41 Quantum states of the electromagnetic field

Slides: Lecture 41a Number states

Text reference: Quantum Mechanics for Scientists and Engineers

Section 15.6 to end of subsection “Representation of time dependence ...”
Quantum states of the electromagnetic field

Quantum mechanics for scientists and engineers

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Quantum states of the electromagnetic field

Number states

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Number states

The Hamiltonian and number operator eigenstates $|n_\lambda\rangle$ correspond to $n_\lambda$ photons in the mode and are known as the number states or Fock states.

In these states, the probability of measuring any particular amplitude $B_{\lambda y}$ in the mode is distributed according to the square of the Hermite-Gaussian harmonic oscillator solutions with quantum number $n_\lambda$.

The $E_{\lambda z}$ amplitudes are similarly distributed.
Number states

The expectation values of the electric and magnetic field amplitudes are both zero for any number state, e.g.,

\[
\langle n_{\lambda} | \hat{E}_{\lambda z} | n_{\lambda} \rangle = i \sqrt{\frac{\hbar \omega_{\lambda}}{LE_o}} \sin kx \langle n_{\lambda} | \hat{a}_{\lambda}^\dagger - \hat{a}_{\lambda} | n_{\lambda} \rangle \\
= i \sqrt{\frac{\hbar \omega_{\lambda}}{LE_o}} \sin kx \left( \sqrt{n_{\lambda} + 1} \langle n_{\lambda} | n_{\lambda} + 1 \rangle - \sqrt{n_{\lambda}} \langle n_{\lambda} | n_{\lambda} - 1 \rangle \right) = 0
\]

because the states \( |n_{\lambda}\rangle, |n_{\lambda} - 1\rangle, \) and \( |n_{\lambda} + 1\rangle \)
are all eigenstates of the same Hamiltonian
and so are orthogonal
and similarly for the magnetic field mode amplitude.
Schrödinger and Heisenberg representations

So far, we have used solutions to the time-independent Schrödinger equation for the electromagnetic mode. Here we use the term “Schrödinger equation” in the generalized sense where we mean that

\[ \hat{H} |\phi\rangle = E |\phi\rangle \]

is a Schrödinger equation for a system in an eigenstate $|\phi\rangle$ with eigenenergy $E$.

Explicitly, for the eigenstates of our electromagnetic mode, we have

\[ \hat{H} |n_{\lambda}\rangle = \left(n_{\lambda} + 1/2\right) \hbar \omega_{\lambda} |n_{\lambda}\rangle \]
Generalizing our earlier postulations, we also postulate here that the time-dependent generalized Schrödinger equation is valid, i.e.,

\[ \hat{H} |\phi\rangle = i\hbar \frac{\partial}{\partial t} |\phi\rangle \]

even if our Hamiltonian is not the one in our original Schrödinger equation for an electron. This postulation does appear to work.
With this approach to describing time-dependence as before, to get the time variation of a given state we multiply the time-independent energy eigenstates by

\[ \exp\left[-i\left(n_\lambda + 1/2\right)\frac{\hbar \omega_\lambda t}{\hbar}\right] = \exp\left[-i\left(n_\lambda + 1/2\right)\omega_\lambda t\right] \]

to make \( \hat{H}\left|n_\lambda\right\rangle = \left(n_\lambda + 1/2\right)\hbar \omega_\lambda \left|n_\lambda\right\rangle \)

consistent with \( \hat{H}\left|\phi\right\rangle = i\hbar \frac{\partial}{\partial t}\left|\phi\right\rangle \)

so including time-dependence the number states become

\[ \exp\left[-i\left(n_\lambda + 1/2\right)\omega_\lambda t\right]\left|n_\lambda\right\rangle \]
41 Quantum states of the electromagnetic field

Slides: Lecture 41b The coherent state

Text reference: Quantum Mechanics for Scientists and Engineers

Section 15.6 subsection “Coherent state”
Quantum states of the electromagnetic field

The coherent state

Quantum mechanics for scientists and engineers

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The coherent state

The state that corresponds most closely to the classical field in an electromagnetic mode is the coherent state introduced previously as an example with the harmonic oscillator.

Using our current notation we can rewrite this as

\[ |\Psi_{\lambda n}\rangle = \sum_{n_{\lambda}=0}^{\infty} c_{\lambda n_{\lambda}} \exp \left[ -i \left( n_{\lambda} + \frac{1}{2} \right) \omega_{\lambda} t \right] |n_{\lambda}\rangle \]

where \( c_{\lambda n_{\lambda}} = \sqrt{\frac{n_{\lambda}^{n_{\lambda}} \exp(-n_{\lambda})}{n_{\lambda}!}} \).
The coherent state

In these expansion coefficients \( c_{\lambda \bar{n}n} = \sqrt{\frac{\bar{n}^{n\lambda} \exp(-\bar{n})}{n^!}} \)

the quantity \( \bar{n} \) will turn out to be the expected value of the number of photons in the mode.

As before, note that

\[
|c_{\lambda \bar{n}n}|^2 = \frac{\bar{n}^{n\lambda} \exp(-\bar{n})}{n^!}
\]

is the Poisson distribution with mean \( \bar{n} \) and standard deviation \( \sqrt{\bar{n}} \)
The coherent state

Note that, in the coherent state
the number of photons in the mode is not determined

The coefficients $|c_{\lambda \bar{m}}|^2$
tell us the probability that
we will find $n_\lambda$ photons in the mode
if we make a measurement

This number is now found to be distributed according to a Poisson distribution
The coherent state

It is in fact the case that the statistics of the number of photons in an oscillating “classical” electromagnetic field are Poissonian.

For example, if we put a photodetector in a laser beam we will measure a Poissonian distribution of the arrival rates of the photons, an effect known as shot noise.
Coherent state

Coherent state oscillations with

\[ |\Psi_{\lambda\bar{n}}\rangle \equiv \Psi_{\lambda\bar{n}}(\xi_\lambda, t) \]

\[ = \sum_{n_\lambda=0}^{\infty} c_{\lambda\bar{n}n} \exp \left[ -i \left( n_\lambda + \frac{1}{2} \right) \omega_\lambda t \right] |n_\lambda\rangle \]

where

\[ c_{\lambda\bar{n}n} = \sqrt{\frac{\bar{n}^{n_\lambda}}{n_\lambda !}} \exp(-\bar{n}) \]

and \( \xi_\lambda \) is the magnetic field amplitude
Coherent state oscillations with

\[ |\Psi_{\lambda \bar{n}}(\xi, t)|^2 \]

where

\[
\Psi_{\lambda \bar{n}}(\xi, t) = \sum_{n_\lambda=0}^{\infty} c_{\lambda \bar{n}} \exp \left[ -i \left( n_\lambda + \frac{1}{2} \right) \omega_\lambda t \right] |n_\lambda\rangle
\]

and \( \bar{n} \) is the magnetic field amplitude.
Coherent state

Coherent state oscillations with

\[ |\Psi_{\lambda\bar{n}}(\xi, t)\rangle \equiv \sum_{n_{\lambda}=0}^{\infty} c_{\lambda\bar{n}} \exp \left[ -i \left( n_{\lambda} + \frac{1}{2} \right) \omega_{\lambda} t \right] |n_{\lambda}\rangle \]

where

\[ c_{\lambda\bar{n}} = \sqrt{\frac{\bar{n}^{n_{\lambda}} \exp(-\bar{n})}{n_{\lambda}!}} \]

and \( \xi_{\lambda} \) is the magnetic field amplitude
Coherent state oscillations with

\[ |\Psi_{\lambda \overline{n}}(\xi, t)\rangle = \Psi_{\lambda \overline{n}}(\xi_\lambda, t) \]

\[ = \sum_{n_\lambda=0}^{\infty} c_{\lambda \overline{n} n} \exp \left[ -i \left( n_\lambda + \frac{1}{2} \right) \omega_\lambda t \right] |n_\lambda\rangle \]

where

\[ c_{\lambda \overline{n} n} = \sqrt{\frac{\overline{n}^{n_\lambda} \exp(-\overline{n})}{n_\lambda!}} \]

and \( \xi_\lambda \) is the magnetic field amplitude
Coherent state oscillations with

\[ |\Psi_{\lambda, n}\rangle = \Psi_{\lambda, n}(\xi_\lambda, t) \]

\[ = \sum_{n_\lambda=0}^{\infty} c_{\lambda, n_\lambda} \exp \left[ -i \left( n_\lambda + \frac{1}{2} \right) \omega_\lambda t \right] |n_\lambda\rangle \]

where

\[ c_{\lambda, n_\lambda} = \sqrt{\frac{\bar{n}^{-n_\lambda} \exp(-\bar{n})}{n_\lambda!}} \]

and \( \xi_\lambda \) is the magnetic field amplitude
41 Quantum states of the electromagnetic field

Slides: Lecture 41c Sets of modes

Text reference: Quantum Mechanics for Scientists and Engineers

Section 15.7 to start of “Multimode photon states”
Quantum states of the electromagnetic field

Sets of modes

Quantum mechanics for scientists and engineers

David Miller
Sets of classical modes

We postulate a set of classical modes each of which has the following form

\[ E_\lambda (r,t) = -p_\lambda (t)D_\lambda u_\lambda (r) \quad B_\lambda (r,t) = q_\lambda (t)\frac{D_\lambda}{c}v_\lambda (r) \]

Here \( E_\lambda, B_\lambda, u_\lambda, \) and \( v_\lambda \) are all in general vectors and \( D_\lambda \) is a constant.

The forms we used for our plane wave example

\[ E_z = p(t)D \sin kx \quad B_y = q(t)\frac{D}{c} \cos kx \]

correspond to these with

\[ u_\lambda (r) = -\hat{z} \sin (kx) \quad v_\lambda (r) = \hat{y} \cos (kx) \]
Sets of classical modes

\[ \mathbf{E}_\lambda (\mathbf{r}, t) = -p_\lambda (t) D_\lambda \mathbf{u}_\lambda (\mathbf{r}) \] and \[ \mathbf{B}_\lambda (\mathbf{r}, t) = q_\lambda (t) \frac{D_\lambda}{c} \mathbf{v}_\lambda (\mathbf{r}) \]
satisfy Maxwell’s equations and the wave equation in free space if we require

\[ \nabla \times \mathbf{u}_\lambda (\mathbf{r}) = \frac{\omega_\lambda}{c} \mathbf{v}_\lambda (\mathbf{r}) \]
\[ \nabla \times \mathbf{v}_\lambda (\mathbf{r}) = \frac{\omega_\lambda}{c} \mathbf{u}_\lambda (\mathbf{r}) \]

\[ \frac{dq_\lambda}{dt} = \omega_\lambda p_\lambda \]
\[ \frac{dp_\lambda}{dt} = -\omega_\lambda q_\lambda \]

We presume the classical electromagnetic problem with its boundary conditions has been solved to give these modes.
Sets of classical modes

We will also presume that

the spatial functions \( u_\lambda (r) \) and \( v_\lambda (r) \)

are normalized over the entire volume

and they are all orthogonal

So, for two (possibly different) modes

with indices \( \lambda_1 \) and \( \lambda_2 \) respectively

\[
\int u_{\lambda_1} (r) \cdot u_{\lambda_2} (r) \, d^3r = \delta_{\lambda_1, \lambda_2}
\]

and

\[
\int v_{\lambda_1} (r) \cdot v_{\lambda_2} (r) \, d^3r = \delta_{\lambda_1, \lambda_2}
\]
Consider a classical superposition of such modes
\[ E(r, t) = \sum_{\lambda} -p_{\lambda}(t)D_\lambda u_{\lambda}(r) \quad B(r, t) = \sum_{\lambda} q_{\lambda}(t)\frac{D_\lambda}{c}v_{\lambda}(r) \]

The total energy of such a field is
\[
H = \int \frac{1}{2}\left( \varepsilon_o E^2 + \frac{1}{\mu_o} B^2 \right)d^3r
\]
\[
= \frac{1}{2} \varepsilon_o \sum D_{\lambda_1}D_{\lambda_2} \int \left[ p_{\lambda_1}p_{\lambda_2} u_{\lambda_1}(r) \cdot u_{\lambda_2}(r) + q_{\lambda_1}q_{\lambda_2} v_{\lambda_1}(r) \cdot v_{\lambda_2}(r) \right]d^3r
\]
where we have also used \( 1/ c^2 = \varepsilon_o \mu_o \)
Classical superpositions and energy

Using the orthonormality of the $u_\lambda$ and of the $v_\lambda$ in

$$H = \frac{1}{2} \epsilon_o \sum_{\lambda_1, \lambda_2} D_{\lambda_1} D_{\lambda_2} \int \left[ p_{\lambda_1} p_{\lambda_2} u_{\lambda_1}(r) \cdot u_{\lambda_2}(r) + q_{\lambda_1} q_{\lambda_2} v_{\lambda_1}(r) \cdot v_{\lambda_2}(r) \right] d^3 r$$

eliminates the "cross terms" with different indices, so

$$H = \frac{1}{2} \epsilon_o \sum_{\lambda} D_{\lambda}^2 \left( p_{\lambda}^2 + q_{\lambda}^2 \right)$$

so we can write a sum of separate Hamiltonians

$$H = \sum_{\lambda} H_\lambda$$

where $H_\lambda = \frac{1}{2} \epsilon_o D_{\lambda}^2 \left( p_{\lambda}^2 + q_{\lambda}^2 \right)$
Classical mode Hamiltonians

In each
\[ H_\lambda = \frac{1}{2} \varepsilon_o D_\lambda^2 \left( p_\lambda^2 + q_\lambda^2 \right) \]

if we now choose
\[ D_\lambda = \sqrt{\frac{\omega_\lambda}{\varepsilon_o}} \]

then we have
\[ H_\lambda = \frac{\omega_\lambda}{2} \left( p_\lambda^2 + q_\lambda^2 \right) \]

and the \( H_\lambda, p_\lambda, \) and \( q_\lambda \) now obey

Hamilton’s equations
\[
\begin{align*}
\frac{dp_\lambda}{dt} &= -\frac{\partial H_\lambda}{\partial q_\lambda} \\
\frac{dq_\lambda}{dt} &= \frac{\partial H_\lambda}{\partial p_\lambda}
\end{align*}
\]

as we could check
Quantization of individual modes

We can proceed for each mode, postulating a “momentum” operator for each mode

\[ \hat{p}_\lambda = -i\hbar \frac{d}{dq_\lambda} \]

hence changing the classical Hamiltonian

\[ H_\lambda = \frac{\omega_\lambda}{2} \left( p_\lambda^2 + q_\lambda^2 \right) \]

to the proposed the quantum mechanical one

\[ \hat{H}_\lambda = \frac{\omega_\lambda}{2} \left[ -\hbar^2 \frac{d^2}{dq_\lambda^2} + q_\lambda^2 \right] \]
Quantization of individual modes

We next rewrite this Hamiltonian as

\[ \hat{H}_\lambda = \frac{\hbar \omega_\lambda}{2} \left[ -\frac{d^2}{d \xi_\lambda^2} + \xi_\lambda^2 \right] = \hbar \omega_\lambda \left( \hat{a}_\lambda^\dagger \hat{a}_\lambda + \frac{1}{2} \right) \]

defining dimensionless units \( \xi_\lambda = q_\lambda / \sqrt{\hbar} \)

and creation and annihilation operators

\[ \hat{a}_\lambda^\dagger \equiv \frac{1}{\sqrt{2}} \left( -\frac{d}{d \xi_\lambda} + \xi_\lambda \right) \quad \hat{a}_\lambda \equiv \frac{1}{\sqrt{2}} \left( \frac{d}{d \xi_\lambda} + \xi_\lambda \right) \]

so the total Hamiltonian for the set of modes is

\[ \hat{H} = \sum_\lambda \hbar \omega_\lambda \left( \hat{a}_\lambda^\dagger \hat{a}_\lambda + \frac{1}{2} \right) \]
41 Quantum states of the electromagnetic field

Slides: Lecture 41d Multimode photon states

Text reference: Quantum Mechanics for Scientists and Engineers

Section 15.7 subsection “Multimode photon states”
Occupation number representation

For example, the state with
one photon in mode $k$, three in mode $m$
and none in any other mode
which would be an example of
a "multimode" state
could be written as

$$|0_a, \ldots, 0_j, 1_k, 0_l, 3_m, 0_n, \ldots\rangle$$

where we have labeled the modes progressively with the lower case letters
Single mode operators with multimode states

Just as before, the annihilation operators will have the property now specific to given mode

\[ \hat{a}_\lambda \left|... , n_\lambda ,... \right> = \sqrt{n_\lambda} \left|... , (n_\lambda - 1)_\lambda ,... \right> \]

with \[ \hat{a}_\lambda \left|... , 0_\lambda ,... \right> = 0 \]

Similarly \[ \hat{a}^\dagger_\lambda \left|... , n_\lambda ,... \right> = \sqrt{n_\lambda + 1} \left|... , (n_\lambda + 1)_\lambda ,... \right> \]

and the number operator for a given mode will still be \[ \hat{N}_\lambda \equiv \hat{a}^\dagger_\lambda \hat{a}_\lambda \]

so \[ \hat{N} \left|... , n_\lambda ,... \right> = n_\lambda \left|... , n_\lambda ,... \right> \]
Writing multimode states using operators

We can create a multimode state by progressively operating with the appropriate creation operators starting with the “zero” state or “empty” state often written simply as $|0\rangle$

For our example state, we could write

$$\left|0_a,\ldots,0_j,1_k,0_l,3_m,0_n,\ldots\right\rangle = \frac{1}{\sqrt{1!3!}} \hat{a}_k^{\dagger} \hat{a}_m^{\dagger} \hat{a}_m^{\dagger} \hat{a}_m^{\dagger} |0\rangle$$

where the factor $1/\sqrt{1!3!}$ compensates for the factors introduced by the creation operators in

$$\hat{a}_\lambda^{\dagger} \left|\ldots,n_\lambda,\ldots\right\rangle = \sqrt{n_\lambda + 1} \left|\ldots,(n_\lambda + 1)_\lambda,\ldots\right\rangle$$

keeping the state normalized
Writing multimode states using operators

In general, we can write a state with

\( n_1 \) particles in mode 1
\( n_2 \) particles in mode 2

and so on

as

\[
|n_1, n_2, \ldots, n_\lambda, \ldots\rangle = \frac{1}{\sqrt{n_1!n_2!\ldots n_\lambda!\ldots}} \left( \hat{a}_1^\dagger \right)^{n_1} \left( \hat{a}_2^\dagger \right)^{n_2} \cdots \left( \hat{a}_\lambda^\dagger \right)^{n_\lambda} \cdots |0\rangle
\]
41 Quantum states of the electromagnetic field

Slides: Lecture 41e Multimode operators

Text reference: Quantum Mechanics for Scientists and Engineers

Section 15.7 subsection “Commutation relations ... “ to end
Quantum states of the electromagnetic field

Multimode operators

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Commutation relations for boson operators

Formally, then, for creation operators operating on any state we must have
$$\hat{a}_j \hat{a}_k = \hat{a}_k \hat{a}_j$$
or, in the form of a commutation relation
$$\hat{a}_j \hat{a}_k^\dagger - \hat{a}_k \hat{a}_j^\dagger = 0$$

Similarly, for annihilation operators it does not matter in what order we destroy particles and so we similarly have
$$\hat{a}_j \hat{a}_k^\dagger - \hat{a}_k \hat{a}_j^\dagger = 0$$
Commutation relations for boson operators

For mixtures of annihilation and creation operators

if we annihilate a boson in one mode
and create one in another

it does not matter in what order we do that either

Only if we are creating and annihilating in the same mode
does it matter what order we do this

with a commutation relation we have previously
deduced (i.e., \( \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1 \))

Hence in general we can write

\[
\hat{a}_j \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_j = \delta_{jk}
\]
Multimode field operators

It is now straightforward to construct the full multimode electric and magnetic field operators

We start from the classical definition of the multimode electric field

\[ E(r, t) = \sum_{\lambda} -p_{\lambda}(t)D_{\lambda}u_{\lambda}(r) \]

as an expansion in classical field modes

We use the relation \[ D_{\lambda} = \sqrt{\omega_{\lambda} / \varepsilon_0} \] we deduced to get Hamilton’s equations, and we substitute the operator \( \hat{p}_{\lambda} \) for the quantity \( p \) in each mode.
Multimode field operators

We therefore use our previously deduced operator

\[ \hat{p}_\lambda = i \sqrt{\frac{\hbar}{2}} (\hat{a}^\dagger_\lambda - \hat{a}_\lambda) \]

in

\[ E(r,t) = \sum_{\lambda} -p_\lambda(t)D_\lambda u_\lambda(r) \]

to obtain the (multimode) electric field operator

\[ \hat{E}(r,t) = i \sum_{\lambda} (\hat{a}_\lambda - \hat{a}^\dagger_\lambda) \sqrt{\frac{\hbar \omega_\lambda}{2 \epsilon_0}} u_\lambda(r) \]
By a similar argument, starting from the classical expression for a multimode magnetic field

\[ B(\mathbf{r}, t) = \sum_\lambda q_\lambda(t) \frac{D_\lambda}{c} \mathbf{v}_\lambda(\mathbf{r}) \]

substituting the operator \( \hat{q}_\lambda \equiv \sqrt{\frac{\hbar}{2}} (\hat{a}_\lambda + \hat{a}_\lambda^\dagger) \) for \( q_\lambda \)

we obtain

\[ \hat{B}(\mathbf{r}, t) = \sum_\lambda (\hat{a}_\lambda + \hat{a}_\lambda^\dagger) \sqrt{\frac{\hbar \omega_\lambda \mu_o}{2}} \mathbf{v}_\lambda(\mathbf{r}) \]