

# On modeling pressure diffusion in non-homogeneous shear flows

By A. O. Demuren,<sup>1</sup> M. M. Rogers,<sup>2</sup> P. Durbin<sup>3</sup> AND S. K. Lele<sup>3</sup>

New models are proposed for the “slow” and “rapid” parts of the pressure diffusive transport based on the examination of DNS databases for plane mixing layers and wakes. The model for the “slow” part is non-local, but requires the distribution of the triple-velocity correlation as a local source. The latter can be computed accurately for the normal component from standard gradient diffusion models, but such models are inadequate for the cross component. More work is required to remedy this situation.

---

## 1. Introduction

In higher-order turbulence models ‘pressure diffusion’ is usually neglected, or at best added to ‘turbulent diffusion’ (Launder 1984) and the two modeled in aggregate. Pressure diffusion refers to the term  $\partial_i \overline{u_j p}$  in the Reynolds stress budget; turbulent diffusion refers to  $\partial_k \overline{u_i u_j u_k}$ . The latter represents the ensemble averaged effect of random convection and can often be modeled as a diffusion process; the former, however, is harder to explain as diffusion. Turbulent diffusion is usually considered to be the dominant diffusion mechanism, and pressure diffusion is considered to be negligible. However, Lumley (1975a) showed that for homogeneous turbulence the application of symmetry and incompressibility constraints to the exact equation for the “slow” or non-linear part of the pressure diffusion led to the result that its magnitude is 20% that of the triple velocity correlation. In addition, it is of opposite sign, so that if turbulent diffusion could be modeled as a gradient transport, pressure diffusion would represent counter-gradient transport. Demuren *et al.* (1994) examined DNS databases for several shear flows, namely: the mixing layer simulation of Rogers and Moser (1994); the wake simulation of Moser *et al.* (1996); the boundary layer simulation of Spalart (1988); and the backward facing step simulation of Le *et al.* (1993). These confirm for the  $\overline{q^2}$ -equation that in simple shear regions pressure diffusion is roughly 20-30% of turbulent diffusion, and it appears to be mostly counter-gradient transport, so that it merely reduces the effect of turbulent diffusion, which is mostly gradient transport. Thus, the current practice of absorbing pressure diffusion and turbulent diffusion into a single model term appears reasonable, as far as the main shear regions are concerned. But the DNS data show that near the edges of the shear layers turbulent diffusion decreases

1 Old Dominion University

2 NASA Ames Research Center

3 Stanford University

rapidly to zero, while pressure diffusion decreases only very gradually, so the latter then becomes dominant. Thus, the budgets show that near the free stream edge the balance is between pressure transport and mean convection, or temporal drift, rather than between turbulent transport and the latter. Further, where shear layer interactions occur, as near the middle of a wake, pressure diffusion no longer follows counter-gradient transport (see Fig. 1), and Lumley’s model becomes inadequate. It in fact becomes additive very close to the center, leading to an overall increase in total transport, in contrast to the effect in the simple shear layers on either side. Both effects cannot be captured by a mere change in model coefficient. Therefore, in order to build a model for total diffusive transport in general shear flows one must look beyond the homogeneous model of Lumley. The pressure diffusion should also be modeled separately. Inhomogeneous effects can be introduced via a non-local model based on the elliptic relaxation concept of Durbin (1991, 1993) as implemented in the previous study of Demuren *et al.* (1994). The local model, required in this formulation, will be based on the “slow” or non-linear part of the pressure diffusion. A splitting of the pressure diffusion into “slow” and “rapid” parts is therefore necessary.

DNS databases for the mixing layer simulation of Rogers and Moser (1994) and the wake simulation of Moser *et al.* (1996) are post-processed to split the velocity-pressure gradient and pressure diffusion terms into “slow” and “rapid” parts. Separate models are then proposed for these terms.

## 2. Governing equations

The Reynolds stress equations can be written for time-developing plane shear flows as:

$$D_t \overline{u_i u_j} = -(\overline{u_i u_k} \partial_k U_j + \overline{u_j u_k} \partial_k U_i) - \partial_k \overline{u_i u_j u_k} - (\overline{u_i \partial_j (p/\rho)} + \overline{u_j \partial_i (p/\rho)}) - 2\nu \overline{\partial_k u_i \partial_k u_j} + \nu \nabla^2 \overline{u_i u_j} \quad (1)$$

( $D_t$  represents the total derivative and  $\partial_k$  the partial derivative in the  $x_k$  coordinate.) Thus, the time-derivative is balanced by the production, turbulent transport, velocity-pressure gradient correlation, dissipation, and viscous diffusion, respectively. For the cases under consideration in this study, only normal ( $x_2$ ) derivatives of turbulent statistics are non-zero. There is also only one non-zero mean-velocity gradient,  $\partial_2 U_1$ .

Budgets for the  $\overline{u_2^2}$  normal component of the Reynolds stress and the shear stress  $\overline{u_1 u_2}$ , obtained from the DNS database for the plane wake of Moser *et al.* (1996), are presented in Figs. 2 and 3, respectively. (In all these figures,  $y_c$  is the center-line,  $\delta$  is the wake half-width, and  $\delta_m$  is the mixing layer momentum thickness.) In both cases, turbulent and pressure transports are of comparable magnitude. Therefore, it would be inconsistent to model one and not the other. And, if one tries to model them together, then the comment in the introduction with respect to the  $\overline{q^2}$ -equation also applies to the  $\overline{u_2^2}$ -equation. On the other hand, in the  $\overline{u_1 u_2}$ -equation,

$\overline{q^2}$  balance

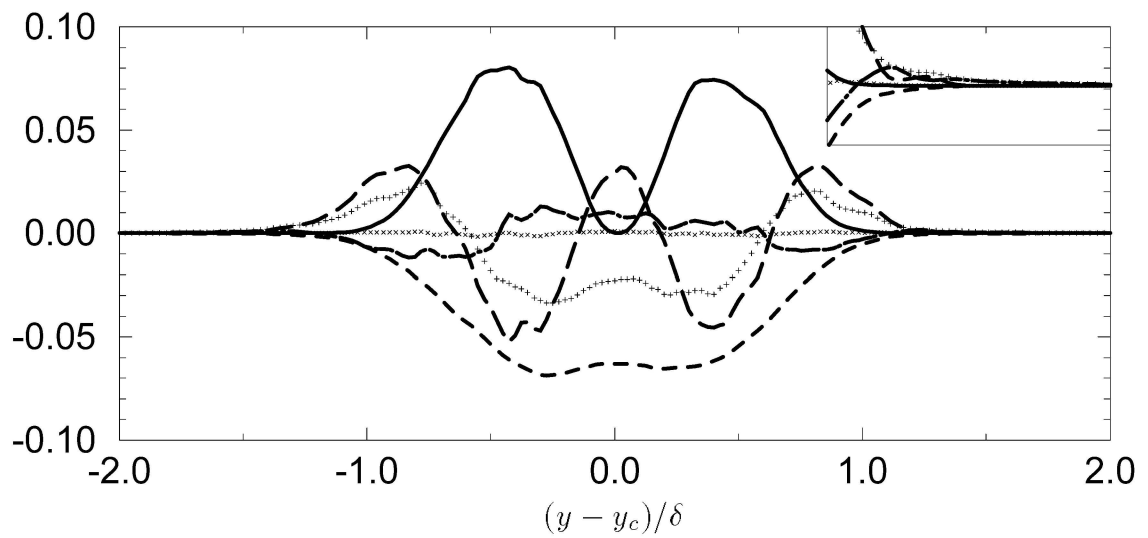


FIGURE 1. Reynolds stress budget of  $\overline{q^2}$  from DNS of a plane wake:—, production;---, turbulent diffusion;— · —, velocity-pressure gradient;·····, pressure diffusion;----, viscous dissipation;×, viscous diffusion;+, temporal drift. (Inset is a blowup of the vertical axis near the edge).

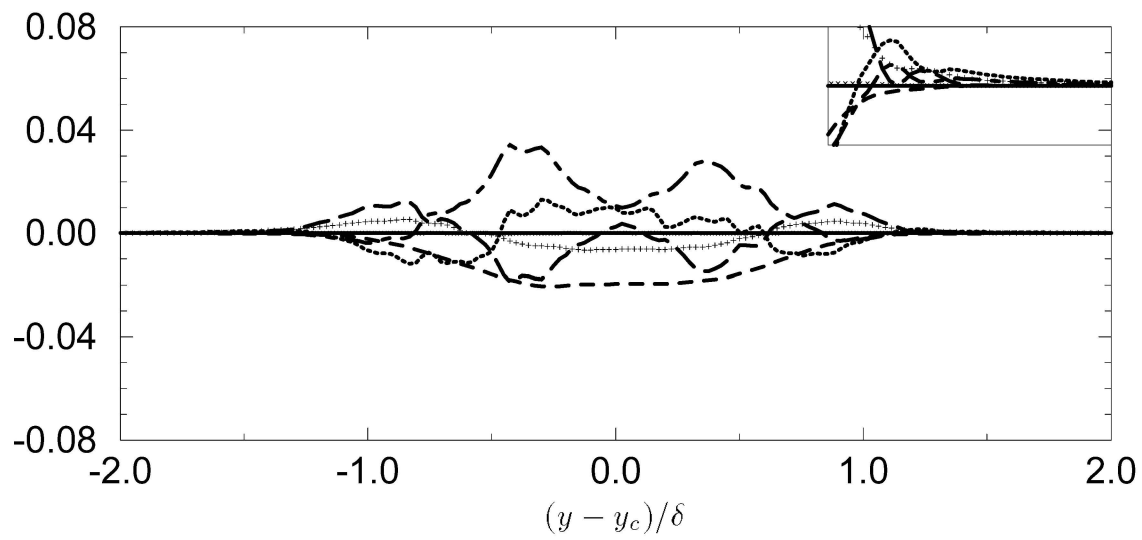
$\overline{u_2^2} \text{ balance}$ 

FIGURE 2. Reynolds stress budget of  $\overline{u_2^2}$  from the DNS of plane wake. (See Fig. 1 for legend.)

these terms virtually cancel each other out, except near the edges of the shear layers. Hence, it appears unlikely that a single composite model could reproduce all these features.

The velocity-pressure gradient correlation can be split into a pressure-strain correlation and a pressure diffusive transport as:

$$-\overline{(u_i \partial_j (p/\rho) + u_j \partial_i (p/\rho))} = \overline{(p/\rho)(\partial_j u_i + \partial_i u_j)} - (1/\rho)(\delta_{j2} \partial_2 \overline{p u_i} + \delta_{i2} \partial_2 \overline{p u_j}) \quad (2)$$

Further, each of these terms can be split into “slow” and “rapid” parts by splitting the pressure  $p$ , used in the correlations, into  $p_s$  and  $p_r$ , respectively. In the present study,  $p_s$  and  $p_r$  are obtained from solution of the equations:

$$\begin{aligned} \nabla^2 p_s &= -\partial_q u_p \partial_p u_q + \partial_{22} \overline{u_2^2} \\ \nabla^2 p_r &= -2\partial_2 U_1 \partial_1 u_2 \end{aligned} \quad (3)$$

Figure 4 shows results of the splittings of the velocity-pressure gradient correlations into “slow” and “rapid” parts for the four non-zero components of the Reynolds stresses. We note that for the diagonal components all energy is produced in the streamwise component  $\overline{u_1^2}$  and then transferred to the normal  $\overline{u_2^2}$  and transverse  $\overline{u_3^2}$  components via pressure scrambling. The transfer mechanism appears to be quite different for these components; transfer to the normal component is solely through the “slow” part, and transfer to the transverse component is through the “rapid” part. It is conjectured that some structural mechanism must be responsible for these, though it could not be identified from the analyzed data. However, these results agree quite well with those for the homogeneous shear flow simulation of Rogers *et al.* (1986). On the other hand, the shear stress results do not agree. Whereas the present results for wakes and mixing layers (not shown) show the velocity-pressure gradient correlation to be mostly in the “rapid” part, the homogeneous shear flow results showed nearly equal distribution between the “slow” and “rapid” parts. Figure 5 shows results of the splittings of the pressure diffusive transport the  $\overline{u_2^2}$  and  $\overline{u_1 u_2}$  components, all others being zero. In both cases, “slow” and “rapid” parts are comparable, and are mostly of opposite sign. Hence, they are, in general, larger in magnitude than the sum. Further, the “slow” part appears to represent counter-gradient transport, and the “rapid” part gradient transport, i.e., more like turbulent diffusion.

### 3. Proposed transport models

It is proposed to model the transport terms in the Reynolds stress equations in three parts, namely, turbulent diffusion, “slow” pressure diffusion, and “rapid” pressure diffusion.

#### 3.1 Turbulent diffusion model

Turbulent diffusive transport (TDIFF) is modeled following the proposal of Mellor and Herring (1973), (hereafter denoted MH) as:

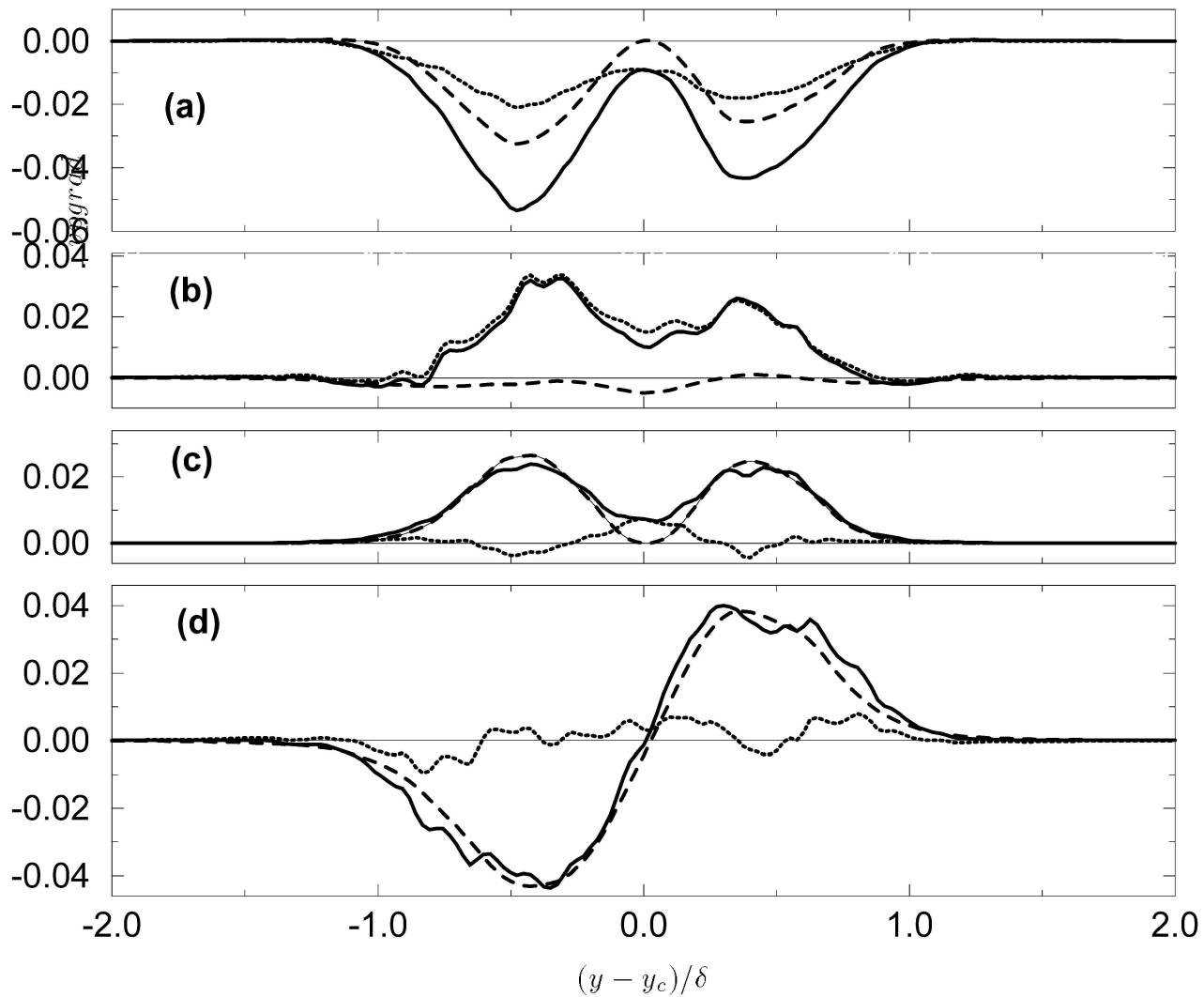


FIGURE 4. Separation of velocity-pressure gradient correlation into “slow” and “rapid” parts, from DNS data for plane wake, (a)  $\overline{u_1^2}$ , (b)  $\overline{u_2^2}$ , (c)  $\overline{u_3^3}$ , (d)  $\overline{u_1 u_2}$  : — , total; ····· , “slow”; ---- , “rapid”.

$$TDIFF_{\overline{u_i u_j}} = -[\overline{u_i u_j u_k}]_{,k} = c_3[(k^3/\varepsilon)[(\overline{u_j u_k})_{,i} + (\overline{u_k u_i})_{,j} + (\overline{u_i u_j})_{,k}]],_{k} \quad (4)$$

(“ $,k$ ” represents derivative with respect to  $x_k$ .) This model is derived from the isotropization of coefficients in the more complex Hanjalic and Launder(1972) diffusion model. It does preserve the symmetry of the indices in the triple-velocity correlation. It was found by Demuren and Sarkar (1993) to yield the correct anisotropy of the Reynolds stress in the wake region of channel flows, and a variant by Mellor and Yamada (1986) is widely used in geophysical flows. The model will also be used to calculate triple-velocity correlations which are required for the modeling of “slow” pressure diffusion in the next section.

### 3.2 “Slow” pressure diffusion model

The “slow” part of the pressure transport (SPDIFF) is modeled using a non-local elliptic relaxation approach as:

$$SPDIFF_{\overline{u_i u_j}} = -[\delta_{jk} \overline{p_s u_i} + \delta_{ik} \overline{p_s u_j}]_{,k} = [\delta_{jk} f_i + \delta_{ik} f_j]_{,k} \quad (5)$$

where

$$L^2 \partial_2^2 f_i - f_i = -f_i^L \quad (6)$$

The local source term  $f^L$  is given by Lumley’s (1975a) model as:

$$f_i^L = \overline{p_s u_i}^L = -0.2 \overline{q^2 u_i} \quad (7)$$

It is assumed that the same length scale which governs the non-locality in the pressure redistribution would also govern the non-locality in the pressure transport. For the  $\overline{u_1 u_2}$  and  $\overline{u_2^2}$  equations, respectively,  $\overline{q^2 u_1}$  and  $\overline{q^2 u_2}$  are obtained from the MH model. This treatment represents a generalization of the previous study by Demuren *et al.* (1994) in which the  $k - \varepsilon$  turbulence model was used.

The principal effect of the non-local model is to “elliptically” spread the influence of the local source over the length scale  $L$ . Figures 6 and 7 present comparisons of the pressure diffusion, based on the local model, to DNS data for mixing layer. In each case, two model computations are made; one assumes that the triple-velocity correlations are known from DNS, and the other computes them from the MH model. For  $\overline{u_2^2}$ , shown in Fig. 6, both computations yield similar results, with peaks that are somewhat higher than in the DNS. Hence, full application of the non-local model would produce quite good agreement. For  $\overline{u_1 u_2}$ , shown in Fig. 7, only the first approach, which assumes a pre-knowledge of the triple-velocity correlation, gives results in agreement with DNS. The MH model grossly underpredicts the  $\overline{q^2 u_1}$ , and hence its derivative. This appears to be a general flaw of gradient-diffusion models for the triple-velocity correlation. They are usually calibrated to reproduce the normal component of the turbulent diffusion in simple shear flows. They fail to reproduce other components, if these are present, as in the 3D boundary layer study of Schwarz and Bradshaw (1994) or the shearless mixing layer study of Briggs *et al.* (1996). This problem will have to be addressed before a reliable, self-contained model for the pressure diffusion of  $\overline{u_1 u_2}$  can be produced.

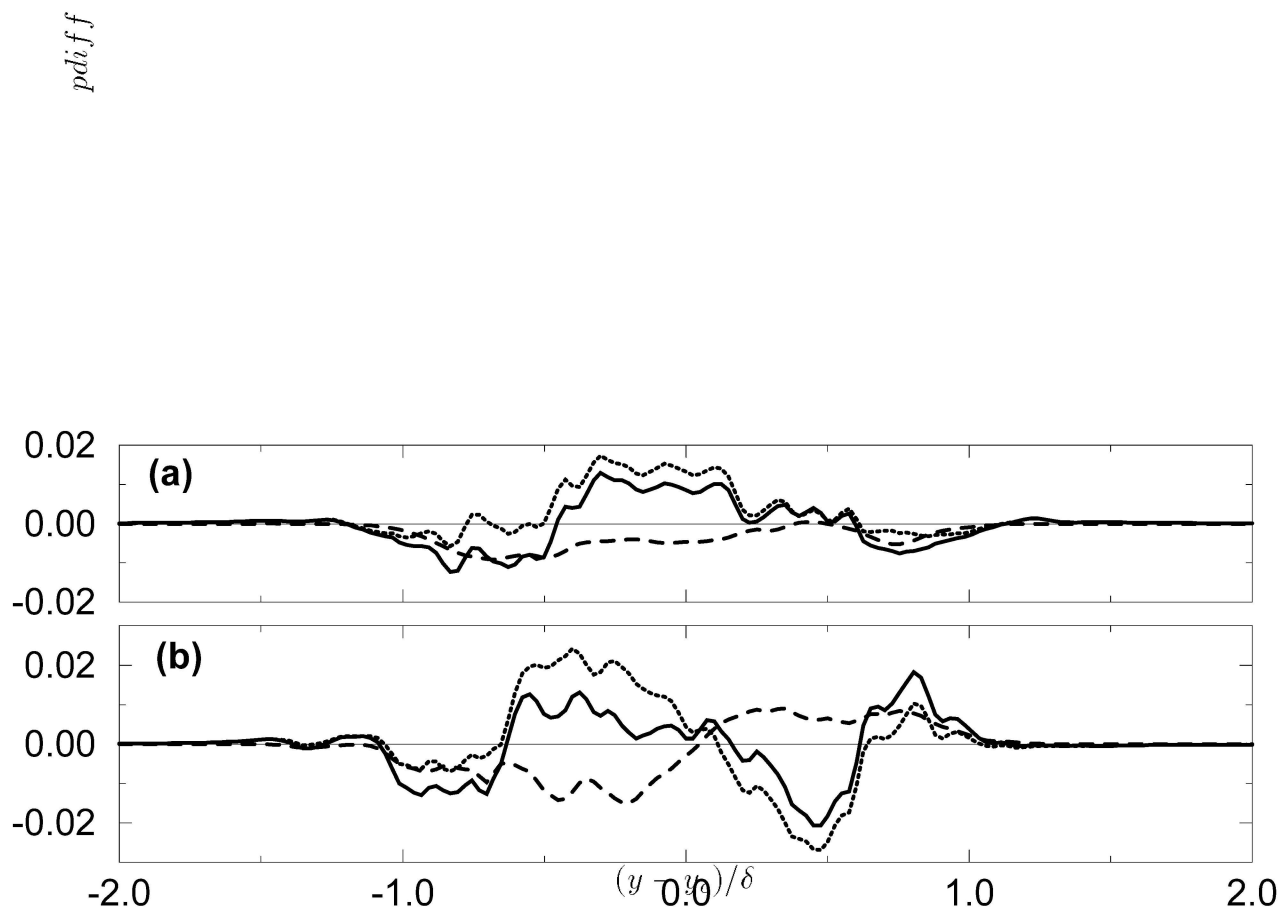


FIGURE 5. Separation of pressure diffusion into slow and rapid parts: DNS data for plane wake, (a)  $\overline{u_2^2}$ , (b)  $\overline{u_1 u_2}$ . (See Fig. 4 for legend.)

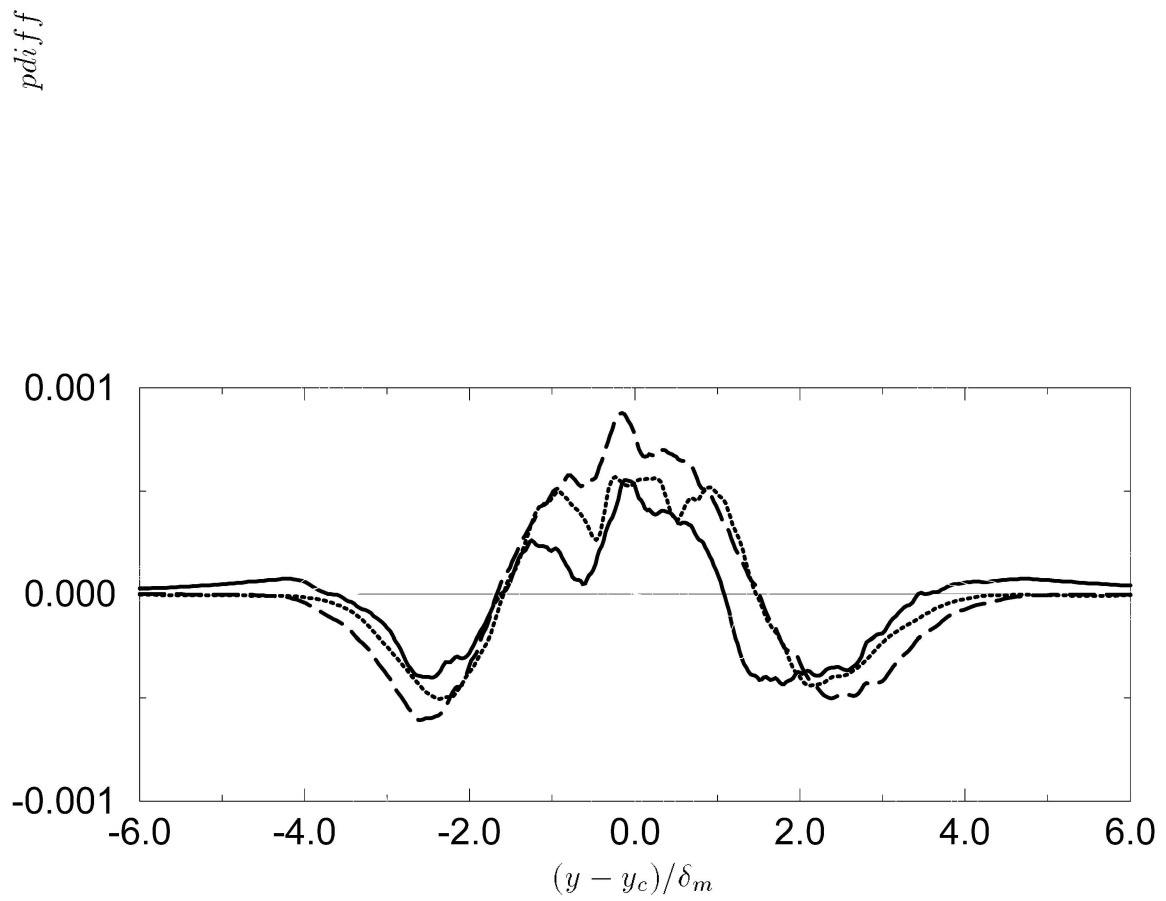


FIGURE 6. Model predictions of pressure diffusion of  $\overline{u_2^2}$  in the plane mixing layer:—, DNS;---, model with triple-correlations from DNS;....., model with triple-correlations from MH.

### 3.3 “Rapid” pressure diffusion model

The “rapid” part of the pressure transport (RPDIFF) is modeled in terms of the Reynolds stresses and mean velocity gradients. The simplest such model has the form:

$$RPDIFF_{\overline{u_i u_j}} = -[\delta_{jk} \overline{p_r u_i} + \delta_{ik} \overline{p_r u_j}]_{,k} = [\delta_{jk} g_i + \delta_{ik} g_j]_{,k} \quad (8)$$

where

$$g_{i,k} = c_r \overline{u_i u_l} U_{l,k} \quad (9)$$

Equation (8) is similar in form to the “rapid” part of some pressure-strain models. This is consistent with the suggestion of Lumley (1975b) that the traditional separation of velocity-pressure gradient correlation into a pressure-strain correlation and a pressure transport is not unique. Preliminary tests show that  $c_r$  should have a value between 0.1 and 0.3. It has been suggested that this part should also be modeled with non-local effects, consistent with the modeling of the slow part and the pressure-strain correlation, but evidence for such behavior could not be discerned from the DNS data. Further testing is desirable.

## 4. Conclusions

The “slow” and “rapid” parts of the velocity-pressure gradient correlations and the pressure transport have been calculated from DNS databases of plane mixing layers and wakes. These show that, in agreement with homogeneous shear flow simulation, the mechanism for the transfer of energy from the streamwise component of the Reynolds stress to the normal component is via the “slow” part, whereas for the transverse component it is through the “rapid” part. But pressure transport is distributed significantly into both “slow” and “rapid” parts, the former being mostly counter-gradient transport, and the latter closer to gradient transport. Models are proposed for both parts, which show qualitative agreement with DNS data for the normal component of the Reynolds stress but have shortcomings when applied to the Reynolds shear stress. The main flaw is the inability of gradient diffusion models to predict other than the normal component of the triple-velocity correlation for which they have been calibrated. Further development and testing is required.

## REFERENCES

- BRIGGS, D. A., FERZIGER, J. H., KOSEFF, J. R., & MONISMITH, S. G. 1996 Entrainment in a shear-free turbulent mixing layer. *J. Fluid Mech.* **310**, 215-241.
- DEMUREN, A. O., LELE, S., & DURBIN, P. 1994 Role of pressure diffusion in non-homogeneous shear flows. *Proc. CTR Summer Prog.* Center for Turbulence Research, NASA Ames/Stanford Univ.
- DEMUREN, A. O., & SARKAR, S. 1993 Perspective: Systematic study of Reynolds stress closure models in computations of plane channel flows. *J. Fluid Engr.* **115**, 5-12.

- DURBIN, P. 1991 Near-wall turbulence closure modeling without ‘damping functions’. *Theoret. Comput. Fluid Dynamics*. **3**, 1-13.
- DURBIN, P. 1993 A Reynolds stress model for near-wall turbulence. *J. Fluid Mech.* **249**, 465-498.
- HANJALIC, K., & LAUNDER, B. E. 1972 A Reynolds stress model of turbulence and its applications to thin shear flows. *J. Fluid Mech.* **52**, 609-638.
- LE, H., MOIN, P., & KIM, J. 1993 Direct numerical simulation of turbulent flow over a backward-facing step. *Rep. TF-58*, Thermosciences Division, Department of Mechanical Engineering, Stanford University.
- LAUNDER, B. E. 1984 Second-moment closure: methodology and practice. In *Turbulence Models and Their Applications*, (Editions Eyrolles, Paris).
- LUMLEY, J. L. 1975a Prediction methods for turbulent flows - Introduction. *Lecture Series 76*, von Karman Inst., Rhodes St. Genese, Belgium.
- LUMLEY, J. L. 1975b Pressure-strain correlation. *Phys. Fluids*. **18**, 750.
- MELLOR, G. L., & HERRING, H. J. 1973 A survey of mean turbulent field closure. *AIAA J.* **11**, 590-599.
- MELLOR, G. L., & YAMADA, T. 1986 Development of a turbulence closure model for geophysical fluid problems. *Rev. Geophys. Space Phys.* **20**, 851-875.
- MOSER, R. D., ROGERS, M. M., & EWING, D. W. 1996 Self-similarity of time-evolving plane wakes. (Submitted).
- ROGERS, M. M., & MOSER, R. D. 1994 Direct simulation of a self-similar turbulent mixing layer. *Phys. Fluids*. **6**, 903-923.
- ROGERS, M. M., MOIN, P., & REYNOLDS, W. C. 1986 The structure and modeling of the hydrodynamic and passive scalar fields in homogeneous turbulent shear flow. *Rep. TF-25*, Thermosciences Division, Department of Mechanical Engineering, Stanford University.
- SCHWARZ, W. R., & BRADSHAW, P. 1994 Term-by-term tests of stress-transport turbulence models in a three-dimensional boundary layer. *Phys. Fluids*. **6**, 986-998.
- SPALART, P. R. 1988 Direct simulation of a turbulent boundary layer up to  $R_\theta = 1410$ . *J. Fluid Mech.* **187**, 61-98.