12} \equiv \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

For notational convenience, the state equation (11a) will be considered as “forced” with a right hand side forcing term  $r$  which shall be set to zero, as this regulation problem simply drives the state towards zero without external command input. The state equation (11a), the performance measure  $z$ , and the state estimate  $\hat{x}$  then may be written

$$\dot{x} = Ax + r + B_2 u \tag{15a}$$

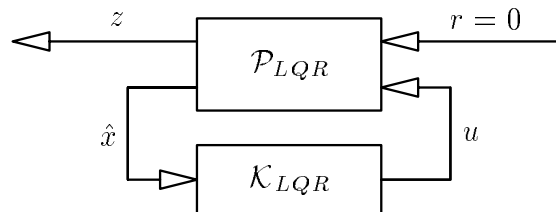
$$z = C_1 x + D_{12} u \tag{15b}$$

$$\hat{x} = x. \tag{15c}$$

The optimal controller  $\mathcal{K}_{LQR}$  is sought to relate the (precise) state estimate  $\hat{x}$  to the control  $u$ , which is applied to control the evolution of the state  $x$  such that the cost  $\mathcal{J}_{LQR}(z)$  is minimized. The important matrices of the system described by (15) may be summarized in the shorthand form

$$\mathcal{P}_{LQR} = \begin{array}{c} \dot{x} \\ z \\ \hat{x} \end{array} \left[ \begin{array}{c|cc} & x & r & u \\ \hline A & I & B_2 \\ \hline C_1 & 0 & D_{12} \\ \hline I & 0 & 0 \end{array} \right].$$

The flow of information is represented by the block diagram



where  $\mathcal{P}_{LQR}$  is the flow system given by (15) and  $\mathcal{K}_{LQR}$  is the optimal controller, which is still to be determined. The system output  $z$  may be used to monitor the

performance of the system. Note that the command input is  $r = 0$  and there are no disturbance inputs; the task of the control  $u$  is simply to regulate the state  $x$  from nonzero initial conditions back to zero. The state  $x = \hat{x}$  is fed back through the controller  $\mathcal{K}_{LQR}$  to control the system.

Given this general setup, a Hamiltonian is defined such that

$$H_2 \equiv \begin{pmatrix} A & -B_2 B_2^* \\ -C_1^* C_1 & -A^* \end{pmatrix}. \quad (16a)$$

As shown in Doyle *et al.* (1989), the Hermetian positive-definite solution  $X_2$  to the algebraic Riccati equation defined by this Hamiltonian

$$A^* X_2 + X_2 A - X_2 (B_2 B_2^*) X_2 + (C_1^* C_1) = 0, \quad (16b)$$

denoted  $X_2 = \text{Ric}(H_2)$ , then yields the optimal LTI state feedback matrix

$$K_2 = -B_2^* X_2. \quad (16c)$$

The optimal LTI controller  $\mathcal{K}_{LQR}$  is then given simply by

$$u = K_2 \hat{x}. \quad (17)$$

This controller minimizes  $\int_0^\infty z^* z dt$  in a system with no disturbances and arbitrary initial conditions. Note that standard numerical techniques to solve equations of the form (16b) are well developed (Laub 1991).

#### 4.1.2 Kalman-Bucy filter (KBF)

When there are disturbances to the system, and thus the state is not precisely known, the state (or some portion thereof) must first be estimated, then the control determined based on this state estimate. The Kalman-Bucy filter, of the form (12), accomplishes the required state estimation by assuming that the state disturbances and the measurement noise are uncorrelated white Gaussian processes. To accomplish this, we introduce two zero-mean white Gaussian processes  $w_1$  and  $w_2$  with covariance matrices  $E[w_1^* w_1] = I$ ,  $E[w_2^* w_2] = I$ , where  $E[\cdot]$  denotes the expectation value. With these new disturbance signals, and with  $G_1$  defined as the square root of the covariance of the disturbances to the state equation and  $G_2$  defined as the square root of the covariance of measurement noise, the system (11) takes the form

$$\dot{x} = Ax + G_1 w_1 + Bu \quad (18a)$$

$$y_m = Cx + G_2 w_2 + Du. \quad (18b)$$

The objective of the Kalman-Bucy filter is to estimate the state  $x$  as accurately as possible based solely on the measurements  $y_m$ . Put another way, the Kalman-Bucy filter attempts to regulate the norm of the state estimation error  $x_E$  to zero, where

$$x_E \equiv x - \hat{x}$$

and where the state estimate  $\hat{x}$  shall be determined by a filter of the form (12). Mathematically, a cost function for this problem may thus be expressed as

$$\mathcal{J}_{KBF} \equiv E[ \|z_E\|^2 ],$$

where  $z_E \equiv x_E$  for notational convenience. (As Gaussian disturbances  $w_1$  and  $w_2$  continually drive this system, an integral on  $t \in [0, \infty)$ , as used to define  $\mathcal{J}_{LQR}$ , is not convergent for this problem, and the expectation value is the relevant measure.)

The mathematical statement of the present control problem, then, is the minimization of  $\mathcal{J}_{KBF}$ . This results in a “best possible” estimate of the state  $x$ . In order to arrive at a form which is easily generalized in later sections, assume  $G_2$  is nonsingular and define

$$B_1 \equiv (G_1 \quad 0) \quad C_2 \equiv G_2^{-1}C \quad D_{21} \equiv (0 \quad I)$$

and the vector of disturbances

$$w \equiv \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Also, define new “observation” vectors  $y$  and  $\hat{y}$  by a simple change of variables such that

$$y \equiv G_2^{-1}(y_m - D u) \quad \hat{y} \equiv G_2^{-1}(\hat{y}_m - D u).$$

Note that this change of variables does not represent any real limitation, for whenever any flow measurement  $y_m$  is made in a physical implementation, the control  $u$  at that moment is also known, so the observation  $y$  is easily determined from the flow measurement  $y_m$ . With this change of variables, (18b) and (12b) may be expressed as

$$y = C_2 x + D_{21} w \tag{19a}$$

$$\hat{y} = C_2 \hat{x}. \tag{19b}$$

As we are developing the equations for an estimator, it is appropriate now to examine the equations for the state estimation error  $x_E$  and the output estimation error  $y_E \equiv y - \hat{y}$ . Subtracting (12a) from (18a) and (19b) from (19a) yields the system

$$\dot{x}_E = A x_E + B_1 w + \hat{u} \tag{20a}$$

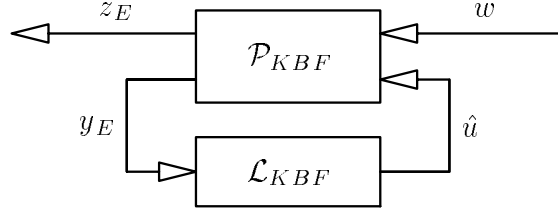
$$z_E = x_E \tag{20b}$$

$$y_E = C_2 x_E + D_{21} w. \tag{20c}$$

The Kalman-Bucy filter  $\mathcal{L}_{KBF}$  is sought to relate the output estimation error  $y_E$  to the output injection term  $\hat{u}$ , which is used to control the evolution of the state estimation error  $x_E$  such that the cost  $\mathcal{J}_{KBF}(z_E)$  is minimized in the presence of Gaussian disturbances  $w$ . The important matrices of the system described by (20) may be summarized in the shorthand form

$$\mathcal{P}_{KBF} = \begin{array}{c} \dot{x}_E \\ z_E \\ y_E \end{array} \left[ \begin{array}{c|cc} x_E & w & \hat{u} \\ \hline A & B_1 & I \\ I & 0 & 0 \\ \hline C_2 & D_{21} & 0 \end{array} \right]$$

The flow of information is represented by the block diagram



where  $\mathcal{P}_{KBF}$  is the flow system given by (18) and  $\mathcal{L}_{KBF}$  is the Kalman-Bucy filter, which is still to be determined. The system output  $z_E$  may be used to monitor the performance of the system. This system accounts for Gaussian disturbances  $w$  and noisy observations  $y_E$  of the system, which are fed back through the filter  $\mathcal{L}_{KBF}$  to produce the state estimate. Note the striking similarity of the structure of  $\mathcal{P}_{KBF}$  to the structure of the conjugate transpose of  $\mathcal{P}_{LQR}$ . For this reason, these two problems are referred to as “duals”, and their solutions are closely related.

Given this general setup, another Hamiltonian is defined such that

$$J_2 \equiv \begin{pmatrix} A^* & -C_2^* C_2 \\ -B_1 B_1^* & -A \end{pmatrix}. \quad (21a)$$

As shown in Doyle *et al.* (1989), the Hermetian positive-definite solution  $Y_2$  to the algebraic Riccati equation defined by this Hamiltonian

$$A Y_2 + Y_2 A^* - Y_2 (C_2^* C_2) Y_2 + (B_1 B_1^*) = 0, \quad (21b)$$

denoted  $Y_2 = \text{Ric}(J_2)$ , then yields the LTI estimator feedback matrix

$$L_2 = -Y_2 C_2^*. \quad (21c)$$

The LTI Kalman-Bucy filter  $\mathcal{L}_{KBF}$  is then simply given by

$$\hat{u} = L_2 y_E,$$

and thus the complete state estimator is given by

$$\dot{\hat{x}} = A \hat{x} + B_2 u - L_2 (y - C_2 \hat{x}) \quad (22)$$

This estimator minimizes  $E[||x - \hat{x}||^2]$  in a system with Gaussian disturbances in the state equation and Gaussian noise in the measurements.



### 4.1.3 $\mathcal{H}_2$ control ( $LQG = LQR + KBF$ )

A controller/estimator of the form (12)—(13) for the complete system described by (18) with Gaussian disturbances may now be constructed. The objective of the control is to minimize

$$\mathcal{J}_2 \equiv E[\|x\|^2 + \ell^2 u^* u],$$

where  $\|\cdot\|$  denotes the standard Euclidian norm, also known as a “2-norm”. Note that minimization of  $\mathcal{J}_2$  is equivalent to minimization of the expectation value of  $z^* z$ , where

$$z \equiv \begin{pmatrix} Q^{1/2} x / \ell \\ u \end{pmatrix},$$

and  $Q$  is a diagonal matrix with diagonal entries  $Q_{jj} = \pi/N$  as required by the appropriate definition of the inner product. As the control objective is the minimization of the expectation value of the square of a 2-norm, this type of controller/estimator is referred to as  $\mathcal{H}_2$ . As the state equation is linear, the cost quadratic, and the disturbances Gaussian, this type of controller/estimator is also referred to as linear quadratic Gaussian (LQG).

Combining the notation developed in the previous two sections

$$\begin{aligned} B_1 &\equiv (G_1 \quad 0) & C_1 &\equiv \begin{pmatrix} Q^{1/2}/\ell \\ 0 \end{pmatrix} & D_{12} &\equiv \begin{pmatrix} 0 \\ I \end{pmatrix} \\ B_2 &\equiv B & C_2 &\equiv G_2^{-1} C & D_{21} &\equiv (0 \quad I), \end{aligned}$$

with the vector of disturbances  $w$  and the observation vectors  $y$  and  $\hat{y}$  defined such that

$$w \equiv \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \begin{aligned} y &\equiv G_2^{-1}(y_m - D u) \\ \hat{y} &\equiv G_2^{-1}(\hat{y}_m - D u), \end{aligned}$$

the system (18) and the control objective for the minimization of  $\mathcal{J}_2$  take the form

$$\dot{x} = A x + B_1 w + B_2 u \quad (23a)$$

$$z = C_1 x + D_{12} u. \quad (23b)$$

$$y = C_2 x + D_{21} w. \quad (23c)$$

An  $\mathcal{H}_2$  controller/estimator is sought to relate the observations  $y$  to the control  $u$ , which is applied to control the evolution of the state  $x$  such that the cost  $\mathcal{J}_2(z)$  is minimized in the presence of Gaussian disturbances  $w$ .

The remarkable result from control theory (Lewis 1995) is that the  $\mathcal{H}_2$  controller/estimator of the form (12)—(13) which minimizes  $\mathcal{J}_2$  for this system is formed by simple combination of the optimal controller and the Kalman-Bucy filter such that

$$u = K_2 \hat{x} \quad (24a)$$

$$\dot{\hat{x}} = A \hat{x} + B_2 u - L_2 (y - C_2 \hat{x}) \quad (24b)$$

where  $K_2$  is given by (16)

$$K_2 = -B_2^* X_2 \quad X_2 = \text{Ric} \begin{pmatrix} A & -B_2 B_2^* \\ -C_1^* C_1 & -A^* \end{pmatrix} \quad (24c)$$

and  $L_2$  is given by (21)

$$L_2 = -Y_2 C_2^* \quad Y_2 = \text{Ric} \begin{pmatrix} A^* & -C_2^* C_2 \\ -B_1 B_1^* & -A \end{pmatrix}. \quad (24d)$$

Note the separation structure of this solution. The computation of  $K_2$  does not depend upon the influence of the disturbances, which are accounted for in  $B_1$  and  $C_2$ . The computation of  $L_2$  does not depend upon the weightings in the cost function, which are accounted for in  $C_1$ , or the manner in which the control  $u$  affects the state, which is accounted for in  $B_2$ . In other words, the problem of control and the problem of state estimation are entirely decoupled.

#### 4.2 $\mathcal{H}_\infty$ control

The  $\mathcal{H}_\infty$  controller/estimator described in this section is very similar to the  $\mathcal{H}_2$  controller/estimator described previously. Consideration is now given to disturbances, which we shall distinguish with a new variable  $\chi$ , of the “worst” possible structure (as made precise below), rather than the Gaussian structure assumed in the  $\mathcal{H}_2$  case. Considered in the frequency domain, the controller/estimators developed in this section provide a system behavior in which the maximum singular value of the closed-loop transfer function, also known as the “ $\infty$ -norm”, is less than some constant, which shall be referred to as  $\gamma$ . As this approach may be interpreted as bounding the  $\infty$ -norm of the transfer function from the disturbances to the performance measure, it is referred to as  $\mathcal{H}_\infty$  control. For further details of the frequency-domain explanation of  $\mathcal{H}_\infty$ , the reader is referred to Doyle *et al.* (1989) and Zhou, Doyle, & Glover (1996).

The governing equations to be considered in this section are identical to (23):

$$\dot{x} = Ax + B_1 \chi + B_2 u \quad (25a)$$

$$z = C_1 x + D_{12} u. \quad (25b)$$

$$y = C_2 x + D_{21} \chi. \quad (25c)$$

An  $\mathcal{H}_\infty$  controller/estimator is sought to relate the observations  $y$  to the control  $u$ , which is applied to control the evolution of the state  $x$  such that the cost  $\mathcal{J}_\infty(z)$  is minimized in the presence of some “worst case” disturbance  $\chi$ . As before, the  $G_1$  and  $G_2$  matrices used to define this system describe any covariance structure of the disturbances known or expected *a priori* (for instance, if one measurement is known to be noisier than another). These matrices are taken as identity matrices if no such structure is known in advance.

Effectively, the cost function considered for  $\mathcal{H}_\infty$  control is

$$\mathcal{J}_\infty \equiv E[x^* Q x + \ell^2 u^* u - \gamma^2 \chi^* \chi]. \quad (26)$$

A  $u$  is sought, through a controller/estimator of the form (12)—(13), to *minimize*  $\mathcal{J}_\infty$ , while *simultaneously* an external disturbance  $\chi$  is sought to *maximize*  $\mathcal{J}_\infty$ . (In this manner,  $\chi$  is the “worst possible” disturbance, as it is exactly that disturbance which increases the relevant cost function the most.) Thus, the  $\mathcal{H}_\infty$  problem is a “min-max” problem. The term involving  $-\gamma^2$  limits the magnitude of the unstructured disturbance in the maximization of  $\mathcal{J}_\infty$  with respect to  $\chi$  in a manner analogous to the term involving  $\ell^2$ , which limits the magnitude of the control in the minimization of  $\mathcal{J}_\infty$  with respect to  $u$ .

The result (Doyle *et al.* 1989) is that an  $\mathcal{H}_\infty$  controller/estimator of the form (12)—(13) which minimizes  $\mathcal{J}_\infty$  in the presence of some component of the worst case unstructured disturbance  $\chi$  for this system is given by

$$u = K_\infty \hat{x} \quad (27a)$$

$$\dot{\hat{x}} = A \hat{x} + B_2 u - L_\infty (y - C_2 \hat{x}) \quad (27b)$$

where  $K_\infty$  is given by

$$K_\infty = -B_2^* X_\infty \quad X_\infty = \text{Ric} \left( \begin{array}{cc} A & \gamma^{-2} B_1 B_1^* - B_2 B_2^* \\ -C_1^* C_1 & -A^* \end{array} \right) \quad (27c)$$

and  $L_\infty$  is given by

$$L_\infty = -Y_\infty C_2^* \quad Y_\infty = \text{Ric} \left( \begin{array}{cc} A^* & \gamma^{-2} C_1^* C_1 - C_2^* C_2 \\ -B_1 B_1^* & -A \end{array} \right). \quad (27d)$$

Note first that, in the  $\gamma \rightarrow \infty$  limit, the  $\mathcal{H}_2$  controller/estimator is recovered, so the set of two Riccati equations in (27) describes both the  $\mathcal{H}_2$  (optimal control + Kalman-Bucy filter) and the  $\mathcal{H}_\infty$  problems.

It may also be shown that, as the upper-right blocks of the Hamiltonians may not be negative definite, a solution to these Riccati problems exists only for sufficiently large  $\gamma$ ; the smallest  $\gamma = \gamma_0$  for which a solution to these equations exists may be found by trial and error (Doyle *et al.* 1989). An  $\mathcal{H}_\infty$  controller/estimator for  $\gamma > \gamma_0$  is referred to as suboptimal.

#### 4.3 Comparison of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control equations

Most of the robustness problems associated with  $\mathcal{H}_2$  stem from the state estimation. Optimal (LQR) controllers themselves, provided with full state information, generally have excellent performance and robustness properties (Dailey *et al.* 1990). Recall from §4.1.3 that the problems of control and state estimation in the  $\mathcal{H}_2$  formulation are decoupled.

An important observation of §4.2 is that the problems of control and state estimation in the  $\mathcal{H}_\infty$  formulation are coupled. Specifically, the computation of  $K_\infty$  depends on the expected covariance of the state disturbances, which are accounted for in  $B_1$ , and the computation of  $L_\infty$  depends on the weightings in the cost function, which are accounted for in  $C_1$ . This is one of the essential features of  $\mathcal{H}_\infty$  control.

By taking into account the expected covariance of the state disturbances, reflected in  $B_1$ , when determining the state feedback matrix  $K_\infty$ , the components of  $\hat{x}$  corresponding to the components of  $x$  that are expected to have the smallest forcing by external disturbances are weighted least in the feedback control relationship  $u = K_\infty \hat{x}$ .

Similarly, by taking into account the weightings in the cost function, reflected in  $C_1$ , when determining the estimator feedback matrix  $L_\infty$ , the components of  $\hat{x}$  corresponding to the components of  $x$  that are least important in the computation of  $\mathcal{J}_\infty$  are forced with the smallest corrections by the output injection term  $L_\infty (y - \hat{y})$  in the equation for the estimator.

By applying strong control only on those components of  $\hat{x}$  significantly excited by external disturbances, and by applying strong estimator corrections only to those components of  $\hat{x}$  important in the computation of the cost function,  $\mathcal{H}_\infty$  feedback gains for components of the system not relevant to the control problem are reduced from those in the  $\mathcal{H}_2$  case. With such feedback gains reduced, the stability properties of  $\mathcal{H}_\infty$  controller/estimators in the presence of state disturbances and measurement noise may be expected to be better than their  $\mathcal{H}_2$  counterparts, at the cost of a (hopefully, small) degradation of performance in terms of the 2-norm of the output  $z$  for the undisturbed system.

#### 4.4 Numerical method

Standard numerical techniques are now applied to all aspects of this problem. In order to simplify both the theory to be presented and the numerical algorithm to be coded, no further manipulation of the equations is used beyond the matrix representations (25) and (27). It was observed that the minimal realization approach (Kailath 1980) is well suited to reduce the computation time necessary to determine effective control algorithms by the present approach; however, such an approach was not found to be necessary in the present case.

The algebraic Riccati equations are solved using the method of Laub (1991), which involves a Schur factorization. This is found to be a stable numerical algorithm for all cases tested. The implementation of Laub's method is written in Fortran-90 and follows closely the algorithm used by the Matlab function `are.m` (Grace *et al.* 1992). A Lyapunov solver, modeled after the Matlab function `lyap.m`, is used to compute the system Gramians.

Two LAPACK routines (Anderson *et al.* 1995), `zgeev.f` and `zgees.f`, are used to compute eigenvalues/eigenvectors and Schur factorizations. These routines are compiled in quad precision (128 bits per real number) to ensure sufficient numerical precision in the eigenvalue computation. All computations are carried out with  $N = 140$  to ensure good resolution of all significant eigenmodes. The eigenvalues of  $A$  match all those tabulated by Orszag (1971) to all eight decimal places, as shown in Table 1, indicating that this numerical method is sufficiently accurate.

### 5. Performance of controlled systems (no disturbances)

We now examine the behavior of the "closed-loop" systems obtained by application of the above controllers and estimators to the "nominal" (*i.e.* no disturbances)

channel flow stability problem. In other words, we examine the behavior of the flow and the controller/estimators operating together as a single dynamical system. By looking at “root locus” plots which map the movement of the eigenvalues of these systems in the complex plane with respect to the relevant parameters, this behavior is well quantified. We shall also examine the control and observation sensitivities defined in §3.2 for two special cases in order to better understand the fundamental limitations of controllers and estimators applied to the present system.

### 5.1 $\mathcal{H}_2$ control

#### 5.1.1 Optimal control (LQR)

In order to investigate the controllability of the closed-loop eigenmodes when all modes are observable, consider the system described in §4.1.1. With  $r = 0$  and examining only the equations for  $\dot{x}$  and  $\hat{x}$ , the plant is given (in the shorthand notation used in §4) by

$$\mathcal{P}_{LQR} = \begin{array}{c} \dot{x} \\ \hat{x} \end{array} \left[ \begin{array}{c|c} x & u \\ \hline A & B_2 \\ \hline I & 0 \end{array} \right]$$

with the control now given by

$$u = K_2 \hat{x} + u',$$

where an additional control term  $u'$  has been added to study the sensitivity of the closed-loop system to further modification of the control. Putting the plant and the controller together, the closed-loop system may be represented by

$$\mathcal{P}_{LQR(\text{closed loop})} = \begin{array}{c} \dot{x} \\ \hat{x} \end{array} \left[ \begin{array}{c|c} x & u' \\ \hline A + B_2 K_2 & B_2 \\ \hline I & 0 \end{array} \right].$$

The eigenmodes of  $A_{K_2} \equiv A + B_2 K_2$  describe the dynamics of the closed-loop system for the unmodified control rule ( $u' = 0$ ). Figure 2 shows the movement of these eigenvalues with respect to the free parameter of the control problem,  $\ell$ , used to determine  $K_2$ . The eigenvalues for  $\ell \rightarrow \infty$  are very near those of the uncontrolled system  $A$  in Fig. 1, with the previously unstable mode moved just to the left of the imaginary axis. The eigenvalues generally move to the left as  $\ell$  is decreased. Comparing Fig. 2b with Fig. 1b, it is seen that the control modifies most those eigenmodes with significant variations near the wall.

The sensitivity of the eigenmodes of the closed loop LQR system to modification of the control rule may be quantified by performing the analysis of §3.2.1, replacing the eigenmodes of  $A$  by the eigenmodes of  $A_{K_2}$ . The result of this analysis for small  $\ell$  is shown in Table 2. This table shows that, in the  $\ell \rightarrow 0$  limit, the system matrix is modified to the point that the eigenmodes are no longer sensitive to further modification of the control. In other words, all the controllable dynamics of the system have been modified by  $K_2$  and are accounted for in the closed loop system in this limit. This is one demonstration that the optimal controller extracts the best possible performance from a given (full-information) system.

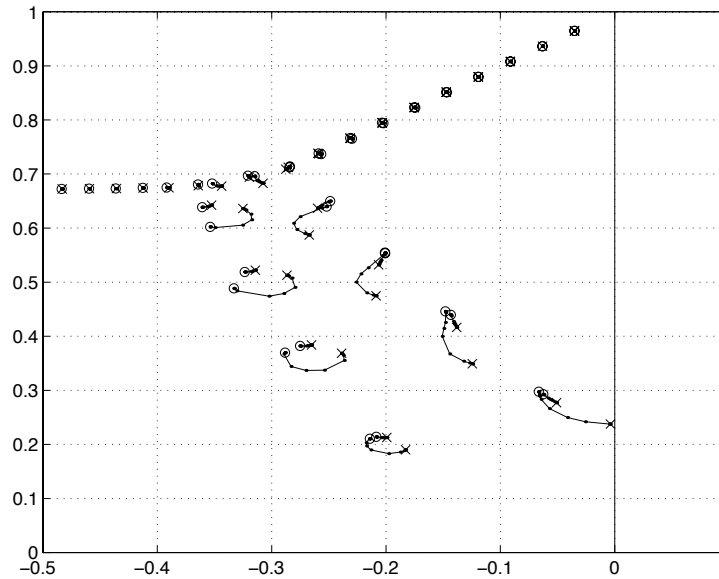


FIGURE 2a. Root locus of least stable eigenvalues of  $A_{K_2}$  as a function of the free parameter of the  $\mathcal{H}_2$  controller,  $\ell$ . The eigenvalues for  $\ell \rightarrow \infty$  are marked with an (x).

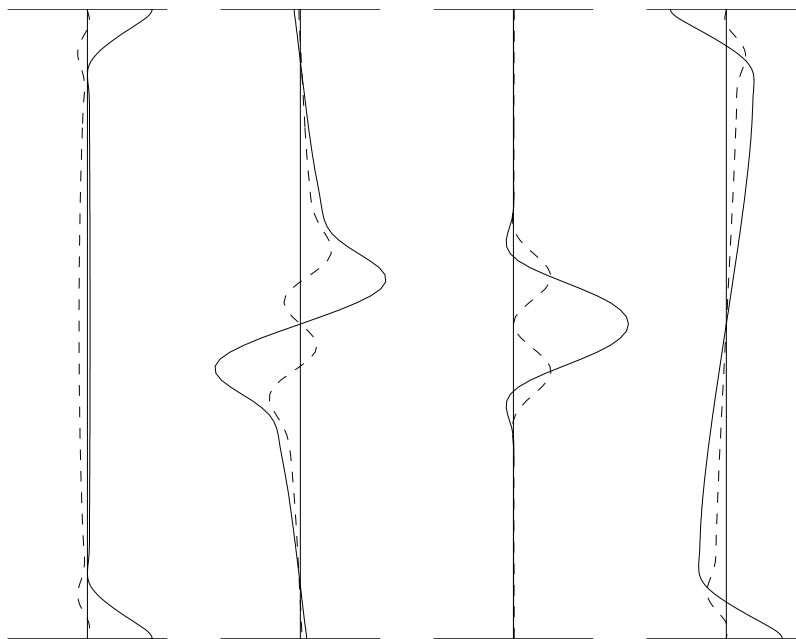


FIGURE 2b. Eigenvectors of  $A_{K_2}$ , with  $\ell = 10^{-4}$ , corresponding to (left to right):  $j = 1$ ,  $j = 3$ ,  $j = 4$ , and  $j = 5$ . Real component of eigenvector is shown solid and imaginary component dashed. Corresponding eigenvalues are reported in Table 2.

### 5.1.2 Kalman-Bucy filter (KBF)

The estimator itself has its own set of dynamics. These dynamics are captured by the equations for the state estimator error, as described in §4.1.2. We now make use of this system in order to investigate the observability of closed-loop eigenmodes

$j$	$\lambda_j$	$f_j$
3	$-0.03513233 - 0.96462128i$	0.000000029
4	$-0.03518652 - 0.96464261i$	0.000000001
<b>5</b>	<b><math>-0.06255259 - 0.29262711i</math></b>	<b>0.000001101</b>
6	$-0.06310358 - 0.93629329i$	0.000000070
7	$-0.06325089 - 0.93635257i$	0.000000003
<b>1</b>	<b><math>-0.06644730 - 0.29721403i</math></b>	<b>0.000001116</b>
8	$-0.09102975 - 0.90793951i$	0.000000129
9	$-0.09130964 - 0.90805917i$	0.000000008
10	$-0.11890731 - 0.87955083i$	0.000000226
11	$-0.11936036 - 0.87976246i$	0.000000020
<b>12</b>	<b><math>-0.14335180 - 0.43962023i</math></b>	<b>0.000002303</b>
14	$-0.14673294 - 0.85111508i$	0.000000414
15	$-0.14739907 - 0.85146161i$	0.000000045
<b>13</b>	<b><math>-0.14803996 - 0.44586838i</math></b>	<b>0.000003081</b>
16	$-0.17450455 - 0.82261690i$	0.000000842

TABLE 2. Least stable eigenmodes of the closed-loop system  $A_{K_2}$  and their sensitivity to control for the optimal controller in the cheap control limit ( $\ell = 10^{-4}$ ). The numbering of the eigenvalues shown is the same as the numbering of the eigenvalues of Table 1 to which they are connected by the root locus of Fig. 2. Note that the control in this limit drives all eigenmodes to positions at which they are insensitive to further modifications of the control, as illustrated by the large reductions in  $f_j$ . Note also that those eigenmodes with the largest values of  $f_j$  in Table 1 (specifically, those in the lower branch) have moved the most.

when all modes are controllable. With  $w = 0$  and examining only the equations for  $\dot{x}_E$  and  $y_E$ , this plant is given by

$$\mathcal{P}_{KBF} = \begin{array}{c} \dot{x}_E \\ y_E \end{array} \left[ \begin{array}{c|c} x_E & \hat{u} \\ \hline A & I \\ C_2 & 0 \end{array} \right]$$

with the output injection now given by

$$\hat{u} = L_2 y_E + \hat{u}',$$

where an additional output injection term  $\hat{u}'$  has been added to study the sensitivity of the closed-loop system to further modification of the output injection rule. Putting the plant and the estimator together, the closed-loop system may be represented by

$$\mathcal{P}_{KBF(\text{closed loop})} = \begin{array}{c} \dot{x}_E \\ y_E \end{array} \left[ \begin{array}{c|c} x_E & \hat{u}' \\ \hline A + L_2 C_2 & I \\ C_2 & 0 \end{array} \right].$$

The eigenmodes of  $A_{L_2} \equiv A + L_2 C_2$  describe the dynamics of the closed-loop system for the unmodified output injection rule ( $\hat{u}' = 0$ ). Figure 3 shows the movement of

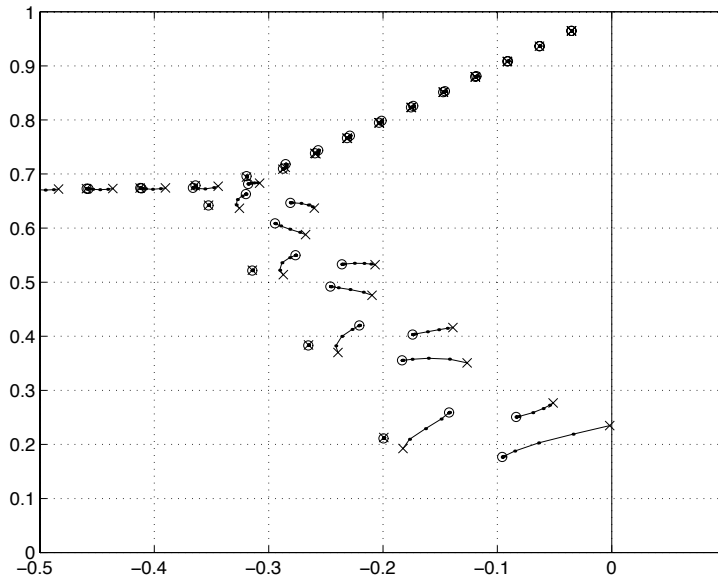


FIGURE 3. Root locus of least stable eigenvalues of  $A_{L_2}$  as a function of the free parameters of the  $\mathcal{H}_2$  estimator,  $g_1$  and  $g_2$  (note that we take  $g_1 = g_2$  for the purpose of drawing the root locus). The eigenvalues for  $g_1 = g_2 \rightarrow 0$ , marked with (x), are very near those of the uncontrolled system  $A$  in Fig. 1, with the previously unstable mode moved just to the left of the imaginary axis. The eigenvalues generally move to the left as  $g_1$  and  $g_2$  are increased.

these eigenvalues with respect to the free parameters of the estimator problem. This is done by assuming that the matrices describing the covariance of the disturbances have the simple form  $G_1 = g_1 I$  and  $G_2 = g_2 I$ , where  $g_1$  and  $g_2$  are real scalars.

The sensitivity of measurements  $y_E$  to the eigenmodes of the closed loop KBF system may be quantified by performing the analysis of §3.2.2, replacing the eigenmodes of  $A$  by the eigenmodes of  $A_{L_2}$ . The result of this analysis for large  $g_1 = g_2$  is shown in Table 3. This table shows that, in the  $g_1 = g_2 \rightarrow \infty$  limit, the system matrix is modified to the point that the measurements are no longer sensitive to the eigenmodes of the closed-loop system. In other words, all the measurable dynamics of the system have been extracted by  $L_2$  and are accounted for in the closed loop system in this limit. This is one demonstration that the Kalman-Bucy filter extracts the best possible state estimate from a given (fully-controllable) state estimator.

### 5.1.3 $\mathcal{H}_2$ control ( $LQG = LQR + KBF$ )

It was mentioned in §4.1.3 that the controller/estimator which minimized the relevant cost functional ( $\mathcal{J}_2$ ) in the presence of Gaussian disturbances could be found by considering the controller and estimator problems separately. In this section, it is shown that the closed-loop performance of a system of the form (23) (without disturbances)

$$\begin{aligned}\dot{x} &= Ax + B_2 u \\ y &= C_2 x\end{aligned}$$



$j$	$\lambda_j$	$g_j$
3	$-0.03505745 - 0.96474093i$	0.000000568
4	$-0.03518656 - 0.96464253i$	0.000000004
6	$-0.06287931 - 0.93668086i$	0.000000644
7	$-0.06325136 - 0.93635193i$	0.000000008
<b>5</b>	<b><math>-0.08362450 - 0.25066856i</math></b>	<b>0.000002858</b>
8	$-0.09059621 - 0.90874817i$	0.000000673
9	$-0.09131196 - 0.90805689i$	0.000000011
<b>1</b>	<b><math>-0.09565183 - 0.17658643i</math></b>	<b>0.000000094</b>
10	$-0.11823779 - 0.88095122i$	0.000000646
11	$-0.11936807 - 0.87975709i$	0.000000014
<b>12</b>	<b><math>-0.14209547 - 0.25910275i</math></b>	<b>0.000000130</b>
14	$-0.14584717 - 0.85329567i$	0.000000549
15	$-0.14741926 - 0.85145223i$	0.000000014
16	$-0.17347707 - 0.82577419i$	0.000000399
<b>13</b>	<b><math>-0.17418920 - 0.40314656i</math></b>	<b>0.000002002</b>

TABLE 3. Least stable eigenmodes of the closed-loop system  $A_{L_2}$  and their sensitivity to observation for the Kalman-Bucy filter in the large disturbance limit ( $g_1 = g_2 = 10^2$ ). The numbering of the eigenvalues shown is the same as the numbering of the eigenvalues of Table 1 to which they are connected by the root locus of Fig. 1. Note that the estimator in this limit modifies all eigenmodes until the measurements are no longer sensitive to them, as illustrated by the large reductions in  $g_j$ . Note also that those eigenmodes with the largest values of  $g_j$  in Table 1 (specifically, those in the lower branch) have moved the most.

combined with an estimator/controller of the form (24)

$$\begin{aligned} u &= K_2 \hat{x} \\ \dot{\hat{x}} &= A \hat{x} + B_2 u - L_2 (y - C_2 \hat{x}) \end{aligned}$$

may also be evaluated by considering the controller and estimator problems separately. To accomplish this, simply combine the above equations into the closed-loop composite system

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & B_2 K_2 \\ -L_2 C_2 & A + B_2 K_2 + L_2 C_2 \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

Gaussian elimination, first on the rows and then on the columns, reveals that the eigenvalues of this system are the same as the eigenvalues of the system

$$\begin{pmatrix} A + B_2 K_2 & B_2 K_2 \\ 0 & A + L_2 C_2 \end{pmatrix}.$$

In other words, the eigenvalues of the closed-loop composite system for the  $\mathcal{H}_2$  problem are simply the union of the eigenvalues of the controlled system  $A_{K_2} = A + B_2 K_2$  and the eigenvalues of the estimated system  $A_{L_2} = A + L_2 C_2$  discussed in the previous two sections and illustrated in Fig. 4.

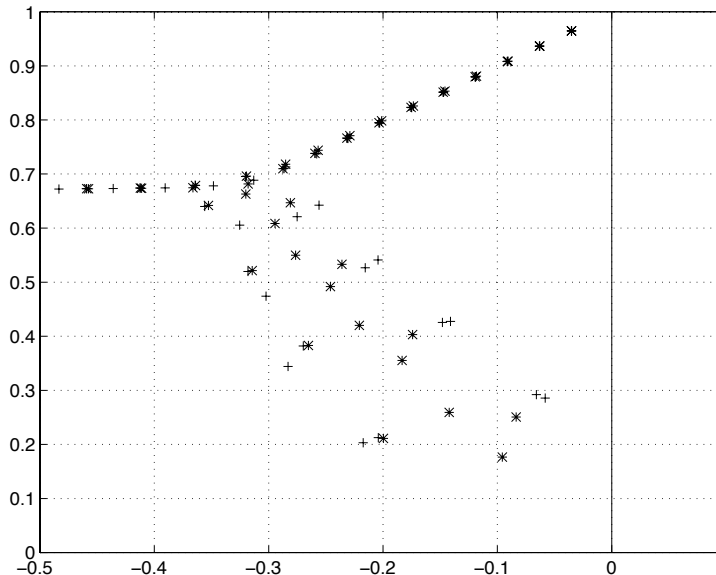


FIGURE 4. Least stable eigenvalues of the composite closed-loop system with the  $\mathcal{H}_2$  controller/estimator, taking  $\ell = g_1 = g_2 = 1$ . Note that the eigenvalues are simply the eigenvalues of the closed loop controller (+) together with those of the closed loop estimator (\*).

### 5.2 $\mathcal{H}_\infty$ control

As with the  $\mathcal{H}_2$  controller/estimator, the performance of the closed loop composite system with the  $\mathcal{H}_\infty$  controller/estimator

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & B_2 K_\infty \\ -L_\infty C_2 & A + B_2 K_\infty + L_\infty C_2 \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

may be evaluated by considering the performance of the controlled system  $A_{K_\infty} = A + B_2 K_\infty$  and the performance of the estimated system  $A_{L_\infty} = A + L_\infty C_2$  separately. The root locus of the eigenvalues of  $A_{K_\infty}$  are plotted with respect to the parameter  $\gamma$  of the  $\mathcal{H}_\infty$  problem in Fig. 5, clearly illustrating the tendency of  $\mathcal{H}_\infty$  controllers to modify only the least stable components of the system, as opposed to the  $\mathcal{H}_2$  controller of Fig. 2, which modifies all controllable modes of the system.

## 6. Conclusions

Optimal and robust control theories have been successfully applied to the Orr-Sommerfeld equation. Given control on the wall-normal component of boundary velocity only, the flow system is shown to be stabilizable but not controllable. Given measurements of wall skin-friction only, the flow system is shown to be detectable but not observable. It is shown that  $\mathcal{H}_2$  controllers/estimators modify all of the controllable/observable modes of the system. In contrast, the  $\mathcal{H}_\infty$  controllers modify the corresponding  $\mathcal{H}_2$  controllers only in the most unstable component, as  $\mathcal{H}_\infty$  targets a bound only on the maximum value of the transfer function.

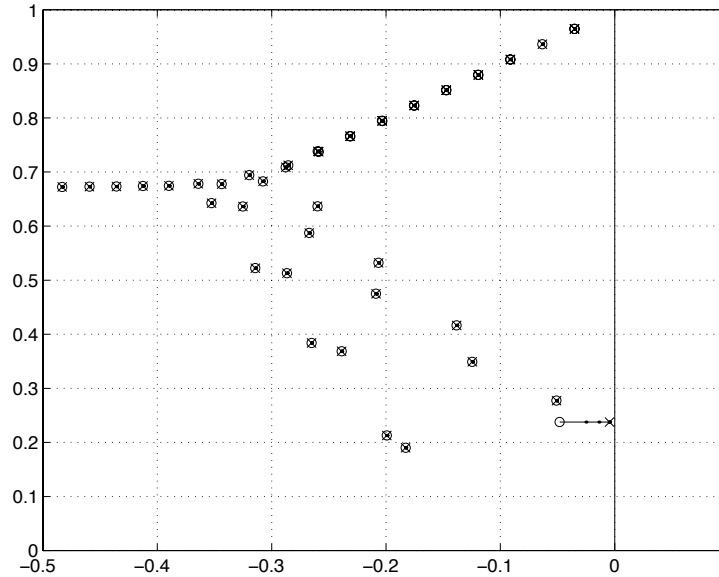


FIGURE 5. Root locus of least stable eigenvalues of the  $\mathcal{H}_\infty$  controller versus  $\gamma$ , taking  $\ell = 100$ ,  $g_1 = g_2 = 0.001$ . The result with  $\gamma \rightarrow \infty$ , marked with the (x), gives the corresponding  $\mathcal{H}_2$  controller. Note that the  $\mathcal{H}_\infty$  controller modifies only the least stable eigenmode of this  $\mathcal{H}_2$  result, without expending any extra control effort to control those eigenmodes not associated with the maximally unstable component of the system. Note also that  $\gamma = \gamma_0$ , marked with the (o), is reached by reducing  $\gamma$  until the least stable eigenvalue corresponds to one of the uncontrollable eigenmodes in the upper branch, which cannot be moved further left; in the present case, this corresponds to a numerical value of  $\gamma_0 = 0.26$ .

In the  $\ell \rightarrow 0$  limit of the  $\mathcal{H}_2$  controller, corresponding to cheap control and thus large values of  $u$ , all eigenmodes of the closed-loop controlled system are shown to be modified to points at which they are no longer sensitive to further modifications of the control. Similarly, in the  $g_1 = g_2 \rightarrow \infty$  limit of the  $\mathcal{H}_2$  estimator, accounting for large disturbances on both the state and the measurements, all eigenmodes of the closed-loop system for the estimator error are shown to be modified to points at which they are not discernible by flow measurements.

These results indicate that  $\mathcal{H}_2$  controllers and estimators are optimal for their desired purposes, but may contain large feedback gains. On the other hand,  $\mathcal{H}_\infty$  controllers only target the least stable components of the system, and thus have smaller feedback gains while still achieving the same worst case performance for the nominal plant. Such reduced feedback gains generally result in improved robustness to inaccuracies in the system model.

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