Evaluation of the statistical Rayleigh-Ritz method in isotropic turbulence decay

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A Rayleigh-Ritz method for calculating the statistics of nonlinear dynamical systems is tested against LES data for homogeneous and isotropic decay of turbulence. The comparisons in this work are of 2-point, 2-time Eulerian velocity correlators. At this level, the Rayleigh-Ritz predictions are formally realized by a linear Langevin model for the fluctuation variables. We study how well standard K-ε models that are adequate to describe the decay of ensemble means can also predict the decay of fluctuations. In addition to such standard *RANS* closures, we also consider some spatially nonlocal and temporally non-Markovian models, which include scale-dependent eddy viscosities and convective sweeping in Fourier space.

1. Introduction

This work investigates a Rayleigh-Ritz variational method to solve for the statistics of nonlinear dynamics, which was earlier proposed (Eyink, 1996). Formally, the method provides an approximate solution of the Liouville-Hopf equation for the time-evolution of probability distributions in phase-space by the method of weighted residuals (Finlayson, 1972). The Rayleigh-Ritz method can be understood most simply as the classical moment-closure method extended to give a description, not merely of averages, but also of fluctuations. The ordinary moment-closure equation for a set of ensemble-averages $m_i(t) = \langle \psi_i(t) \rangle$, which we may write as

$$\dot{m}_i(t) = V_i(\mathbf{m}, t), \tag{1}$$

can be obtained as an Euler-Lagrange equation in a variational solution of the Liouville-Hopf equation by the method of weighted residuals. On the other hand, the Euler-Lagrange equation for a *constrained variation* under a constraint on the mean moment histories is a *perturbed closure equation* of the form

$$\dot{m}_i(t) = V_i(\mathbf{m}, t) + \sum_j C_{ij}(\mathbf{m}, t) h_j(t). \tag{2}$$

Here, $h_j(t)$ is a Lagrange multiplier function to incorporate the constraint. The function $\mathbf{C}(\mathbf{m}, t)$ represents the statistical *covariance matrix* of the closure variables at a single time t, or $C_{ij}(\mathbf{m}, t) := \langle \psi_i'(t) \psi_j'(t) \rangle_{\mathbf{m}}$, given as a function of the moment averages $\mathbf{m}$. As usual, $\psi_i' = \psi_i - m_i$ represents a fluctuation variable.

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Any of the closure schemes ordinarily applied to the modeling of Eq. (1) for the averages can also be applied to Eq. (2), providing thereby predictions for fluctuations. Intuitively, the imposition of constraints should give information about fluctuations by the principle that the probability of a fluctuation is determined by the cost to produce it under a constraint. This may be called the "Cramér principle", see (Frisch, 1995). In fact, it may be shown (Eyink, 1996) that the solution of Eq. (2) contains complete information about the statistical distribution of the random histories $\psi_i'(t)$. In particular, it can be shown that the *2-time covariance matrix* $C_{ij}(t,t_0) := \langle\psi_i'(t)\psi_j'(t_0)\rangle$ is obtained in the Rayleigh-Ritz method from a simple *fluctuation-response relation*:

$$C_{ij}(t,t_0) = R_{ij}(t,t_0) + R_{ji}(t_0,t) \tag{3}$$

where $\mathbf{R}(t,t_0)$ is the response $R_{ij}(t,t_0) := \left.\frac{\delta m_i(t)}{\delta h_j(t_0)}\right|_{\mathbf{h}=0}$ of the solution $m_i(t)$ of Eq. (2) to an infinitesimal change of the control field $h_j(t)$, see (Eyink, 1998).

It is the purpose of this work to investigate the success of standard closures to predict fluctuations when employed in the constrained variational equation (2). We shall consider here only the simplest situation for a turbulent fluid governed by the Navier-Stokes equations, namely, homogeneous, isotropic decay at high Reynolds number. It is well-known that *RANS* closures such as the standard K-ε model are adequate to reproduce the ensemble-averages in such an equilibrium turbulence. The main issue to be addressed here is the success of such closures to predict the lowest-order statistic for the fluctuations, the 2-time covariance matrix of the closure variables. The Rayleigh-Ritz method also gives predictions for all higher-order multi-time statistics, e.g. transition probabilities, but we shall confine ourselves here to a check only at the lowest-order. The 2-time correlations already have some direct interest in terms of the *predictability problem* for meteorology and climatology since they give the statistical correlation between successive states, e.g. the correlation between the weather today and weather tomorrow. In addition, information about fluctuations is essential in engineering problems such as the turbulence *control problem* or the LES-RANS *matching problem* for wall-bounded flows.

The method of our investigation is to compare the Rayleigh-Ritz predictions with those of an Ensemble LES calculation for isotropic decay. The use of LES rather than DNS allows our study to be made, in principle, at infinite Reynolds number and, hence, to avoid the issue of viscous corrections to the closures. The Ensemble LES method is discussed in detail in (Carati, 1997). Our study uses a dynamical Smagorinsky subgrid stress model with the dynamical coefficient calculated for a sharp spectral filter by an average over 64 different ensemble realizations on a 64^3 lattice. The single-time spectrum is obtained for the resolved velocity field $\mathbf{v}(\mathbf{x},t)$ by averaging over the space domain as well as the ensemble realizations. In addition to the single-time statistics, we obtain the *2-time cospectra* of the velocities $\mathbf{v}(\mathbf{x},t)$ at each time t with the velocities $\mathbf{v}(\mathbf{x},t_0)$ at the initial time t_0. This allows us to make a direct check on the Rayleigh-Ritz predictions. To avoid issues of modeling the single-time correlations, we directly input the LES results for the single-time correlations at the initial time t_0. Then, the Rayleigh-Ritz method is used to integrate these input correlations forward in one time variable to obtain the 2-time correlations at the pair of times t, t_0. These predictions are finally compared with the LES results for the same quantities as the basic check on the method. The Rayleigh-Ritz calculations are carried out primarily for the K-ε closure. It is a main objective of this work to determine what temporal statistics may be correctly predicted with such a standard 1-point closure. However, based upon the results of the comparison of those predictions with the LES results, improved Rayleigh-Ritz approximations are also developed exploiting more refined closure assumptions.

2. Comparison of Rayleigh-Ritz with LES

2.1 The single-time LES results

Let us first describe the results of our LES calculations for the velocity spectrum $E(k,t)$, which is graphed in Fig. 1 for several times t over the LES run, including the initial time $t_0 = 0.2295$. As may be seen, the peak wavenumber at the initial time t_0 is approximately $k_P = 9$, but this decreases in time to a value of about $k_P = 6$ at the end of the run. At each time t, the spectrum beyond the peak is an inertial-range power-law. Graphing compensated spectra reveals that the power-law is, to a reasonably good approximation, given by the Kolmogorov $\frac{5}{3}$ law. As time advances, the spectra are degraded rapidly in the high wavenumber range, while at wavenumbers well below the peak the spectrum is nearly unchanged in time. The low-wavenumber range can also be reasonably well fit by a power-law $\propto k^m$ with $m \approx 6$. This power-law is transient due to backscatter of energy into the low-wavenumbers, which leads to a slow decrease in the effective m value over time.

2.2 Comparison of the K-ε Rayleigh-Ritz with LES

The most popular engineering closure for the ensemble-averages is $\overline{v}_i, \overline{K}$, and $\overline{\varepsilon}$ is the K-ε RANS model. As a reminder, its equations have the form:

$$\partial_t \overline{v}_i + (\overline{\mathbf{v}} \cdot \nabla)\overline{v}_i = -\nabla_i \overline{p} - \nabla_j \tau_{ij} + \nu \triangle \overline{v}_i, \tag{4}$$

$$\partial_t \overline{K} + (\overline{\mathbf{v}} \cdot \nabla)\overline{K} = -\tau_{ij} \nabla_j \overline{v}_i - \overline{\varepsilon} - \nabla_i J_i + \nu \triangle \overline{K}, \tag{5}$$

$$\partial_t \overline{\varepsilon} + (\overline{\mathbf{v}} \cdot \nabla)\overline{\varepsilon} = \nu \triangle \overline{\varepsilon} + \nabla_i \mathcal{D}_i + \mathcal{P} - \Phi. \tag{6}$$

Here, $\tau_{ij} = \overline{v_i' v_j'}$ is the Reynolds stress. J_i is the space transport of kinetic energy by turbulent diffusion and molecular viscosity. Likewise, $\mathcal{D}_i$ is the space transport of dissipation, and $\mathcal{P}, \Phi$ are production and destruction of dissipation, respectively. The equations are exact as written, but the terms $\tau_{ij}, J_i, \mathcal{D}_i, \mathcal{P}$, and Φ are all higher-order moments that must be modeled. In the standard K-ε closure, the Reynolds stress is modeled as:

$$\tau_{ij} = \frac{2}{3}\overline{K}\delta_{ij} - \nu_T(\nabla_j \overline{v}_i + \nabla_i \overline{v}_j), \tag{7}$$

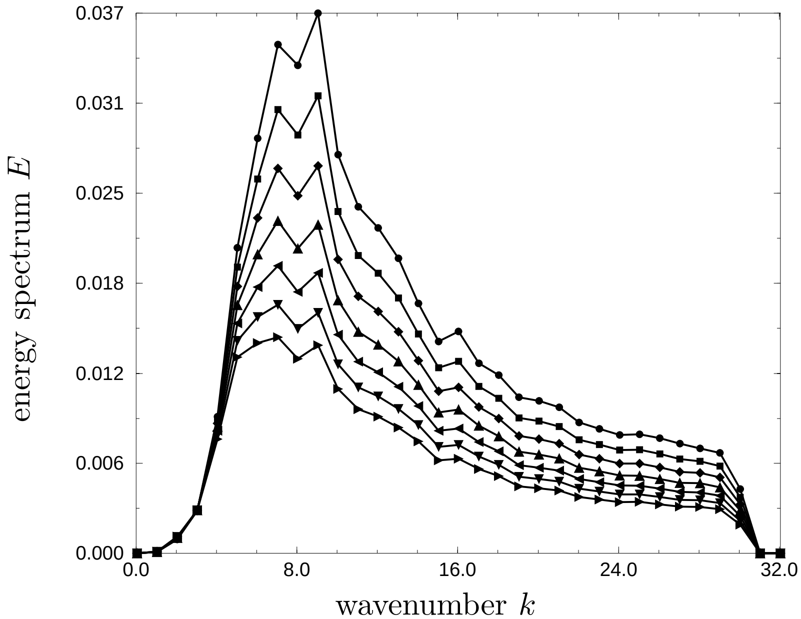

FIGURE 1. Velocity spectrum $E(k,t)$. ● : t=0.2295, ■: t=0.3299, ◆: t=0.4183, ▲: t=0.5262, ◀: t=0.6407, ▼: t=0.7647, ▶: t=0.8940

where the eddy viscosity is given by $\nu_T = C_\mu \frac{\overline{K}^2}{\overline{\varepsilon}}$, with $C_\mu = 0.09$ the conventional value of the constant. Similar gradient-diffusion models are made for the other transport terms in the K-ε closure equations, but these do not concern us here.

As reviewed in the Introduction, the Rayleigh-Ritz method gives predictions for statistics of the fluctuations v_i' by means of a perturbation of the moment-equations (4)-(6). At the level of 2nd-order statistics, the predictions are the same as those of a linear Langevin model (Eyink, 1998). For the K-ε Rayleigh-Ritz, the corresponding Langevin equations are obtained by linearizing (4)-(6) around the homogeneous state $\overline{v}_i \equiv 0, \overline{K}(t), \overline{\varepsilon}(t)$ and by then adding suitable white-noise forces. In the case of isotropic decay, the Langevin models for the two sets of variables v_i' and K', ε' are completely uncoupled, and they can be analyzed separately. Results on the K-ε fluctuations will be given elsewhere. The equation for the velocity fluctuations is $\partial_t v_i' = \nu_T \triangle v_i' + q_i$. Because all of the coefficients in the Langevin model are independent of space, it is advantageous to take a Fourier transform. The fluctuations at distinct wavenumbers then also completely decouple. In the Fourier representation

$$\frac{d}{dt}\widehat{v}_i(\mathbf{k},t) = -\nu_T k^2 \widehat{v}_i(\mathbf{k},t) + \widehat{q}_i(\mathbf{k},t), \qquad (8)$$

It is clear that this equation cannot be accurate at all wavenumbers. In particular, K-ε modeling is not intended to apply to inertial-range wavenumbers and higher. The principle that the dynamics of fluctuations should be governed by the same equations which determine the evolution of the mean values is known in statistical physics as the "Onsager regression hypothesis" (Onsager, 1931). For the hydrodynamic variables of molecular dynamical systems, the hypothesis is asymptotically exact in the limit of wavenumbers small compared to the inverse mean-free-length and frequencies small compared to the inverse mean-free-time. Likewise, we anticipate here that a K-ε Rayleigh-Ritz can be accurate—at most—for some range of low wavenumber modes.

In general, one should expect that a model superior to standard K-ε will allow for a wavenumber- and time-dependent dynamics as in the velocity sector

$$\frac{d}{dt}\widehat{v}_i(\mathbf{k},t) = A(k,t)\widehat{v}_i(\mathbf{k},t) + \widehat{q}_i(\mathbf{k},t). \tag{9}$$

Without loss of generality, this may be represented by a scale-dependent eddy-viscosity as $A(k,t) = -\nu_T(k,t)k^2$. Standard Kolmogorov dimensional reasoning would give $\nu_T(k,t) \propto \varepsilon^{1/3}(t)k^{-4/3}$ in the inertial range. In fact, we may note that a Langevin model for the velocity of the form of (24) was proposed by Kraichnan in his "Distant-Interaction Algorithm" (DSTA) (Kraichnan, 1987). However, Kraichnan's model was proposed for *Lagrangian* time-correlations rather than the *Eulerian* ones considered here. There is nothing that prevents the Rayleigh-Ritz method being applied to Lagrangian variables of the fluid system. However, for the moment we wish to study the ability of the standard K-ε closure to make correct predictions for Eulerian statistics within the variational apparatus.

Our basic test of the K-ε Rayleigh-Ritz scheme is to calculate the 2-time spectrum $E(k;t,t_0)$. Although the predictions are the same as those of the Langevin model in Eq. (8), we shall not calculate the 2-time correlations directly from that stochastic equation. Instead, we make use of the fluctuation-response relation (3). A direct application would involve calculating a numerical derivative with respect to the h-field of the solution of the perturbed closure equation, for $h(t) = h\delta(t-t_0)$. However, this algorithm turns out to be numerically unstable, and its accuracy degrades rapidly in time. Instead, our numerical procedure, for each wavenumber k, is to solve in conjunction with the ODE for the ensemble-means a linearized K-ε closure equation for the 2-time cospectra with the LES 1-time cospectra at time t_0 as initial data. Although the Langevin equations are not directly employed to calculate the 2-time cospectra, it is still important to determine whether these Rayleigh-Ritz predictions have a model realization. To investigate this we have also calculated the noise spectra of the Langevin white-noise forces $Q(k,t)$. These are obtained from a *fluctuation-dissipation relation* (Eyink, 1998)

$$Q(k,t) = \nu_T k^2 E(k,t) + \frac{1}{2}\dot{E}(k,t), \tag{10}$$

by inputting for each time t the single-time spectra discussed in Section 2.1. Realizability requires that $Q(k,t)$ be nonnegative for each wavenumber k.

In Fig. 2 the 2-time spectra $E(k;t,t_0)$ are graphed as functions of wavenumber k, for both the LES and the K-ε Rayleigh-Ritz, at a sequence of times t starting with t_0. Several points of comparison become immediately apparent. At wavenumbers much less than the peak, the LES and Rayleigh-Ritz spectra are almost indistinguishable. Around the peak wavenumber there is a close agreement for a short time, but later on the Rayleigh-Ritz spectrum decays slower than the LES. At wavenumbers higher than the peak, the opposite is initially true: the LES spectrum decays

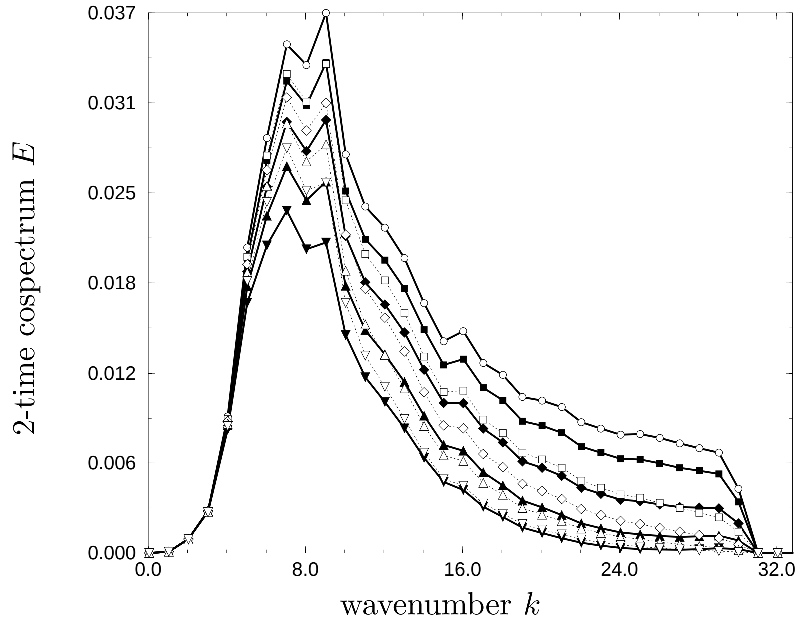

FIGURE 2. Comparison of 2-time velocity cospectra $E(k;t,t_0)$. t=0.2295: • - LES, ○ - $K\varepsilon$RR; t=0.3299: ■- LES, □- $K\varepsilon$RR; t=0.4183: ♦- LES, ◇- $K\varepsilon$RR; t=0.5262: ▲- LES, △- $K\varepsilon$RR; t=0.6407: ▼- LES, ▽- $K\varepsilon$RR.

slower initially and persists at larger magnitude than the Rayleigh-Ritz spectrum. However, as time passes, the decay rate of the LES spectrum increases, and it first equals in magnitude and then dips below the Rayleigh-Ritz spectrum.

These observations are reasonably explained. At the low wavenumbers, the rate of change of both spectra is very slight so that the close agreement is automatic. The agreement at early times near the energy peak is presumably an indication that the K-ε model with the standard choice of constants is an excellent model of the intrinsic dynamics at those scales. This is the range of wavenumbers in which the "regression hypothesis" appears to be valid. In other words, for early times the decay of correlations is due primarily to the space diffusion of fluctuations by turbulent viscosity, well-described by the standard K-ε model. However, since we are dealing with *Eulerian* time-correlations, this decay mechanism is overwhelmed at later times by *convective dephasing* (Kraichnan, 1964b). The phase shift $e^{-i\mathbf{v}\cdot\mathbf{k}(t-t_0)}$ in Fourier amplitudes due to advection by a random velocity $\mathbf{v}$ leads to a decay rate of correlations which goes roughly as $\sim v_0^2k^2(t-t_0)$, where v_0 is an rms velocity. This grows faster in k than an eddy-damping rate such as appears in Eq. (9), which goes as $\nu(k,t)k^2 \sim k^{2/3}$ in the inertial range. At the same time, the decay rate from convection grows in time proportional to $t-t_0$. Hence, it is negligible at short times but rapidly grows to dominate the intrinsic decay from eddy-damping. This leads to the later overestimation of the spectrum around the peak wavenumber by K-ε Rayleigh-Ritz since the latter incorporates no such convective effects. The same physical considerations explain the observations in the high wavenumber range beyond the peak. At early times when convective effects are negligible, the constant eddy-viscosity in the standard K-ε model overestimates the eddy-damping in the higher wavenumbers. Hence, the early decay of the Rayleigh-Ritz spectrum is too rapid. At the same time, the K-ε modeling includes no convective dephasing effects, which rapidly grow to dominate the decay of the Eulerian time-correlations. Hence, the early decay of the LES 2-time spectra lags behind the Rayleigh-Ritz predictions, but at later times it exceeds the Rayleigh-Ritz decay.

We do not have space here for a complete discussion of the realizability of the K-ε Rayleigh-Ritz, but it is important to make a few remarks. The noise spectrum $Q(k,t)$ of the Langevin force in Eq. (12) calculated from the FDT Eq. (10) is not strictly realizable. Indeed, it takes on negative values for large wavenumbers $k > 36$ and also slightly negative values for a range of small wavenumbers $k = 5-8$. The breakdown at large wavenumbers is not really a great surprise because the K-ε modeling is not expected to be valid there. However, the breakdown for the wavenumbers $k = 5 - 8$ is more serious. The reason for the realizability violation, as we shall see below, is the K-ε model's underestimation of the eddy-viscosity in those wavenumber modes. On the other hand, the wavenumber which corresponds to the spectral peak at the initial time, $k = 9$, has a *marginally realizable* noise covariance. That is, to within numerical precision, the noise vanishes at that wavenumber initially. This is not an accident, as we see below.

2.3 Improved Rayleigh-Ritz and comparison with LES

We now consider various strategies to develop an improved Rayleigh-Ritz. As a first step, we shall carry out a *POP analysis*, which, as described in (Penland, 1989), is a technique to obtain models for $A(k,t_0)$ and $Q(k,t_0)$ directly from empirical 2-time data. The time-dependent POP method we use is a "zero-lag" prescription discussed in (Eyink, 1998). In principle, the POP analysis gives the best possible such linear Langevin model although there is an important issue about the "optimum lag" to be used in this analysis. If it is realizable, then the best one could hope is that the Rayleigh-Ritz *Ansatz* should reproduce the POP model. This is always possible if the the Rayleigh-Ritz *Ansatz* goes beyond the standard K-ε model by allowing a wavenumber and time-dependent eddy-viscosity $\nu_T(k,t)$, as in Eq. (9). In addition to the POP analysis, we shall also determine a "zero-lag K-ε model" by insisting that the Rayleigh-Ritz produce $Q(k,t) \equiv 0$ for all k,t. We may enforce in this way some agreement with the zero-lag POP result since the latter always has vanishing noise initially when the input 2-time covariance is continuously differentiable (Eyink, 1998). Thus, we may use this condition as a means to extract k,t-dependent values of the K-ε closure constant $C_\mu(k,t)$. This "zero-lag" Rayleigh-Ritz procedure differs from POP in using only single-time LES data.

We shall determine below both zero-lag POP models and "zero-lag" K-ε Rayleigh-Ritz models and compare these to one another. From the discussion in Section 3.2, we may anticipate that these models should give a good short-time description of the Eulerian 2-time correlations but a much better long-time description of Lagrangian 2-time correlations. Of course, there is nothing to prevent application of both the POP and Rayleigh-Ritz methods to Lagrangian dynamical variables. Furthermore, one should not expect the zero-lag POP models to differ substantially between the two cases. The reason has to do with the physics of the convective dephasing.

We have seen above that the decay rate from convective dephasing should vanish $\propto (t-t_0)$ for short times. In that case, any difference in the time derivative of Eulerian and Lagrangian correlators should *vanish* in the zero-lag limit $t \to t_0$. Needless to say, while the POP models are not expected to differ for the two sets of variables, the *validity* of the POP models will depend upon the choice. In agreement with earlier workers such as (Kraichnan, 1987), we expect the POP Langevin models to give a much better representation of the Lagrangian dynamics.

In principle, therefore, we should compare the predictions of our POP and "zero-lag" Rayleigh-Ritz models to Lagrangian 2-time data from the LES. We hope to do so later on, but, at the moment, such data are not available. Below we have attempted instead to add into the Rayleigh-Ritz equations the "convective dephasing" in order to make a more meaningful comparison with the Eulerian 2-time data at our disposal. We have added the convection effects in a relatively crude way by supplying to the derivative of the 2-time correlations in each wavenumber shell a new term

$$\partial_t E(k;t,t_0) = \cdots - \left[2C_\eta k^2 \int_{t_0}^{t} ds \int_0^k dq\ E(q,s) \right] E(k;t,t_0). \tag{11}$$

Note that $2\int_0^k dq\ E(q,t)$ represents the mean-square velocity $v_{\rm rms}^2(k,t)$ of all the wavenumbers smaller than the given wavenumber. This is natural for a term to represent advection by the larger eddies. The constant C_η represents the "efficiency" of the dephasing. Clearly, this should in reality be wavenumber dependent. Because the smaller eddies are more random and more rapidly evolving, the phase shifts they induce in Fourier amplitudes will suffer much destructive interference before averaging over the ensemble of velocities. Hence, their contribution to correlation decay will be reduced. On the contrary, the larger eddies are much more coherent and slowly evolving so that there will be mostly constructive addition to the Fourier phase shift. The constant value $C_\eta \equiv 1$ would hold for perfect "efficiency" of the dephasing, as is true for a frozen-in-time, uniform Gaussian velocity, with perfect coherence in space-time (Kraichnan, 1964b). Thus, we adopt a value, somewhat arbitrarily, of an order of magnitude less than unity: $C_\eta = 0.1$. This is likely to be an overestimate at high wavenumbers and an underestimate at low wavenumbers. The above crude model can be regarded as a simplified form of the DIA model for the convection effects as discussed in (Kraichnan, 1959), Section 5, and (Kraichnan, 1964a).

Our POP results are presented in the form of a "dimensionless eddy-viscosity" $C_\mu(k,t_0) := \frac{\nu_T(k,t_0)}{\overline{K}^2(t_0)/\overline{\varepsilon}(t_0)}$.

In agreement with our earlier observations, there is a "negative eddy-viscosity" in the lowest wavenumbers $k = 1-3$, the most negative value -1.9333 occurring at $k = 1$. Thereafter, the eddy viscosity grows to a maximum 1.1760 at wavenumber $k = 6$ and beyond the maximum decays in a roughly power-law fashion, consistent with expectations for an inertial range. The value at the energy peak wavenumber $k = 9$ is 0.0858, remarkably close to the standard K-ε value of $C_\mu = 0.09$. We

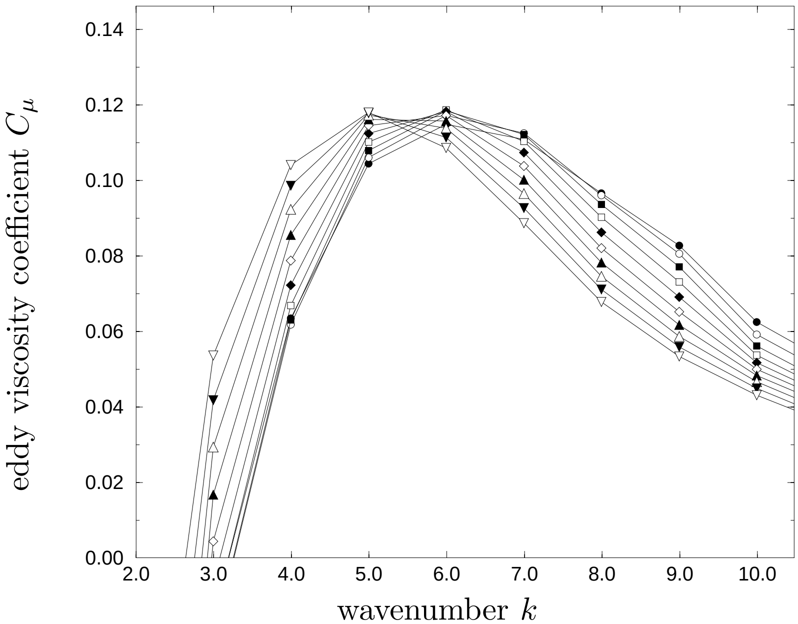

FIGURE 3. Dimensionless eddy-viscosity spectrum $C_\mu(k,t)$ at several times. t=0.2295: ● , t=0.3299: ○ , t=0.4183: ■, t=0.5262: □, t=0.6407: ◆, t=0.7647: ◇, t=0.8940: ▲, t=1.0318: △, t=1.1788: ▼, t=1.3354: ▽.

emphasize that this value is obtained here from the decay of *fluctuations* and not, as usual, from the diffusive decay of *mean* values. The close agreement supports the idea that something like the "regression hypothesis" should hold for wavenumbers around the energy peak.

The results of the "zero-lag" Rayleigh-Ritz analysis are entirely consistent with those for the zero-lag POP. In fact, the results for $C_\mu(k,t_0)$ are so close numerically that a plot of them together would show no difference between them. The "zero-lag" Rayleigh-Ritz values $C_\mu(k,t)$ are plotted in Fig. 3 for the low wavenumbers $k = 2-10$ at a sequence of times. There is seen to be a slight drift to the left in time. Remarkably, this is consistent with the slow decrease of the energy peak wavenumber k_P over that same time from $k_P = 9$ initially to $k_P = 6$ at the final time. The wavenumber at which $C_\mu(k,t) \approx 0.09$ tracks along with the peak wavenumber k_P over the whole period of the decay. This a further verification of the "regression hypothesis" for the energy peak wavenumbers.

A realizability check on the POP model helps to explain the close agreement of the POP and "zero-lag" Rayleigh-Ritz results at early times. Shown in Fig. 4 are the POP noise spectral values for the first fourteen nonzero wavenumber modes, $k = 1-14$, plotted as a function of time up to $t = 1.4$. It may be seen that the noise spectra all start at zero at the initial time t_0, to numerical precision, and thereafter rise to positive values. Most importantly, this result establishes the realizability of the Langevin model with the POP coefficients. In agreement with the results of the previous section, the standard value $C_\mu = 0.09$ leads to a noise spectrum indistinguishable from zero initially at the peak wavenumber.

We are now in possession of a fully realizable Langevin model of the form of Eq. (9)

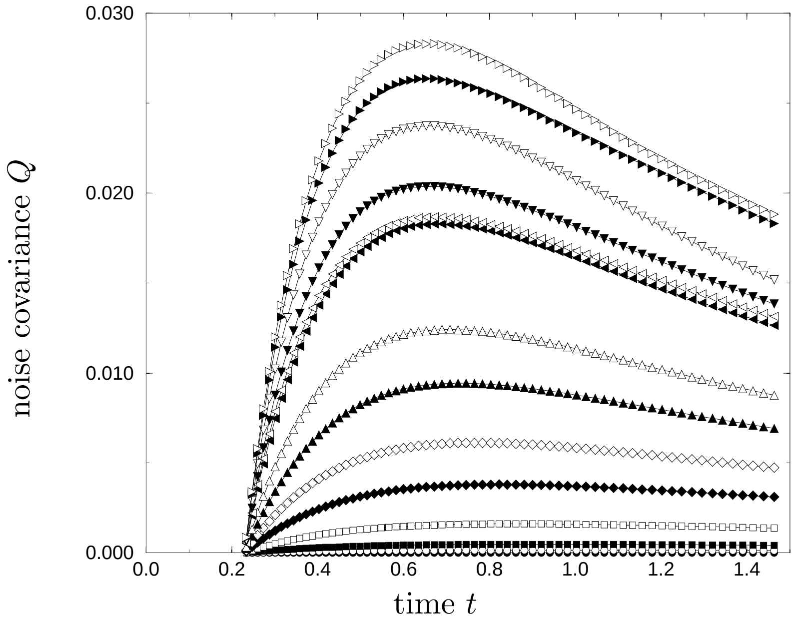

FIGURE 4. POP noise covariance $Q(k,t)$ in low wavenumbers. k=1: ● , k=2: ○ , k=3: ■, k=4: □, k=5: ♦, k=6, ◇, k=7: ▲, k=8: △, k=9: ◀, k=10: ◁, k=11: ▼, k=12: ▽, k=13: ▶, k=14: ▷.

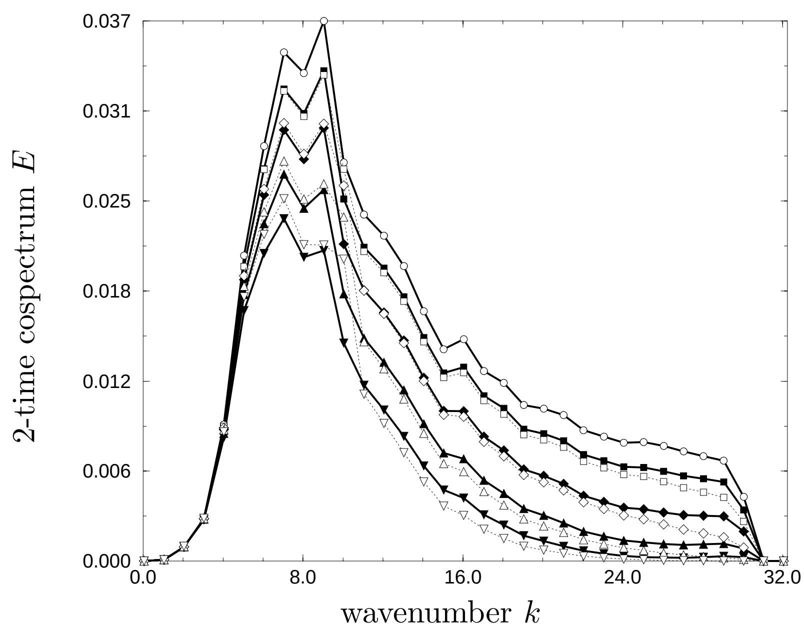

FIGURE 5. Comparison of 2-time velocity cospectra $E(k;t,t_0)$. t=0.2295: ● - LES, ○ - RRC; t=0.3299: ■- LES, □- RRC; ; t=0.4183: ♦- LES, ◇- RRC; t=0.5262: ▲- LES, △- RRC; t=0.6407: ▼- LES, ▽- RRC;

with the coefficients therein determined either from POP or "zero-lag" Rayleigh-Ritz. The model is very similar in form to the DTSA model (Kraichnan, 1987). As has been discussed earlier, the same model will presumably arise for either Eulerian or Lagrangian variables. However, it is probably a much more accurate, long-term predictor for Lagrangian time-correlations than it is for Eulerian. Unfortunately, we have no such Lagrangian data with which to compare. Instead, we shall correct the "zero-lag" Rayleigh-Ritz model to include convective dephasing effects as described previously. In Fig. 5 we show plotted together as functions of wavenumber k the LES results for the 2-time velocity spectrum $E(k;t,t_0)$ and those obtained from the "zero-lag" Rayleigh-Ritz with the convective correction in Eq. (11), for several times over the run. As may be seen, there is a much improved agreement for all times. The only defect is at the later times when the Rayleigh-Ritz calculation gives a slightly too great decay at high wavenumbers and a slightly too slow decay at low wavenumbers. This is in agreement with our earlier remark that the "efficiency" $C_\eta = 0.1$ we used in the convective correction is likely too large at high wavenumbers and too small at low wavenumbers.

4. Conclusions

The results of this work allow us to draw the following tentative, general conclusions:

(i) The standard K-ε model gives a very good quantitative account of fluctuations at the energy peak wavenumber and a rather good one at lower wavenumbers. This supports the idea that a proper application of Onsager's "regression hypothesis" for turbulent flow is to fluctuations in the peak wavenumber range. Presumably, an "optimal" POP analysis would recover something very close to the standard K-ε model in that regime. Because this range of wavenumbers makes a dominant contribution to integrations over k, the single-point, 2-times statistics will be rather well captured by such modeling.

(ii) However, the agreement of the predicted and measured statistics is restricted to a short time for Eulerian correlations. The success in reproducing those correlations at longer time separations by making a simple convective correction suggests that much better long-term predictability will be obtained for Lagrangian variables with a standard $RANS$-type model.

(iii) Further improved predictions can be obtained in the Rayleigh-Ritz framework by going beyond the K-ε modeling. Even the simple expedient of taking the $RANS$ model coefficients to be functions of wavenumber k—and thus nonlocal kernels in physical space—can give much better results. In the case of the velocity sector, a "zero-lag" model of this type is very similar to the DSTA model of Kraichnan (1987). Such models can presumably give good long-term predictions for Lagrangian variables. To predict Eulerian time-correlations at large time-separations seems to require a non-Markovian or history-dependent *Ansatz* in the Rayleigh-Ritz formalism to properly capture the convective dephasing effects.

The final verification of these conclusions will require some more work. In particular, a POP analysis based upon a complete set of 2-time correlation functions ought to be performed. This would allow a systematic investigation of lag-dependence, which is particularly important in the K-ε sector. Furthermore, a proper comparison of the predictions with LES or other data would require Lagrangian time-correlations.

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