Comparison of dynamic Smagorinsky and anisotropic subgrid-scale models

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1. Introduction

LES models using dynamically computed coefficient values have been extensively used since the pioneering work of Germano (Germano *et al.* 1991). Starting from the Smagorinsky subgrid-scale model, the common idea to all these models is to extrapolate the information on the resolved fields at two scale levels to compute optimal coefficient values.

A simple solution to the integral equations which follow from this approach can be obtained by restricting the test filter action to certain directions and assuming that the model coefficient only varies along the orthogonal directions. The coefficient values are then derived from simple least-square formulas.

The resulting global dynamic model has been successfully used for a number of flows having at least one direction of homogeneity. For more general flows, the brute force solution of the integral equations may lead to persistent local negative values, which have a destabilizing effect on the numerical solver. Clipping these values lead to discard up to 50% of the coefficients. Moreover, local large values of the coefficient require an implicit treatment of the eddy viscosity term to avoid prohibitively small time-steps. Ghosal *et al.* (1995) have proposed two models to address this issue. In the first one the positivity of the coefficient is rigorously constrained in the integral equation. In the second one, negative coefficient values are allowed, but the model is supplemented by a transport equation for the subgrid-scale kinetic energy. These techniques do alleviate the restrictions of the global dynamic model, but at the expanse of a significant computational overhead (Cabot, 1994).

Our work here has been motivated by practical considerations. It seems clear that the difficulties associated with the solution of integral equations in the dynamic Smagorinsky model would be avoided if the variations of the coefficient over a scale of the order of the grid-size could be assumed to be small. This, in turn, requires that the underlying subgrid-scale model is well-conditioned for the dynamic procedure, or, in other words, has good correlation properties. It is well known that this is not the case for the Smagorinsky model. Leonard's expansion of the subgrid-scale residual stress, by contrast, is known to have very good correlation properties. Its drawback in actual implementations is that it contains backscatter as well as dissipation. The backscatter control strategies which have been proposed so far do

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not allow it to retain its anisotropic features and thus are inefficient (Vreman *et al.* 1997). This explains why this model is in general complemented by Smagorinsky model in so-called mixed models.

In Cottet (1997) we have proposed a simple grid-implementation of Leonard's expansion which enables a truly anisotropic backscatter control. Tests with constant coefficients for homogeneous isotropic turbulence as well as in channel flow geometry (Cottet and Wray, 1997) showed that this model performs better than the Smagorinsky model. In particular, unlike the Smagorinsky model, it has the property of vanishing in laminar regimes and solid boundaries. Moreover, the specific backscatter control enabled by the grid formulas overcomes the lack of dissipation in general observed in the implementation of Leonard's expansion (Vreman *et al.* 1997).

In view of these results, we believe that this model is a good candidate for simple local dynamic implementations. In the following section we summarize the key properties of the anisotropic subgrid scale formulas and present our approach for dynamic coefficient calculations. In Section 3 we then discuss results obtained for channel flows and draw some preliminary conclusions.

2. The dynamic anisotropic model

2.1 Anisotropic formulas

They are derived from Leonard's expansion of the self-similarity model, using integral approximations. Let ζ be a filter function satisfying

$$\int x_k x_l \zeta(\mathbf{x})\, d\mathbf{x} = \delta_{kl}\ , \qquad k,l = 1,\ldots,3. \tag{1}$$

If Δ is the filter width, then the formal Taylor series expansion gives

$$\Delta^2 D_{ik}\overline{\mathbf{u}}(\mathbf{x}) D_{jk}\overline{\mathbf{u}}(\mathbf{x}) = \frac{1}{\Delta^3}\left\{\int [\overline{u}_j(\mathbf{y}) - \overline{u}_j(\mathbf{x})][\overline{u}_i(\mathbf{y}) - \overline{u}_i(\mathbf{x})]\zeta(\frac{\mathbf{y}-\mathbf{x}}{\Delta})\, d\mathbf{y} + O(\Delta^2)\right\}, \tag{2}$$

where the notation $D_{ik}\overline{\mathbf{u}}$ stands for $\frac{\partial \overline{u}_i}{\partial x_k}$. Differentiating this expression we obtain the following SGS formula:

$$\partial_j \tau_{ij}(\mathbf{x}) \simeq -\frac{C}{\Delta^4}\int [u_j(\mathbf{y}) - u_j(\mathbf{x})][u_i(\mathbf{y}) - u_i(\mathbf{x})]\partial_j\zeta(\frac{\mathbf{y}-\mathbf{x}}{\Delta})\, d\mathbf{y}. \tag{3}$$

For the details of derivation we refer to Cottet (1997). The final SGS model is obtained by approximating Eq. (3), using numerical integration over the grid points close to $\mathbf{x}$. In other words, the filter function ζ is approximated by a discrete grid-based filter.

In the case of a non-uniform grid, a discrete grid-based filter cannot have the same filter widths in all three directions, and thus Eq. (2) can not be formally derived.

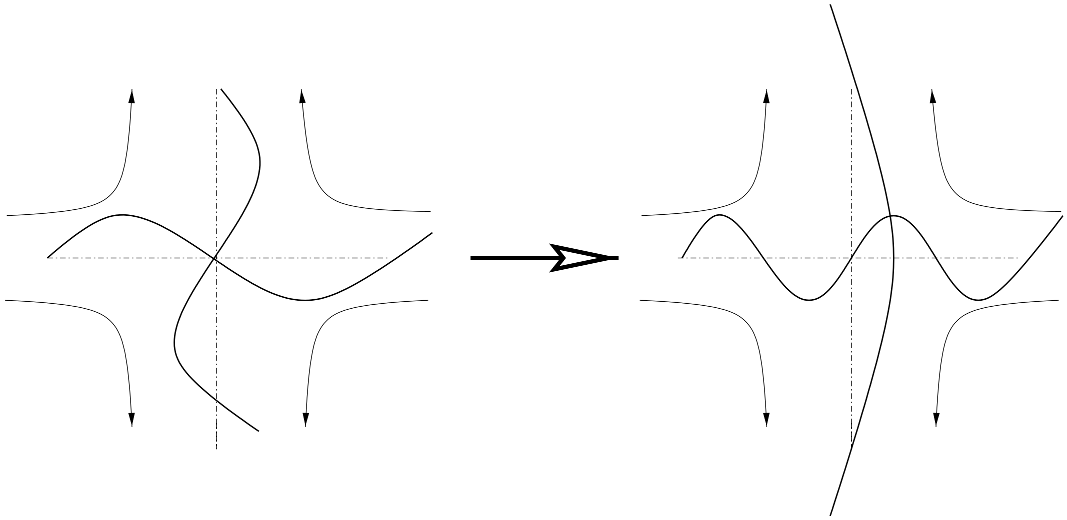

FIGURE 1. Flow around an hyperbolic point and production of small or large scale along the strain directions.

However, the integral in the right-hand side of Eq. (2) can be derived directly from the similarity model. It can be easily shown that

$$\frac{1}{\Delta^3}\int[\overline{u}_j(\mathbf{y})-\overline{u}_j(\mathbf{x})][\overline{u}_i(\mathbf{y})-\overline{u}_i(\mathbf{x})]\zeta(\frac{\mathbf{y}-\mathbf{x}}{\Delta})\,d\mathbf{y}=\widehat{\overline{u}_i\overline{u}_j}-\hat{\overline{u}}_i\hat{\overline{u}}_j+[\hat{\overline{u}}_i-\overline{u}_i][\hat{\overline{u}}_j-\overline{u}_j],$$

where $\widehat{(\cdot)}$ denotes the filtering operation with the filter function ζ. This observation, which shows that the anisotropic formulas are $O(\Delta^4)$ corrections over the similarity model, is true for any anisotropic discrete grid-based filter.

A nice feature of the integral formulas (2) and (3) is that they give a simple way to distinguish the backscatter and dissipation subgrid contributions. Multiplying Eq. (3) by $\overline{\mathbf{u}}$ leads to

$$\int\partial_j\tau_{ij}(\mathbf{x})\overline{u}_i(\mathbf{x})\,d\mathbf{x}\simeq\frac{C}{2}\Delta^{-4}\int[\overline{\mathbf{u}}(\mathbf{y})-\overline{\mathbf{u}}(\mathbf{x})]\cdot\nabla\zeta(\frac{\mathbf{y}-\mathbf{x}}{\Delta})|\overline{\mathbf{u}}(\mathbf{y})-\overline{\mathbf{u}}(\mathbf{x})|^2\,d\mathbf{x}\,d\mathbf{y}. \quad (4)$$

We illustrate the meaning of the Eq. (4) by considering spherically symmetric filters. In this case, $\nabla\zeta(\mathbf{x})=\mathbf{x}\zeta'(|\mathbf{x}|)$ with $\zeta'\leq 0$. Therefore, dissipation or backscatter in the subgrid-scale model occurs respectively in the directions of compression or dilatation. The sketch in Fig. 1 shows the distinction among strain directions in the flow and illustrates the mechanism of small scales production. As a result, the following clipped model prevents backscatter and only dissipates in the direction of flow compression:

$$\partial_j\tau_{ij}\simeq\Delta^{-4}\int\left\{[\overline{\mathbf{u}}(\mathbf{x})-\overline{\mathbf{u}}(\mathbf{y})]\cdot\nabla\zeta(\frac{\mathbf{x}-\mathbf{y}}{\Delta})]\right\}_+[\overline{\mathbf{u}}(\mathbf{x})-\overline{\mathbf{u}}(\mathbf{y})]\,d\mathbf{y}, \quad (5)$$

where $a_+=\max(0,a)$.

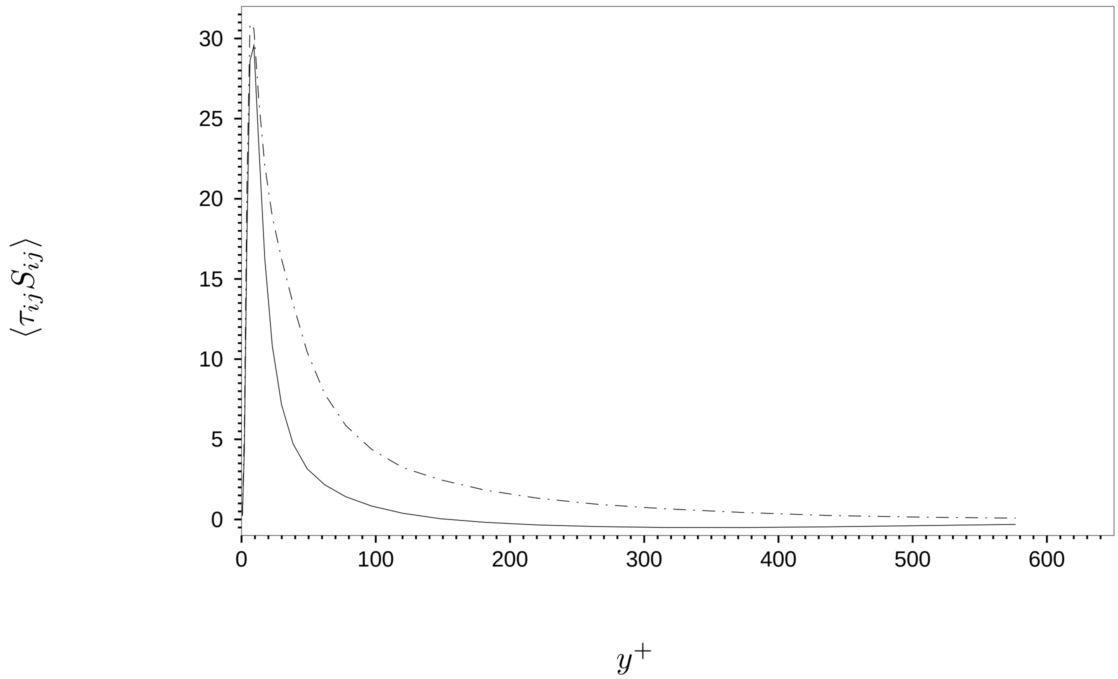

FIGURE 2. LES of plane channel flow at $Re_\tau = 650$. Subgrid-scale dissipation. $-\cdot-$: global dynamic Smagorinsky model; ——: global dynamic anisotropic model.

2.2 Dynamic procedure

Given their relationship with the similarity model, it is natural to view our subgrid-scale formulas as a model of the energy transfer between the scales Δ and $\Delta/2$. Note that this scale range is in general recognized to contain the essential part of the transfer between resolved and unresolved scales. Thus, in order to determine the model coefficient C, it is natural to compute the residual stress between the scales $\hat{\Delta} = 2\Delta$ and Δ explicitly and to match it with the model evaluated on the filtered field $\hat{\overline{\mathbf{u}}}$. In other words, the coefficient C should satisfy

$$\widehat{\overline{u}_i\overline{u}_j} - \hat{\overline{u}}_i\hat{\overline{u}}_j = C\{\sum_{\mathbf{y}\sim\mathbf{x}}[\hat{\overline{u}}_j(\mathbf{y}) - \hat{\overline{u}}_j(\mathbf{x})][\hat{\overline{u}}_i(\mathbf{y}) - \hat{\overline{u}}_i(\mathbf{x})]\zeta(\frac{\mathbf{y}-\mathbf{x}}{\hat{\Delta}}) \tag{6}$$

Following the standard dynamic formulation, a least square solution has to be sought to solve this system of six equations and, if one desired, to constrain variations of the coefficients only along specific directions. Note, that the approach we just described bypasses the Germano identity and instead uses the assumption that the models act in a limited scale range. One may object that replacing an exact identity by an assumption is not satisfactory. However, in our view the subgrid-scale models can never be expected to be very accurate (not speaking of the least square procedure used to adjust the coefficient to a number of equations); therefore, the exactness of the Germano identity is not of crucial importance in the dynamic determination of the coefficient.

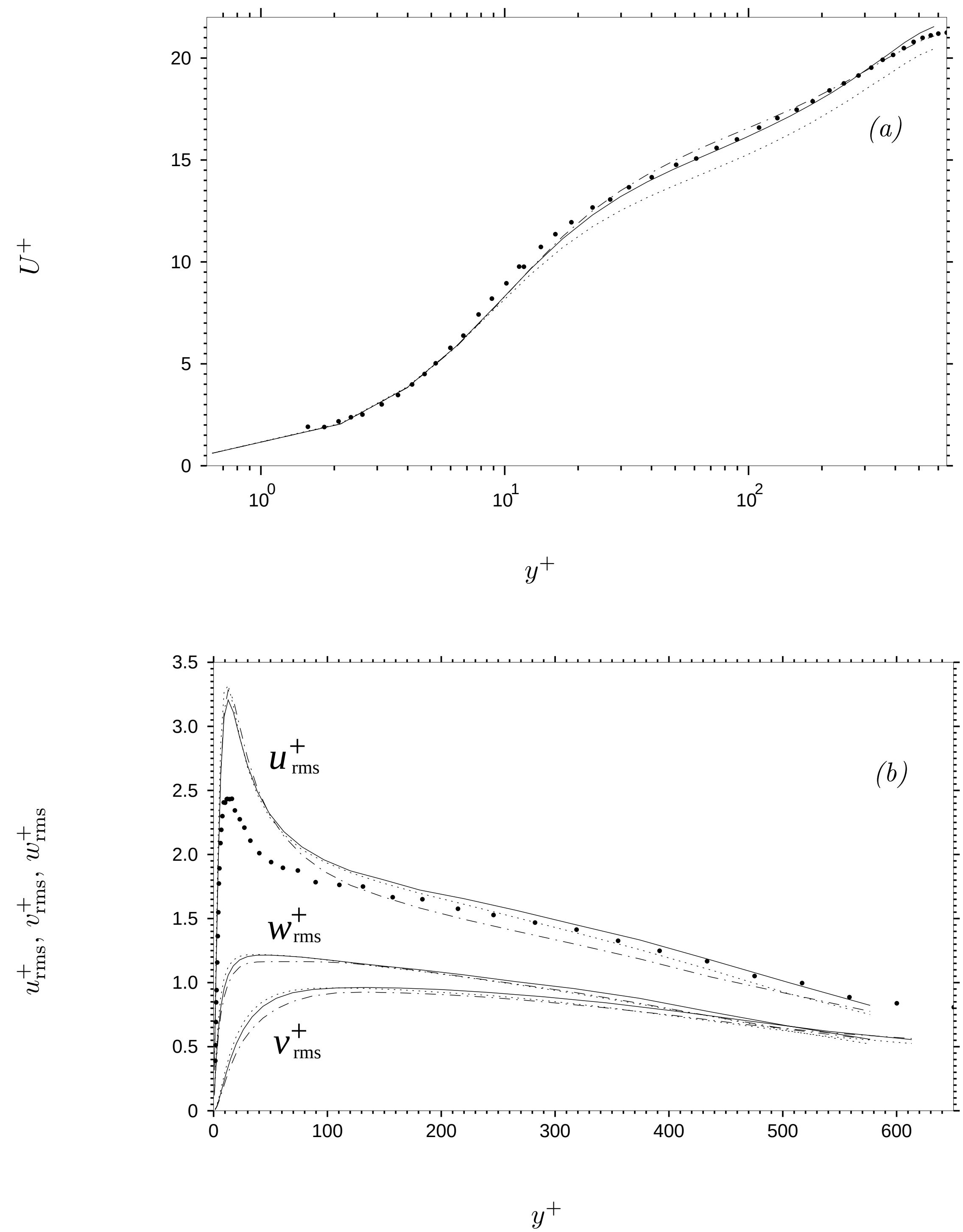

FIGURE 3. LES of plane channel flow at $Re_\tau = 650$. (a) Mean streamwise velocity; (b) Turbulence intensities. ·······: no-model; $-\cdot-$: global dynamic Smagorinsky model; ——: global dynamic anisotropic model; $\bullet$: experiment (Hussain and Reynolds, 1970).

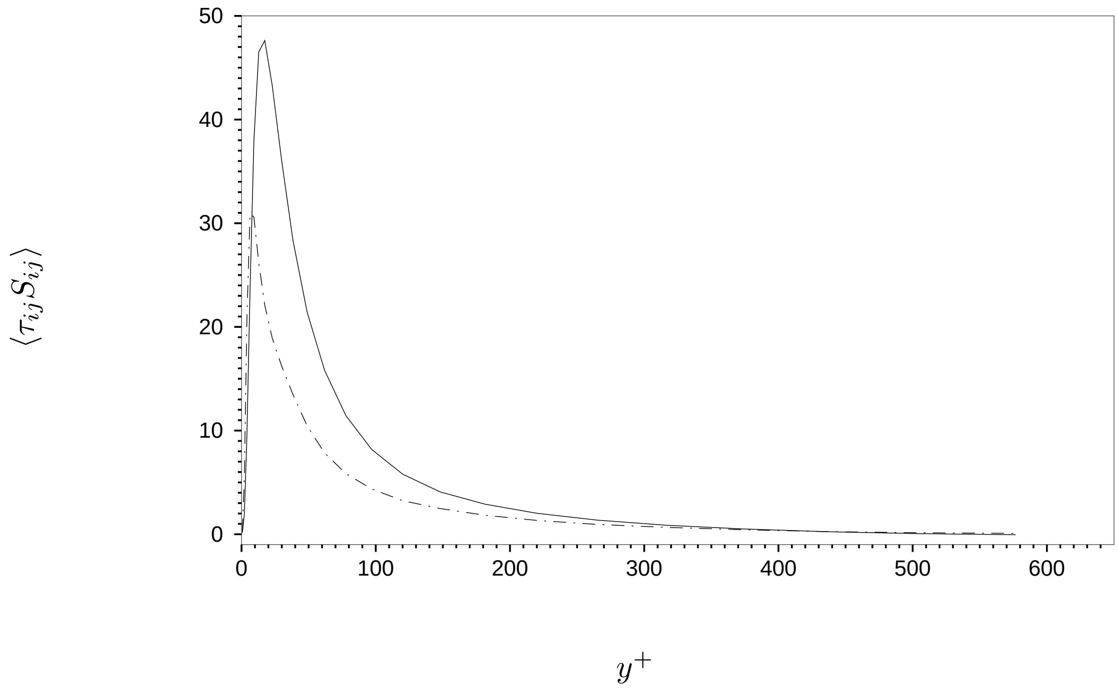

FIGURE 4. LES of plane channel flow at $Re_\tau = 650$. Subgrid-scale dissipation. $- \cdot -$: global dynamic Smagorinsky model; ——: local dynamic anisotropic model.

3. Preliminary results

To validate our dynamic procedure, we have first implemented a *global* dynamic model. In this case the coefficient is constrained to vary only along the direction normal to the channel walls. A least square solution of Eq. (6) under this additional constraint is then given by

$$C = \frac{\langle L_{ij} M_{ij} \rangle}{\langle M_{kl} M_{kl} \rangle}$$

where L_{ij}, M_{ij} receptively denote the left- and right-hand side of Eq. (6), $\langle \cdot\ , \cdot \rangle$ means averaging in the wall parallel directions, and repeated indices mean the summation. This model has been compared to the classical global dynamic Smagorinsky model for a channel flow at Reynolds number of 650. The grid resolution is $48 \times 49 \times 48$. The numerical method is a fourth order finite-difference scheme on a staggered grid system; the grid is refined in the wall normal direction according to a hyperbolic tangent law (see Morinishi *et al.*, 1998, for details). In both methods, a sharp cut-off test filter was used along the homogeneous directions. Figure 2 shows the subgrid-scale dissipation produced by the models. They peak at about the same value. However, the anisotropic model seems to dissipate in a narrower region around the walls.

Figure 3a shows the mean velocity profiles. Both methods give a fair agreement with the experimental data when compared to the case where no-model is used. The similar fair agreement of the two models for the turbulent intensity profiles is

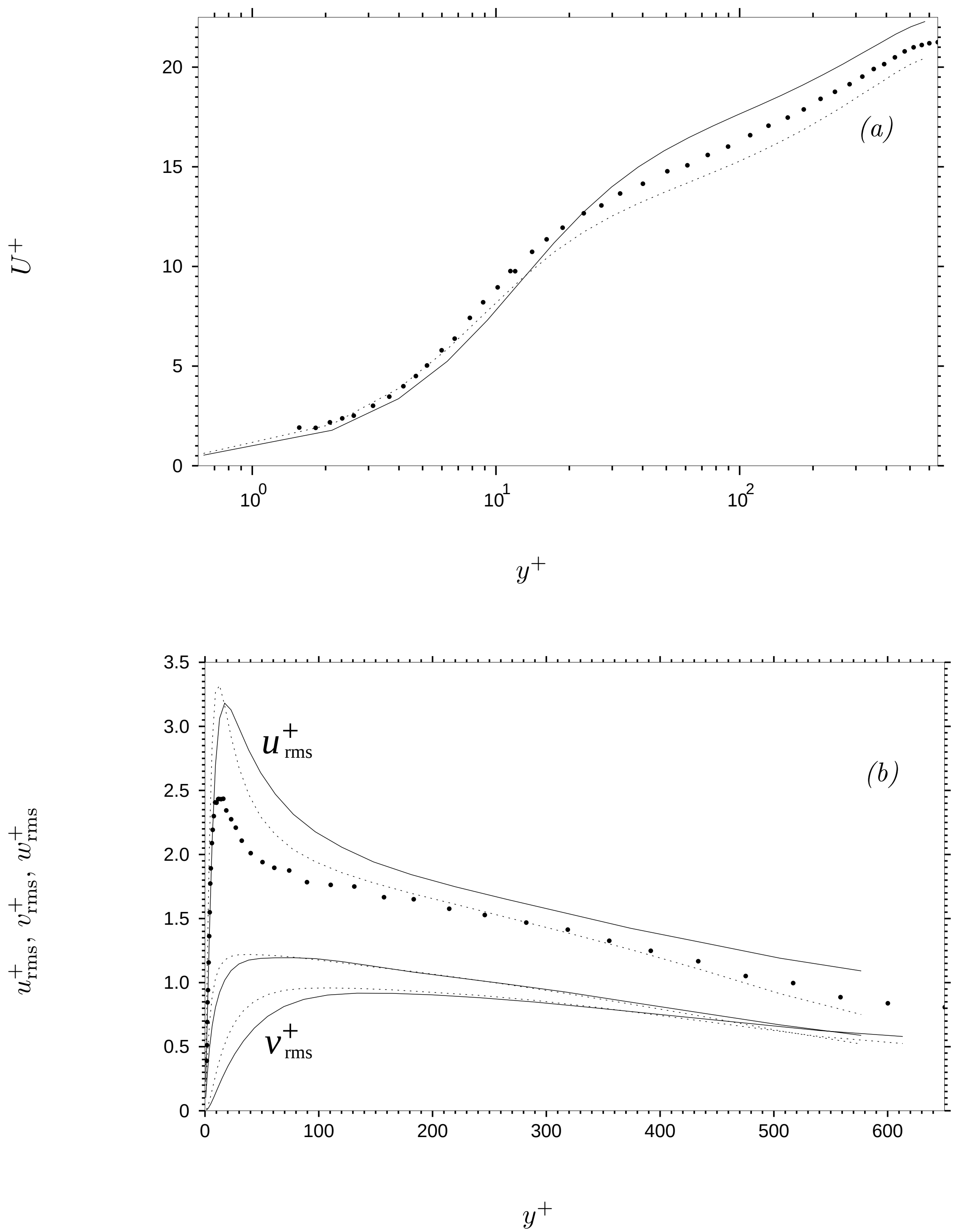

FIGURE 5. LES of plane channel flow at $Re_\tau = 650$. (a) Mean streamwise velocity; (b) Turbulence intensities. ⋯⋯: no-model; ——: local dynamic anisotropic model; •: experiment (Hussain and Reynolds, 1970).

observed in Fig. 3b.

Next we will discuss the implementation of *local* dynamic procedure for the anisotropic subgrid-scale model. There are several ways to implement local dynamic procedure. We decided to focus on the following approach: compute local coefficient C using Eq. (6), discard negative values of C, and use clipped formula (5) to evaluate subgrid-scale contribution.

Our first observation is that this model does not produce large local values of C and thus does not require an implicit treatment of the eddy viscosity term. This already contrasts with the behavior of local dynamic Smagorinsky model where the local model coefficient is found without solving an integral equation. In this case an explicit treatment of the eddy-viscosity term would lead to prohibitively small time-steps.

Figure 4 shows the comparison of the subgrid-scale dissipation of the local dynamic anisotropic model and global dynamic Smagorinsky model. Mean velocity and turbulence intensity profiles for local dynamic anisotropic model are presented in Fig. 5. The results of Fig. 5 show a reasonable agreement between the local dynamic anisotropic model and the experimental results. However, the excess dissipation in the near wall region causes the velocity profile to be overestimated in the middle of the channel.

We have also implemented a local dynamic model with a test filter in the physical space identical to the discrete grid-based filter. This option proved to produce too much dissipation. However, this model practically did not require any clipping: less then 1% of the model coefficients had negative values. This somehow substantiates our expectations that the anisotropic formulas are better conditioned than the Smagorinsky model for local dynamic coefficient calculations. However, more numerical experiments are certainly needed to fully assess the usefulness of these models in general geometries. In particular, we believe that a local model with a test filter adapted to the computational grid is the most natural extension of the method. It is well known that the choice of the test filter is a critical parameter in all dynamic formulations and has to be taken into account in the definition of the filter width. Note that even global dynamic models fail to give good results if the filter width issue is not properly defined (Lund 1997). This issue and its impact on local dynamic anisotropic calculations will be more systematically addressed in a future work.

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