Overhead Slides for

Chapter 2

of

Fundamentals of Atmospheric Modeling

by

Mark Z. Jacobson

Department of
Civil & Environmental Engineering
Stanford University
Stanford, CA 94305-4020

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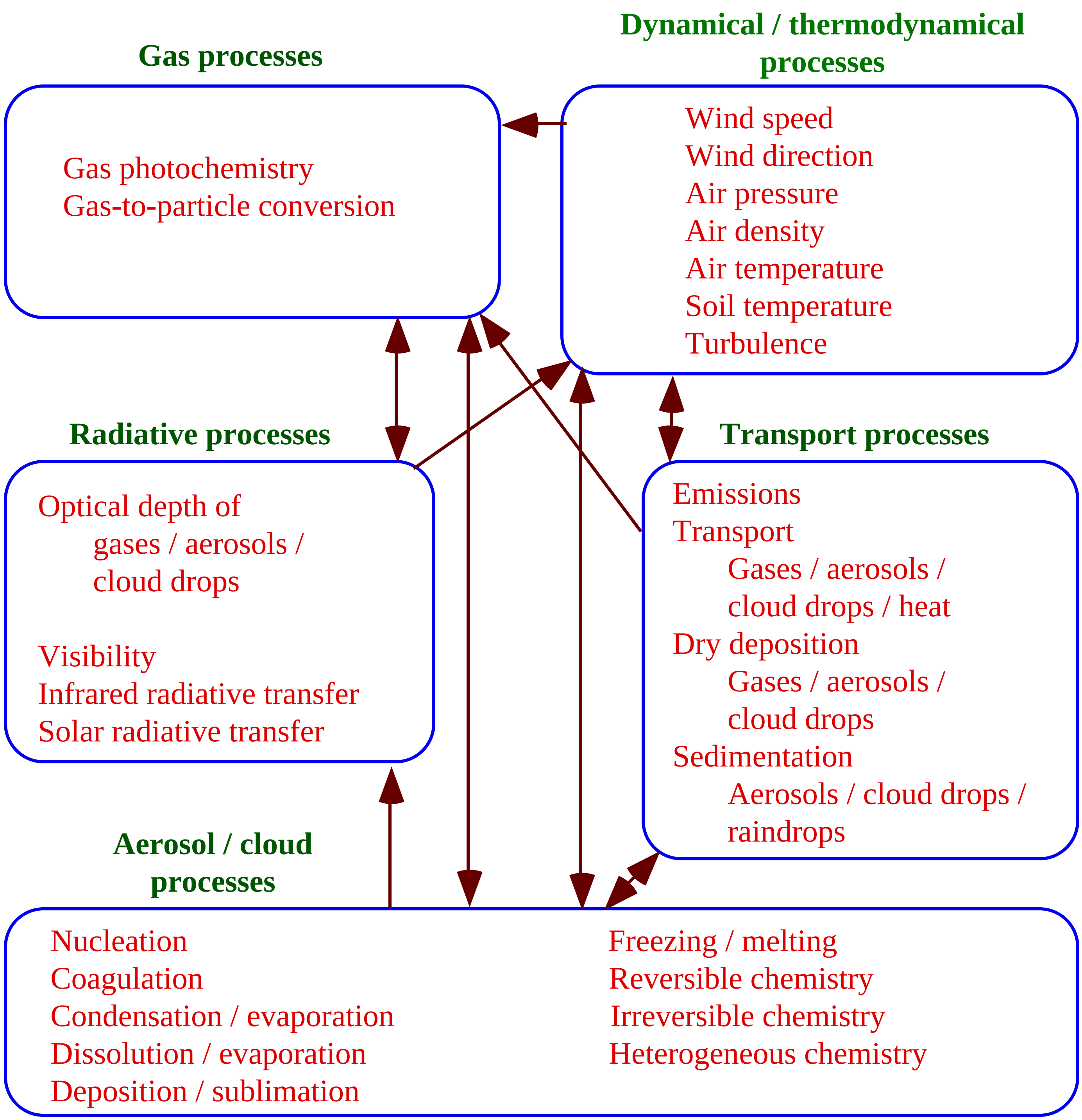
Scales of Motion

Table 1.1.

Scale Name	Scale Dimension	Examples
Molecular scale	« 2 mm	Molecular diffusion Molecular viscosity
Microscale	2 mm - 2 km	Eddies Small plumes Car exhaust Cumulus clouds
Mesoscale	2 - 2,000 km	Gravity waves Thunderstorms Tornados Cloud clusters Local winds Urban air pollution
Synoptic scale	500 - 10,000 km	High / low pressure systems Weather fronts Tropical storms Hurricanes Antarctic ozone hole
Planetary scale	> 10,000 km	Global wind systems Rossby (planetary) waves Stratospheric ozone loss Global warming

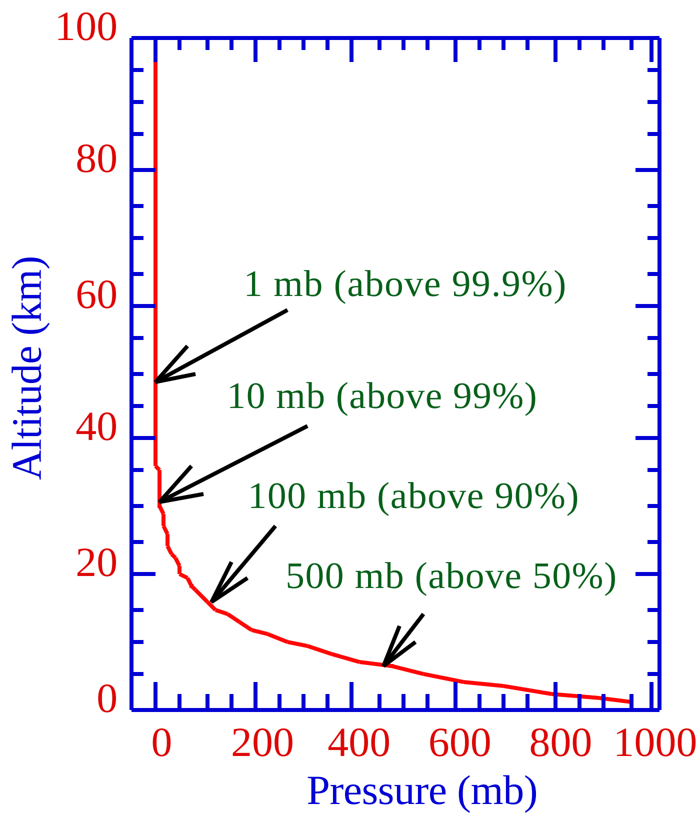
Processes in an Atmospheric Model

Figure 1.1



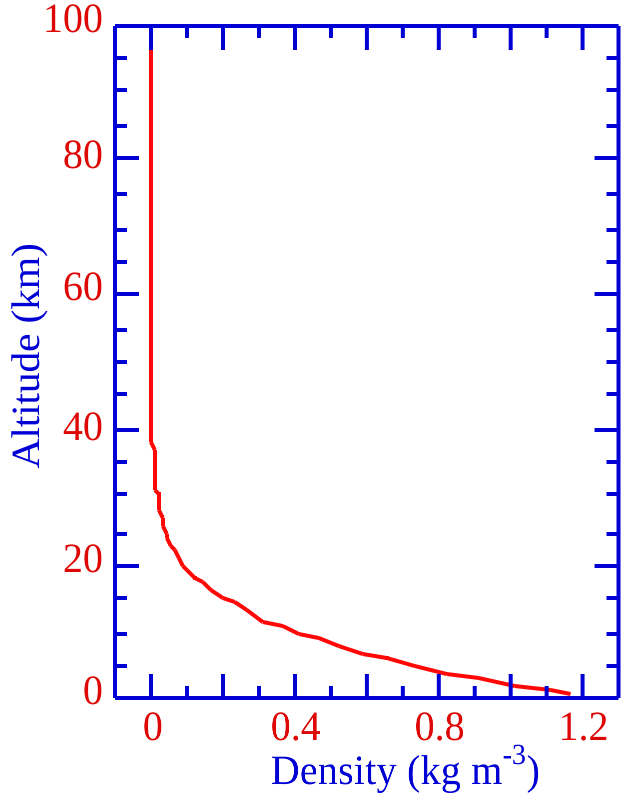
Pressure Versus Altitude

Figure 2.1 a.



Density Versus Altitude

Figure 2.1 b.



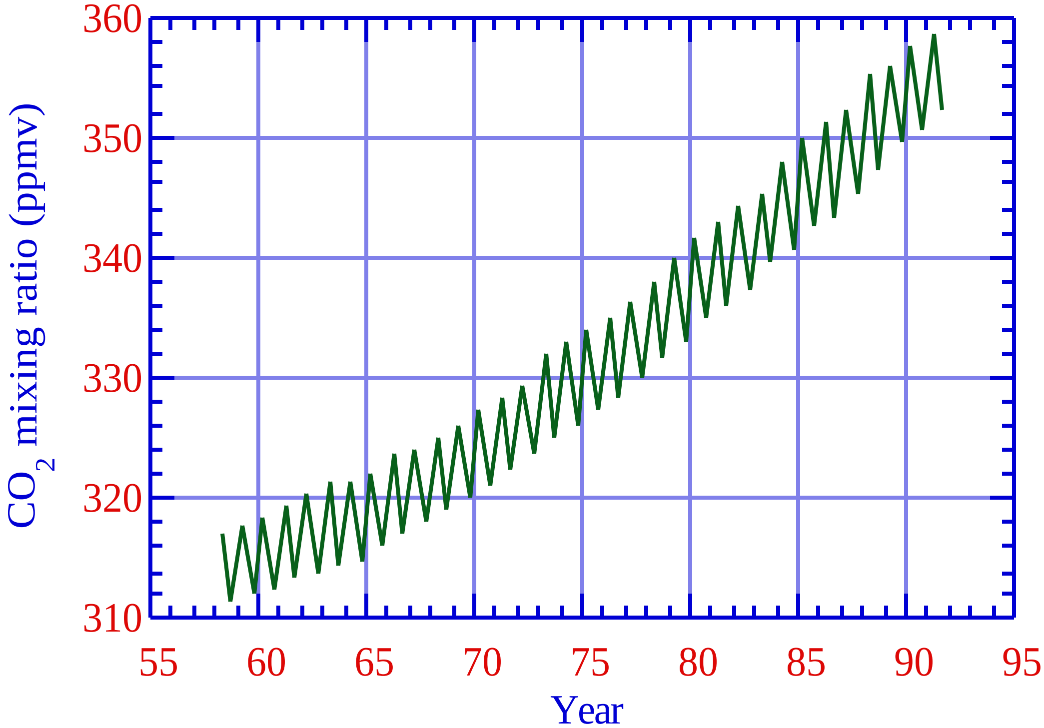
Composition of the Lower Atmosphere

Table 2.1.

Gas	Volume Mixing Ratio (percent)	(ppmv)
Fixed Gases		
Nitrogen (N_2)	78.08	780,000
Oxygen (O_2)	20.95	209,500
Argon (Ar)	0.93	9,300
Neon (Ne)	0.0015	15
Helium (He)	0.0005	5
Krypton (Kr)	0.0001	1
Xenon (Xe)	0.000005	0.05
Variable Gases		
Water vapor (H_2O)	0.00001-4.0	0.1-40,000
Carbon dioxide (CO_2)	0.0360	360
Methane (CH_4)	0.00017	1.7
Ozone (O_3)	0.000003-0.001	0.03-10

Fluctuations in Atmospheric CO_2

Figure 2.2.



Specific Heat and Thermal Conductivity

Specific heat

Energy required to increase the temperature of 1 g of a substance 1 ^{o}C

Thermal conductivity

Rate of conduction of energy through a medium

Thermal conductivity of dry air (J m^{-1} s^{-1} K^{-1})

$$\kappa_d \approx 0.023807 + 7.1128 \times 10^{-5}(T - 273.15) \qquad (2.3)$$

Table 2.2.

Substance	Specific Heat (J kg^{-1} K^{-1})	Thermal Conductivity at 298 K. (J m^{-1} s^{-1} K^{-1})
Dry air at constant pressure	1004.67	0.0256
Liquid water	4185.5	0.6
Clay	1360	0.920
Dry sand	827	0.298

Conductive Heat Flux Equation

$$H_c = -\kappa_d \frac{\Delta T}{\Delta z}$$ (J m^{-2} s^{-1})

Example 2.1.

Near the surface (T = 298 K, κ_d = 0.0256 J m^{-1} s^{-1} K^{-1})

	ΔT	= 12 K
	Δz	= 1 mm
---->	$H_{f,c}$	= 307 W m^{-2}

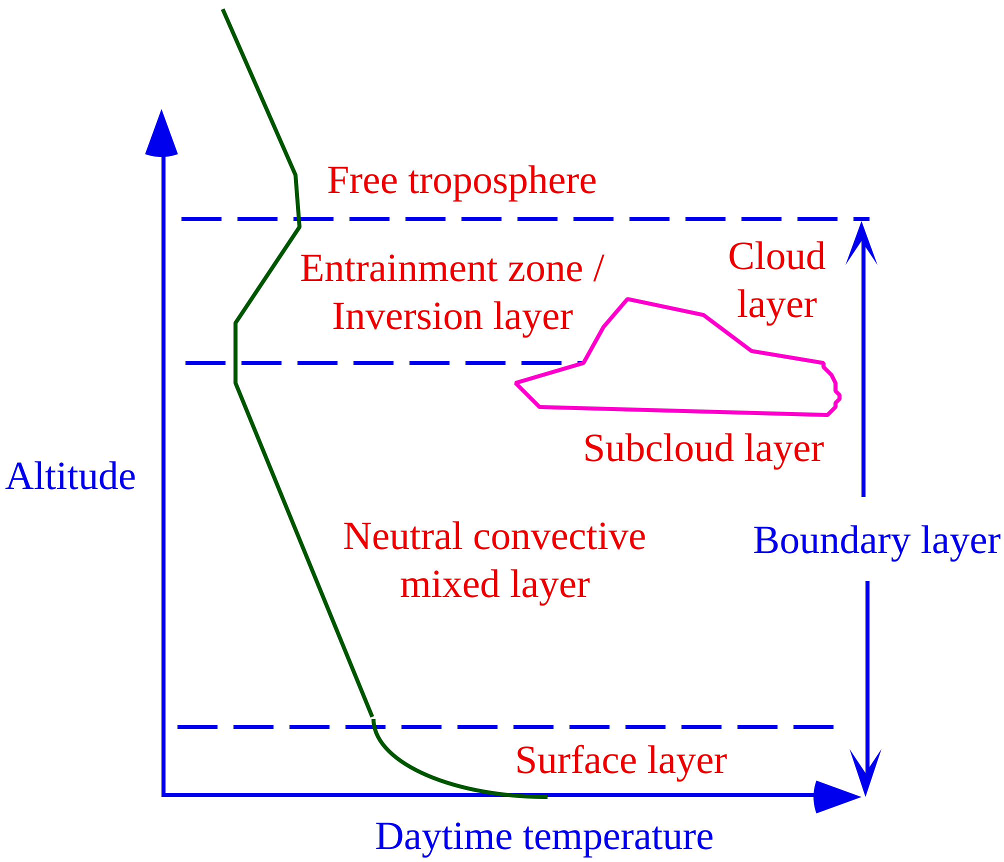
Free troposphere (T = 273 K, κ_d = 0.0238 J m^{-1} s^{-1} K^{-1})

	ΔT	= -6.5 K
	Δz	= 1 km
---->	$H_{f,c}$	= 1.5 x 10^{-4} W m^{-2}

Consequently, air conductivity is an effective energy transfer process only at the immediate ground surface.

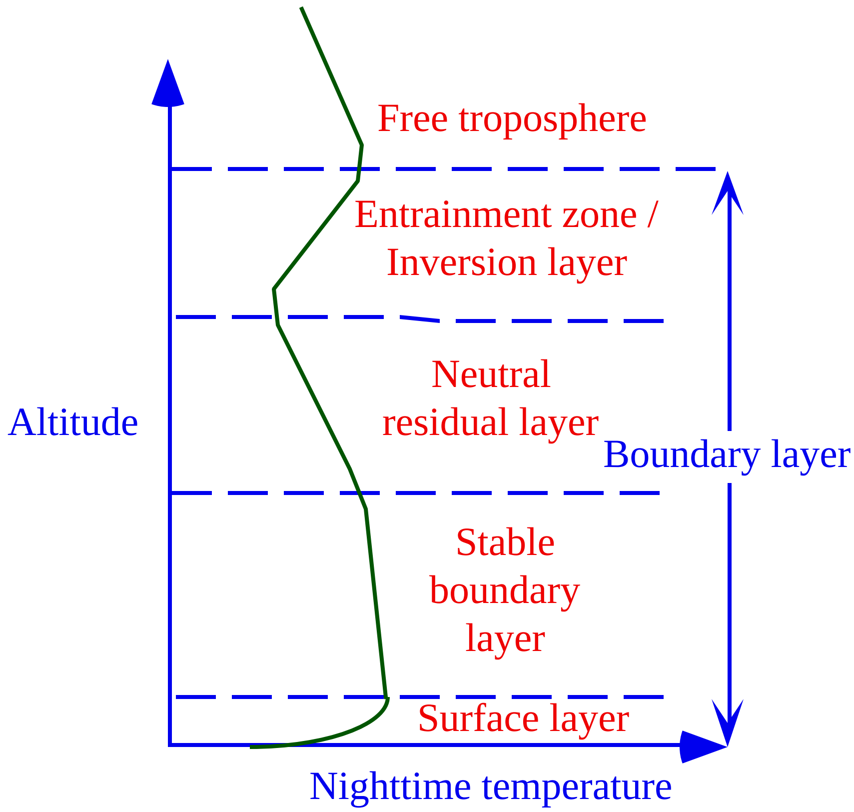
Daytime Boundary Layer

Figure 2.3 a.



Nighttime Boundary Layer

Figure 2.3 b.

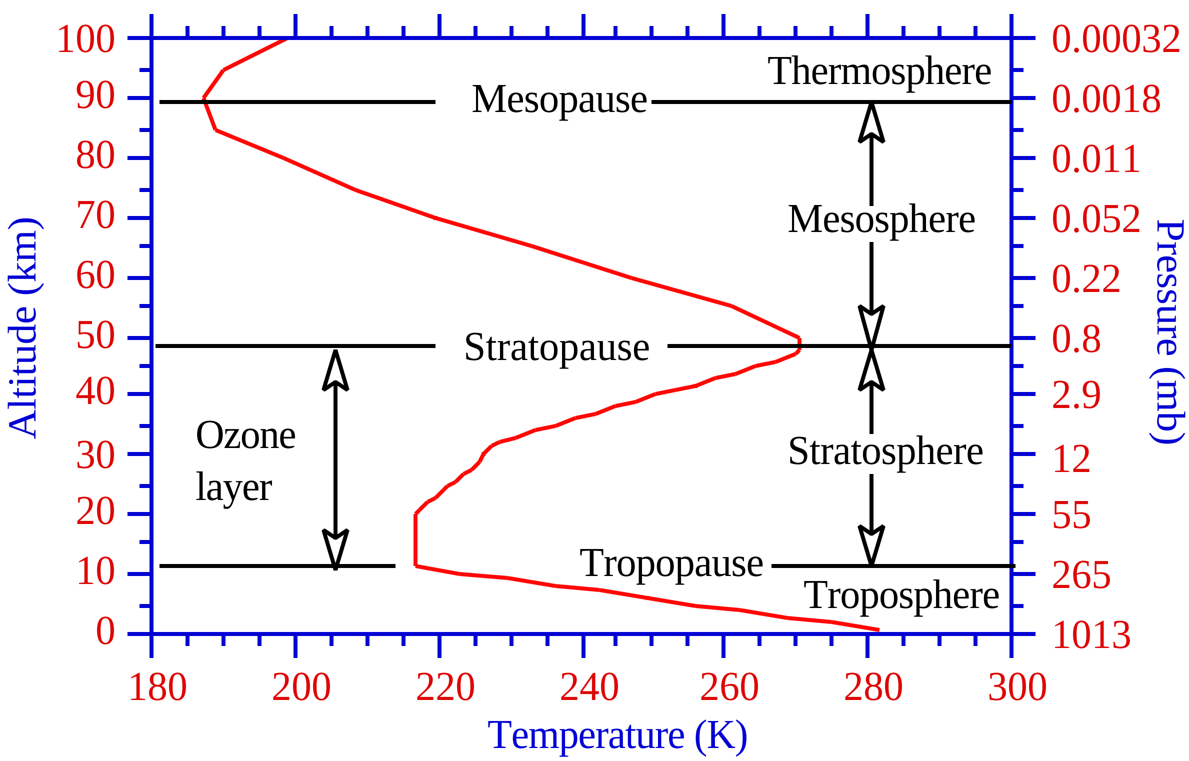


Temperature Structure of the Lower Atmosphere

Temperature

$$\frac{4}{\pi} k_B T = \frac{1}{2} \bar{M} \bar{v}_a^2 \qquad (2.2)$$

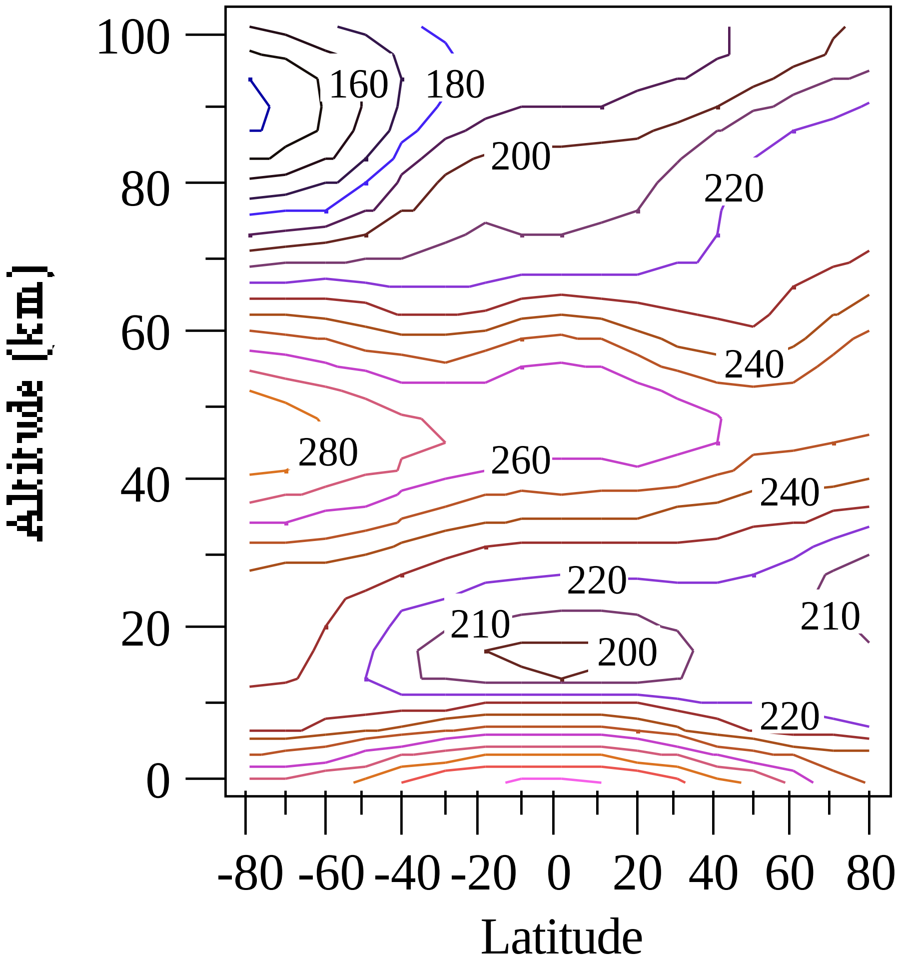
Figure 2.4



Zonally-/Monthly-Averaged Temperatures

Figure 2.5 a

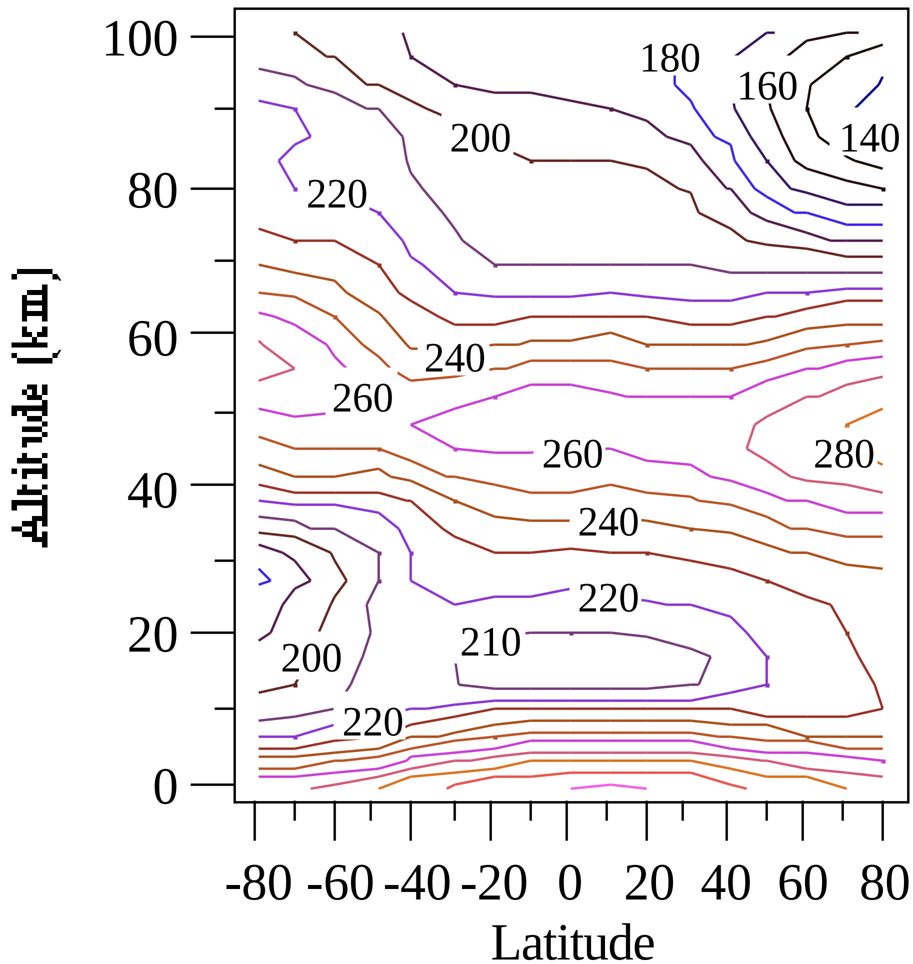
January



Zonally-/Monthly-Averaged Temperatures

Figure 2.5 b

July



Ozone Production / Destruction in the Stratosphere

Natural ozone production

$$O_2 + h\nu \longrightarrow \cdot\dot{O}(^1D) + \dot{O}\cdot \qquad \lambda < 175 \text{ nm} \qquad (2.4)$$

$$O_2 + h\nu \longrightarrow \dot{O}\cdot + \dot{O}\cdot \qquad 175 < \lambda < 245 \text{ nm} \qquad (2.5)$$

$$\cdot\dot{O}(^1D) + M \longrightarrow \cdot\dot{O} + M \qquad (2.6)$$

$$\dot{O}\cdot + O_2 + M \longrightarrow O_3 + M \qquad (2.7)$$

Natural ozone destruction

$$O_3 + h\nu \longrightarrow O_2 + \cdot\dot{O}(^1D) \qquad \lambda < 310 \text{ nm} \qquad (2.8)$$

$$O_3 + h\nu \longrightarrow O_2 + \dot{O}\cdot \qquad \lambda > 310 \text{ nm} \qquad (2.9)$$

$$\dot{O}\cdot + O_3 \longrightarrow 2O_2 \qquad (2.10)$$

Equation of State

Boyle's Law

$$p \quad \frac{1}{V}$$ at constant temperature (2.12)

Charles' Law

$$V \quad T$$ at constant pressure (2.13)

Avogadro's Law

$$V \quad n$$ at constant pressure and temperature (2.14)

Ideal gas law (simplified equation of state)

$$p = \frac{nR^* T}{V} = \frac{nA}{V}\left(\frac{R^*}{A}\right)T = Nk_B T \qquad (2.15)$$

Equation of State

$$p = \frac{nR^* T}{V} = \frac{nA}{V}\left(\frac{R^*}{A}\right)T = Nk_B T \qquad (2.15)$$

Example 2.2.

Surface

	p	= 1013 mb
	T	= 288 K
	k_B	= 1.3807 x 10^{-19} cm^3 mb K^{-1}
---->	N	= 2.55 x 10^{19} molec. cm^{-3}

At 48 km altitude

	p	= 1 mb
	T	= 270 K
---->	N	= 2.68 x 10^{16} molec. cm^{-3}

Dalton's Law of Partial Pressure

Total atmospheric pressure equals the sum of the partial pressures of all the individual gases in the atmosphere.

Total atmospheric pressure (mb)

$$p_a = \sum_q p_q = k_B T \sum_q N_q = N_a k_B T \qquad (2.17)$$

Partial pressures of individual gas (mb)

$$p_q = N_q k_B T \qquad (2.16)$$

Dry and Moist Air

Total air pressure (mb)

$$p_a = p_d + p_v$$

Number concentration air molecules (molec. cm^{-3})

$$N_a = N_d + N_v$$

Equation of State for Dry Air

$$p_d = \frac{n_d R^* T}{V} = \frac{n_d m_d}{V}\left(\frac{R^*}{m_d}\right)T = \rho_d R' T = \frac{n_d A}{V}\left(\frac{R^*}{A}\right)T = N_d k_B T \quad (2.18)$$

Dry air mass density (g cm^{-3})

$$\rho_d = \frac{n_d m_d}{V} \quad (2.19)$$

Dry air number concentration (molec. cm^{-3})

$$N_d = \frac{n_d A}{V} \quad (2.19)$$

Dry air gas constant (Appendix *A*)

$$R' = \frac{R^*}{m_d} \quad (2.19)$$

Equation of State Examples

Examples 2.3 and 2.4

Dry air, at sea level

$$p_d = \frac{n_d R^* T}{V} = \frac{n_d m_d}{V}\left(\frac{R^*}{m_d}\right)T = \rho_d R' T \qquad (2.18)$$

	p_d	= 1013 mb
	T	= 288 K
	R'	= 2.8704 m^3 mb kg^{-1} K^{-1}
---->	ρ_d	= 1.23 kg m^{-3}

Water vapor, at sea level

$$p_v = \frac{n_v R^* T}{V} = \frac{n_v m_v}{V}\left(\frac{R^*}{m_v}\right)T = \rho_v R_v T \qquad (2.20)$$

	p_v	= 10 mb
	T	= 298 K
	R_v	= 4.6189 m^3 mb kg^{-1} K^{-1}
---->	ρ_v	= 7.25 x 10^{-3} kg m^{-3}

Equation of State for Water Vapor

$$p_v = \frac{n_v R^* T}{V} = \frac{n_v m_v}{V}\left(\frac{R^*}{m_v}\right)T = \rho_v R_v T = \frac{n_v A}{V}\left(\frac{R^*}{A}\right)T = N_v k_B T$$

(2.20)

Water-vapor mass density (kg m^{-3})

$$\rho_v = \frac{n_v m_v}{V} \qquad (2.21)$$

Water-vapor number concentration (molec. cm^{-3})

$$N_v = \frac{n_v A}{V} \qquad (2.21)$$

Gas constant for water vapor

$$R_v = \frac{R^*}{m_v} \qquad (2.21)$$

Volume and Mass Mixing Ratios

Volume mixing ratio of gas *j* (molec. gas per molec. dry air)

$$\chi_q = \frac{N_q}{N_d} = \frac{p_q}{p_d} = \frac{n_q}{n_d} \quad (2.24)$$

Mass mixing ratio of gas *q* (g of gas per g of dry air)

$$\omega_q = \frac{\rho_q}{\rho_d} = \frac{m_q N_q}{m_d N_d} = \frac{m_q p_q}{m_d p_d} = \frac{m_q n_q}{m_d n_d} = \frac{m_q}{m_d}\chi_q \quad (2.25)$$

Example 2.5.

Ozone

	χ	= 0.10 ppmv
	m_q	= 48.0 g mole^{-1}
---->	ω	= 0.17 ppmm
	T	= 288 K
	p_d	= 1013 mb
---->	N_d	= 2.55 x 10^{19} molec. cm^{-3}
---->	N_q	= 2.55 x 10^{12} molec. cm^{-3}
---->	p_q	= 0.000101 mb

Mass Mixing Ratio of Water Vapor

Equation of state for water vapor

$$p_v = \rho_v R_v T = \rho_v \left(\frac{R_v}{R'} \right) R' T = \frac{\rho_v R' T}{\varepsilon} \qquad (2.22)$$

$$\varepsilon = \frac{R'}{R_v} = \frac{R^*}{m_d} \left(\frac{m_v}{R^*} \right) = \frac{m_v}{m_d} = 0.622 \qquad (2.23)$$

Mass mixing ratio of water vapor (kg-vapor kg^{-1}-dry air)

$$\omega_v = \frac{\rho_v}{\rho_d} = \frac{m_v p_v}{m_d p_d} = \varepsilon \frac{p_v}{p_d} = \frac{\varepsilon p_v}{p_a - p_v} = \varepsilon \chi_v \qquad (2.26)$$

Example 2.6.

	p_v	= 10 mb
	p_a	= 1010 mb
---->	ω_v	= 0.00622 kg kg^{-1} = 0.622%.

Specific Humidity

= Moist-air mass mixing ratio (kg-vapor kg^{-1}-moist air)

$$q_v = \frac{\rho_v}{\rho_a} = \frac{\rho_v}{\rho_d + \rho_v} = \frac{\dfrac{p_v}{R_v T}}{\dfrac{p_d}{R' T} + \dfrac{p_v}{R_v T}} = \frac{\dfrac{R'}{R_v} p_v}{p_d + \dfrac{R'}{R_v} p_v} = \frac{\varepsilon p_v}{p_d + \varepsilon p_v}$$

(2.27)

Example 2.7.

	p_v	= 10 mb
	p_a	= 1010 mb
---->	p_d	= 1000 mb
---->	q_v	= 0.00618 kg kg^{-1} = 0.618%.

Equation of State for Moist Air

Total air pressure

$$p_a = p_d + p_v = \rho_d R'T + \rho_v R_v T = \rho_a R'T \frac{\rho_d + \rho_v R_v / R'}{\rho_a}$$

(2.28)

Gather terms, multiply numerator / denominator by density

$$p_a = \rho_a R'T \frac{\rho_d + \rho_v/\varepsilon}{\rho_d + \rho_v} = \rho_a R'T \frac{1 + \rho_v/(\rho_d \varepsilon)}{1 + \rho_v/\rho_d} = \rho_a R'T \frac{1 + \omega_v/\varepsilon}{1 + \omega_v}$$

(2.29)

Equation of state for moist (or total) air

$$\boxed{p_a = \rho_a R_m T = \rho_a R' T_v} \qquad (2.30)$$

Gas constant for moist air (2.31)

$$R_m = R' \frac{1 + \omega_v/\varepsilon}{1 + \omega_v} = R'\left(1 + \frac{1-\varepsilon}{\varepsilon} q_v\right) = R'(1 + 0.608 q_v)$$

Virtual temperature (2.32)

Temperature of dry air having the same density as a sample of moist air at the same pressure as the moist air.

$$T_v = T \frac{R_m}{R'} = T \frac{1 + \omega_v/\varepsilon}{1 + \omega_v} = T\left(1 + \frac{1-\varepsilon}{\varepsilon} q_v\right) = T(1 + 0.608 q_v)$$

Molecular Weight of Moist Air

Gas constant for moist air

$$R_m = R' \frac{1 + \omega_v / \varepsilon}{1 + \omega_v} = R' \left(1 + \frac{1 - \varepsilon}{\varepsilon} q_v \right) = R' \left(1 + 0.608 q_v \right)$$

--> Molecular weight of moist air (> than that of dry air)

$$m_a = \frac{m_d}{1 + 0.608 q_v} \qquad (2.33)$$

Moist Air Example

Example 2.8.

p_d = 1013 mb

p_v = 10 mb

T = 298 K

$$q_v = \frac{\varepsilon p_v}{p_d + \varepsilon p_v} = 0.0061 \text{ kg kg}^{-1}$$

$$m_a = \frac{m_d}{1 + 0.608 q_v} = 28.86 \text{ g mole}^{-1}$$

$$R_m = R' \left(1 + 0.608 q_v \right) = 2.8811 \text{ mb m}^3 \text{ kg}^{-1} \text{ K}^{-1}$$

$$T_v = T \left(1 + 0.608 q_v \right) = 299.1 \text{ K}$$

$$\rho_a = \frac{p_a}{R_m T} = 1.19 \text{ kg m}^{-3}$$

Hydrostatic Equation

Vertical pressure gradient is exactly balanced by the downward force of gravity.

Hydrostatic equation

$$\frac{\partial p_a}{\partial z} = -\rho_a g \qquad (2.34)$$

Pressure at a given altitude

$$\frac{\partial p_a}{\partial z} \approx \frac{p_{a,1} - p_{a,0}}{z_1 - z_0} = -\rho_{a,0} g \qquad (2.35)$$

Example 2.9.

Sea level

$p_{a,0}$	= 1013 mb
$\rho_{a,0}$	= 1.23 kg m^{-3}

100 m altitude

----> $p_{a,100\text{m}}$ = 1000.2 mb

Pressure Altimeter

Combine hydrostatic equation with equation of state

$$\frac{\partial p_d}{\partial z} = -\frac{p_d}{R'T} g \qquad (2.36)$$

Assume temperature decrease with altitude is constant

$$T = T_{a,s} - \Gamma_s z$$

Free-tropospheric lapse rate (K km^{-1})

$$\Gamma_s = -\frac{\partial T}{\partial z}$$ assume ≈ 6.5 K km^{-1}

Substitute lapse rate, temperature profile into (2.36) and integrate

$$\ln\left(\frac{p_d}{p_{d,s}}\right) = \frac{g}{\Gamma_s R'} \ln\left(\frac{T_{a,s} - \Gamma_s z}{T_{a,s}}\right) \qquad (2.37)$$

Rearrange

$$z = \frac{T_{a,s}}{\Gamma_s}\left[1 - \left(\frac{p_d}{p_{d,s}}\right)^{\frac{\Gamma_s R'}{g}}\right] \qquad (2.38)$$

Example 2.10

p_d = 850 mb = pressure altimeter reading

----> z = 1.45 km

Scale Height

Height above a reference height at which pressure decreases to 1/e of its value at the reference height.

Density of air from equation of state for moist air

$$\rho_a = \frac{p_a}{R' T_v} = \frac{m_d}{R^*}\frac{p_a}{T_v} = \frac{p_a}{T_v}\left(\frac{A}{R^*}\right)\frac{m_d}{A} \approx \frac{p_a}{T_v}\left(\frac{1}{k_B}\right)\overline{M} = \frac{p_a \overline{M}}{k_B T_v} \qquad (2.39)$$

Mass of one air molecule (4.8096 x 10^{-23} g)

$$\overline{M} \approx \frac{m_d}{A}$$

Combine (2.39) with hydrostatic equation

$$\frac{\mathrm{d}p_a}{p_a} = -\frac{\overline{M} g}{k_B T_v}\mathrm{d}z = -\frac{\mathrm{d}z}{H} \qquad (2.40)$$

Scale height of the atmosphere

$$H = \frac{k_B T_v}{\overline{M} g} \qquad (2.41)$$

Scale Height Equation

Integrate (2.40) at constant temperature

$$p_a = p_{a,ref}\, e^{-\left(z - z_{ref}\right)/H} \qquad (2.42)$$

Example 2.11.

	T	= 298 K
---->	H	= 8.72 km
	$p_{a,ref}$	= 1000 mb
	z_{ref}	= 0 km
	z	= 1 km
---->	p_a	= 891.7 mb

Energy

Capacity of a physical system to do work on matter

Kinetic energy

Energy within a body due to its motion.

Potential energy

Energy of matter that arises due to its position, rather than its motion.

Gravitational potential energy

Potential energy obtained when an object is raised vertically.

Internal energy

Kinetic and/or potential energy of atoms or molecules within an object.

Work

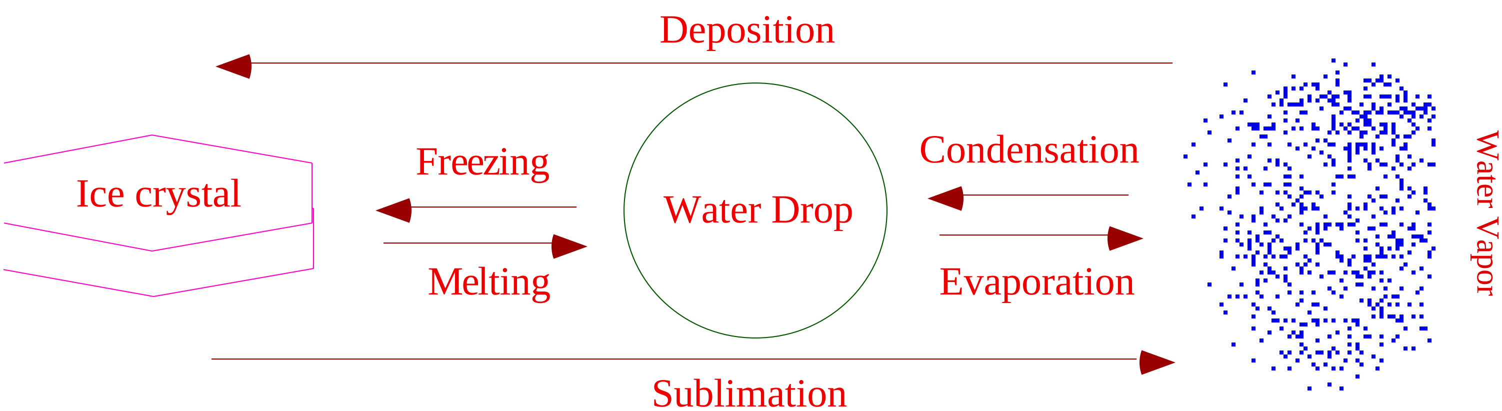
Energy added to a body by the application of a force that moves the body in the direction of the force.

Radiation

Energy transferred by electromagnetic waves.

Phase Changes of Water

Figure 2.6.



Latent Heat of Evaporation

Latent heat of evaporation (J kg^{-1})

$$\frac{dL_e}{dT} = c_{p,V} - c_W \qquad (2.43)$$

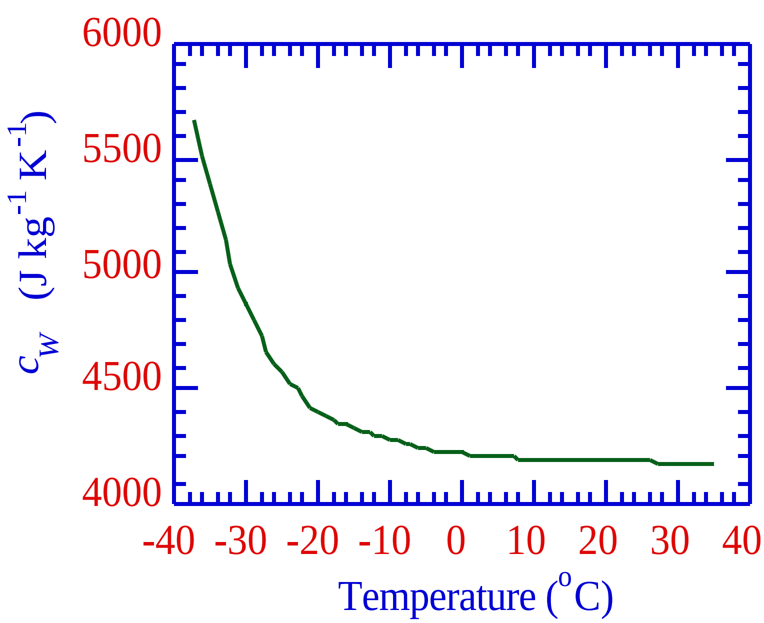
$$L_e = L_{e,0} - \left(c_W - c_{p,V}\right)\left(T - T_0\right) \qquad (2.47)$$

$$L_e \approx 2.501 \times 10^6 - 2370 T_c \qquad (2.48)$$

Example 2.12.

	T	= 273.15 K
---->	L_e	= 2.5 x 10^6 J kg^{-1} (about 600 cal g^{-1})
	T	= 373.15 K
---->	L_e	= 2.264 x 10^6 J kg^{-1} (about 540 cal g^{-1})

Variation of c_W with temperature (Figure 2.7)



Latent Heats of Melting, Sublimation

Latent heat of melting (J kg^{-1})

$$\frac{dL_m}{dT} = c_W - c_I \qquad (2.44)$$

$$L_m \approx 3.3358 \times 10^5 + T_c\left(2030 - 10.46T_c\right) \qquad (2.49)$$

Example 2.13.

----> T = 273.15 K
L_f = 3.34 x 10^5 J gk^{-1} (about 80 cal g^{-1})

----> T = 263.15 K
L_f = 3.12 x 10^5 J kg^{-1} (about 74.6 cal g^{-1})

Supercooled water

Water that exists as a liquid when T < 273.15 K

Latent heat of sublimation (J kg^{-1})

$$\frac{dL_s}{dT} = c_{p,V} - c_I \qquad (2.44)$$

$$L_s = L_e + L_m \approx 2.83458 \times 10^6 - T_c\left(340 + 10.46T_c\right) \qquad (2.50)$$

Clausius-Clapeyron Equation

Clausius-Clapeyron equation

$$\frac{dp_{v,s}}{dT} = \frac{\rho_{v,s}}{T} L_e \qquad (2.51)$$

Density of water vapor over the particle surface (kg m^{-3})

$$\rho_{v,s} = \frac{p_{v,s}}{R_v T}$$

Combine Clausius-Clapeyron equation and density

$$\frac{dp_{v,s}}{dT} = \frac{L_e p_{v,s}}{R_v T^2} \qquad (2.52)$$

Substitute latent heat of evaporation

$$\frac{dp_{v,s}}{p_{v,s}} = \frac{1}{R_v}\left(\frac{A_h}{T^2} - \frac{B_h}{T}\right)dT \qquad (2.53)$$

Integrate

$$p_{v,s} = p_{v,s,0} \exp\left[\frac{A_h}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right) + \frac{B_h}{R_v}\ln\left(\frac{T_0}{T}\right)\right] \qquad (2.54)$$

A_h = 3.15283 x 10^6 J kg^{-1}

B_h = 2390 J kg^{-1} K^{-1}

$p_{s,0}$ = 6.112 mb at T_0 = 273.15 K

Saturation Vapor Pressure over Liquid Water

Derived saturation vapor pressure

$$p_{v,s} = 6.112\exp\left[6816\left(\frac{1}{273.15} - \frac{1}{T}\right) + 5.1309\ln\left(\frac{273.15}{T}\right)\right]$$

(2.55)

Example 2.14.

	T	= 253.15 K (253.15 K)
---->	$p_{v,s}$	= 1.26 mb

	T	= 298.15 K (98.15 K)
---->	$p_{v,s}$	= 31.6 mb

Alternative parameterization

$$p_{v,s} = 6.112\exp\left(\frac{17.67T_c}{T_c + 243.5}\right) \quad (2.56)$$

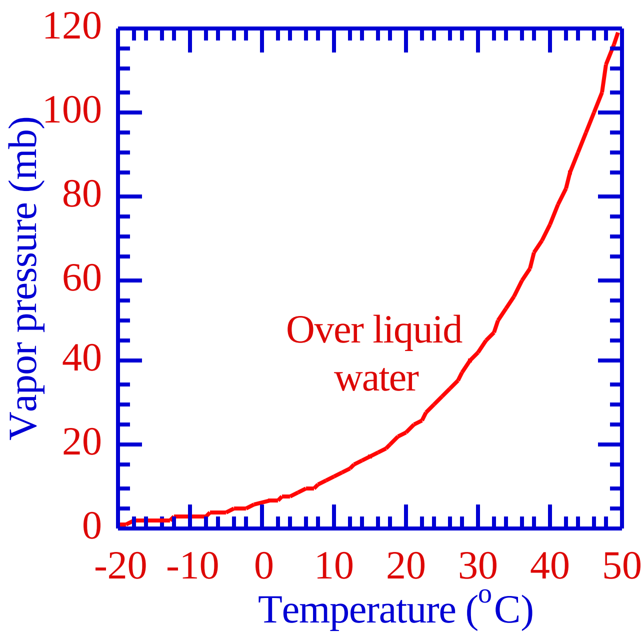
Example 2.15.

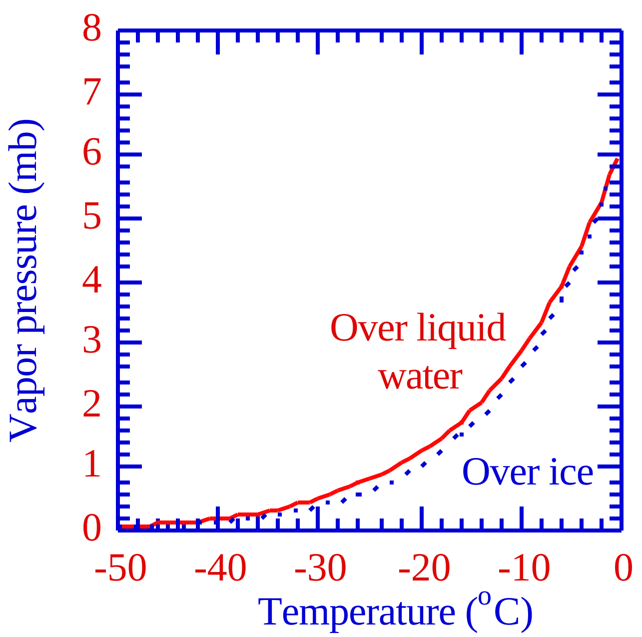
	T_c	= -20 ºC (253.15 K)
---->	$p_{v,s}$	= 1.26 mb

	T_c	= 25 ºC (298.15 K)
---->	$p_{v,s}$	= 31.67 mb

Saturation Vapor Pressure Over Liquid Water / Ice

Figure 2.8 a and b





Saturation Vapor Pressure Over Ice

Clausius-Clapeyron equation

$$\frac{dp_{v,I}}{dT} = \frac{L_s p_{v,I}}{R_v T^2} \qquad (2.57)$$

Substitute latent heat of sublimation and integrate

$$p_{v,I} = p_{v,I,0} \exp\left[\frac{A_I}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right) + \frac{B_I}{R_v}\ln\left(\frac{T_0}{T}\right) + \frac{C_I}{R_v}\left(T_0 - T\right)\right] \qquad (2.58)$$

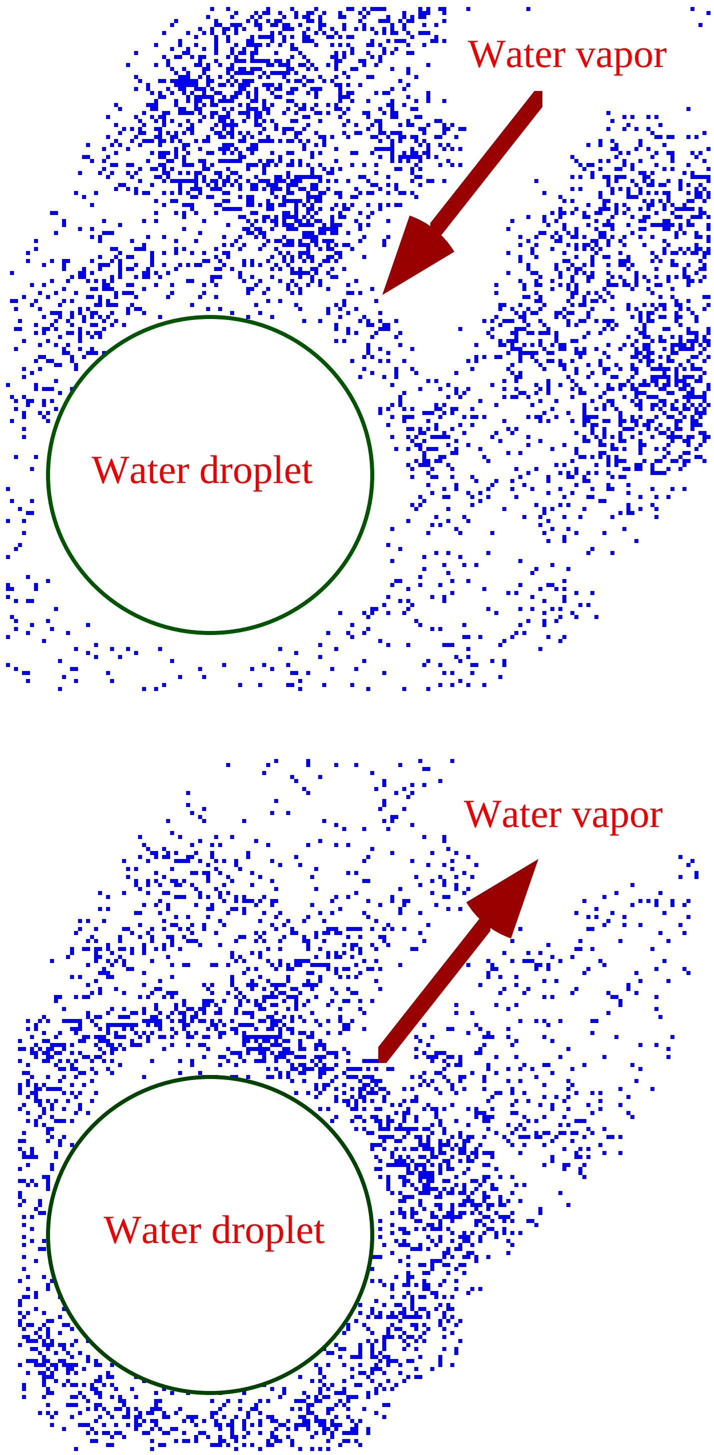
A_I	$= 2.1517 \times 10^6$ J kg^{-1}
B_I	= -5353 J kg^{-1} K^{-1}
C_I	= 10.46 J kg^{-1} K^{-2}
$p_{I,0}$	= 6.112 mb at T_0 = 273.15 K
T	≤ 273.15 K

Example 2.16.

	T	= 253.15 K (-20 ^{o}C)
---->	$p_{v,I}$	= 1.04 mb
---->	$p_{v,s}$	= 1.26 mb

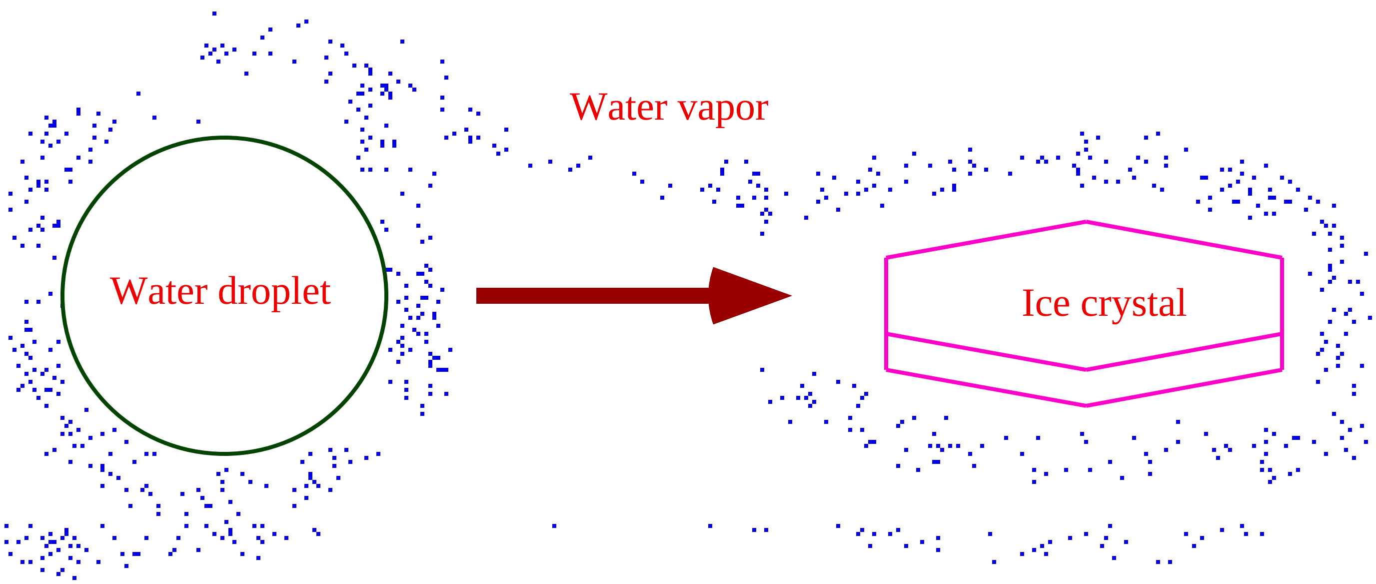
Condensation / Evaporation

Figure 2.9 a and b.



Bergeron Process

Figure 2.10



Relative Humidity

Relative humidity

$$f_r = 100\% \times \frac{\omega_v}{\omega_{v,s}} = 100\% \times \frac{p_v\left(p_a - p_{v,s}\right)}{p_{v,s}\left(p_a - p_v\right)} \approx 100\% \times \frac{p_v}{p_{v,s}} \qquad (2.60)$$

Saturation mass mixing ratio of water vapor

$$\omega_{v,s} = \frac{\varepsilon p_{v,s}}{p_a - p_{v,s}} \approx \frac{\varepsilon p_{v,s}}{p_d} \qquad (2.61)$$

Example 2.17

	T	= 288 K
	p_v	= 12 mb
---->	$p_{v,s}$	= 17.04 mb
---->	f_r	= 100% x 12 mb / 17.04 mb = 70.4%

Dew Point

Dew point

$$T_D = \frac{4880.357 - 29.66 \ln p_v}{19.48 - \ln p_v} = \frac{4880.357 - 29.66 \ln(\omega_v p_d / \varepsilon)}{19.48 - \ln(\omega_v p_d / \varepsilon)}$$

(2.62)

Ambient mass mixing ratio of water vapor

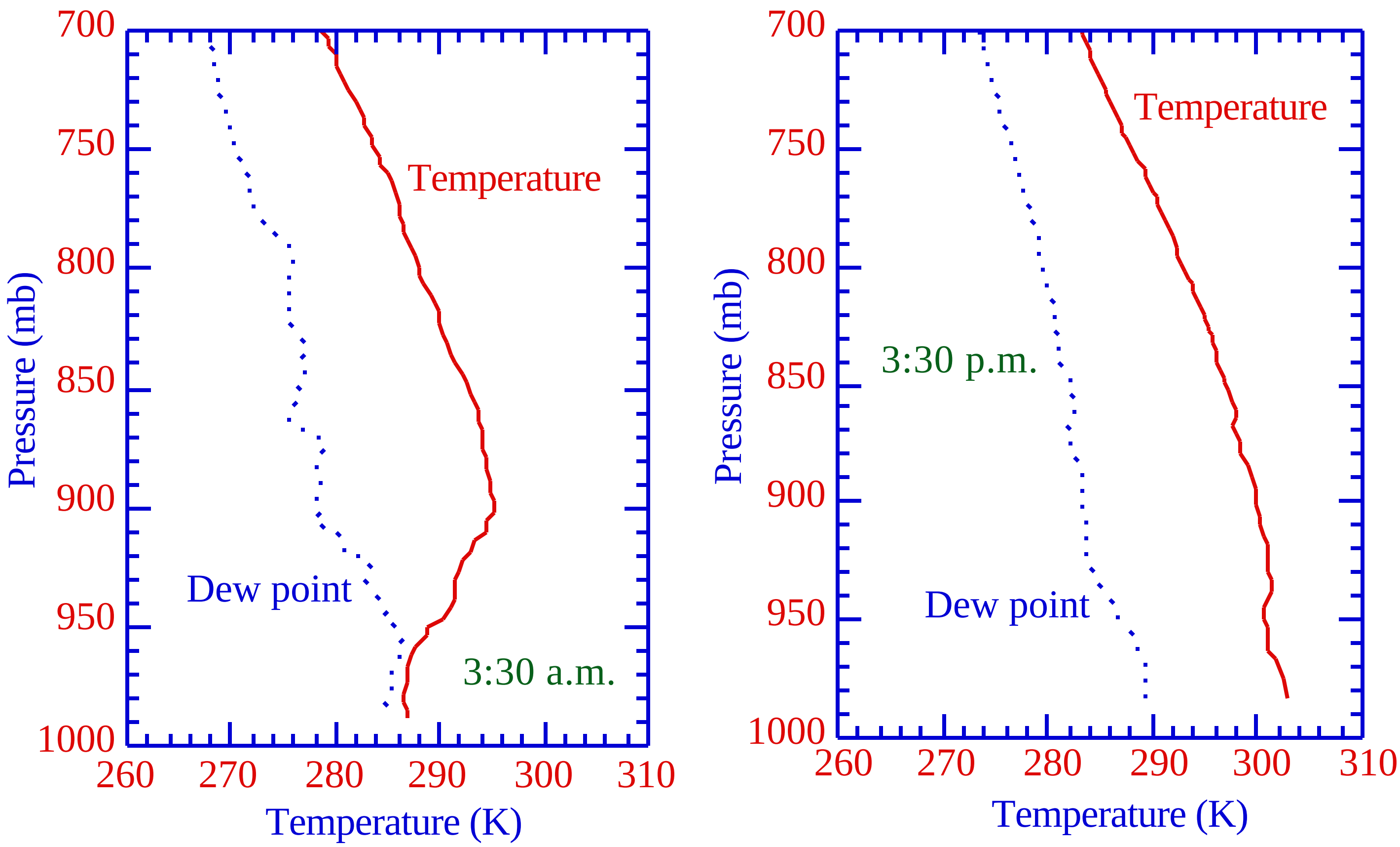
$$\omega_v = \frac{\varepsilon p_v}{p_d}$$

Example

	p_v	= 12 mb
---->	T_D	= 282.8 K

Morning and Afternoon Dew Point and Temperature Profiles at Riverside

Figure 2.11 a and b



First Law of Thermodynamics

$$dQ^* = dU^* + dW^* \quad (2.63)$$

dQ* = change in energy due to energy transfer (J)
dU* = change in internal energy of the air (J)
dW * = work done by (+) or on (-) the air (J)

In terms of energy per unit mass of air (J kg^{-1})

$$dQ = dU + dW \quad (2.65)$$

where

$$dQ = \frac{dQ^*}{M_a} \qquad dU = \frac{dU^*}{M_a} \qquad dW = \frac{dW^*}{M_a} \quad (2.64)$$

M = mass of a parcel of air (kg).

Work and Energy Transfer

Work done by air during adiabatic expansion ($dV > 0$)

$$dW^* = p_a \, dV$$

Work done per unit mass of air

$$dW = \frac{dW^*}{M_a} = \frac{p_a \, dV}{M_a} = p_a \, d\alpha_a . \qquad (2.66)$$

Specific volume of air ($cm^3 \, g^{-1}$)

$$\alpha_a = \frac{V}{M_a} = \frac{1}{\rho_a} \qquad (2.67)$$

Diabatic energy sources ($dQ > 0$)

- condensation
- deposition
- freezing
- solar heating
- infrared heating

Diabatic energy sinks ($dQ < 0$)

- evaporation
- sublimation
- melting
- infrared cooling

Internal Energy

Change in temperature of the gas multiplied by the energy required to change the temperature 1 K, without affecting the work done by or on the gas and without changing its volume.

$$dU = \left(\frac{\partial Q}{\partial T}\right)_{\alpha_a} dT = c_{v,m} \; dT \qquad (2.68)$$

Specific heat of moist air at constant volume (J kg^{-1} K^{-1})

Change in energy required to raise the temperature of 1 g of air 1 K at constant volume.

$$c_{v,m} = \left(\frac{\partial Q}{\partial T}\right)_{\alpha_a} = \frac{M_d c_{v,d} + M_v c_{v,V}}{M_d + M_v} = \frac{c_{v,d} + c_{v,V}\,\omega_v}{1 + \omega_v} = c_{v,d}\left(1 + 0.955 q_v\right)$$

(2.70)

derived from

$$\left(M_d + M_v\right) dQ = \left(M_d c_{v,d} + M_v c_{v,V}\right) dT \qquad (2.69)$$

Variations of First Law

First law of thermodynamics

$$dQ = c_{v,m}dT + p_a d\alpha_a \qquad (2.71)$$

Equation of state for moist air

$$p_a \alpha_a = R_m T$$

Differentiate

$$p_a d\alpha_a + \alpha_a dp_a = R_m dT \qquad (2.72)$$

Combine (2.72) with (2.70)

$$dQ = c_{p,m}dT - \alpha_a dp_a \qquad (2.73)$$

Specific heat of moist air at constant pressure (J kg^{-1} K^{-1})

Energy required to increase the temperature of one gram of air one degree Kelvin without affecting the pressure of air

$$c_{p,m} = \left(\frac{dQ}{dT}\right)_{p_a} = \frac{M_d c_{p,d} + M_v c_{p,V}}{M_d + M_v} = \frac{c_{p,d} + c_{p,V}\omega_v}{1+\omega_v} = c_{p,d}\left(1 + 0.856 q_v\right)$$

(2.74)

First Law in Terms of Virtual Temperature

Change in internal energy in terms of virtual temperature

$$dU = \left(\frac{\partial Q}{\partial T_v}\right)_{\alpha_a} dT_v = c_{v,d}\, dT_v$$

Change in work in terms of virtual temperature

$$dW = p_a\, d\alpha_a = R'\, dT_v - \alpha_a\, dp_a$$

Relationship between $c_{p,d}$ and $c_{v,d}$

$$c_{p,d} = c_{v,d} + R'$$

Substitute into dQ = dU + dW

$$dQ \approx c_{p,d}\, dT_v - \alpha_a\, dp_a \qquad (2.76)$$

Applications of First Law

Isobaric process ($dp_a = 0$)

$$dQ = c_{p,m} dT = \frac{c_{p,m}}{c_{v,m}} dU \quad (2.77)$$

Isothermal process ($dT = 0$)

$$dQ = -\alpha_a dp_a = p_a d\alpha_a = dW \quad (2.78)$$

Isochoric process ($d\alpha_a = 0$)

$$dQ = c_{v,m} dT = dU \quad (2.79)$$

Adiabatic process ($dQ = 0$)

$$c_{v,m} dT = -p_a d\alpha_a \quad (2.80)$$

$$c_{p,m} dT = \alpha_a dp_a \quad (2.81)$$

$$c_{p,d} dT_v = \alpha_a dp_a \quad (2.82)$$

Dry Adiabatic Lapse Rate

Rearrange (2.82)

$$dT_v = \left(\frac{\alpha_a}{c_{p,d}}\right) dp_a \,.$$

Differentiate with respect to altitude

--> Dry adiabatic lapse rate in terms of virtual temperature

$$\Gamma_d = -\left(\frac{\partial T_v}{\partial z}\right)_d = -\left(\frac{\alpha_a}{c_{p,d}}\right)\frac{\partial p_a}{\partial z} = \left(\frac{\alpha_a}{c_{p,d}}\right)\rho_a g = \frac{g}{c_{p,d}} = +9.8 \text{ K km}^{-1} \qquad (2.83)$$

Rearrange (2.81)

$$dT = \left(\frac{\alpha_a}{c_{p,m}}\right) dp_a \,.$$

Differentiate with respect to altitude

--> Dry adiabatic lapse rate in terms of temperature

$$\Gamma_d = -\left(\frac{\partial T}{\partial z}\right)_d = \frac{g}{c_{p,m}} = \frac{g}{c_{p,d}}\left(\frac{1+\omega_v}{1+c_{p,V}\,\omega_v / c_{p,d}}\right) \qquad (2.84)$$

Potential Temperature

Substitute $\alpha_a = R_m T / p_a$ into (2.81)

$$\frac{dT}{T} = \left(\frac{R_m}{c_{p,m}}\right)\frac{dp_a}{p_a} \quad \text{-->} \qquad (2.85)$$

Integrate

$$T = T_0\left(\frac{p_a}{p_{a,0}}\right)^{\frac{R_m}{c_{p,m}}} = T_0\left(\frac{p_a}{p_{a,0}}\right)^{\frac{R'(1+0.608q_v)}{c_{p,d}(1+0.859q_v)}} \approx T_0\left(\frac{p_a}{p_{a,0}}\right)^{\kappa(1-0.251q_v)}$$

(2.86)

Exponential term

$$\kappa = \frac{R'}{c_{p,d}} = \frac{c_{p,d} - c_{v,d}}{c_{p,d}} = 0.286 \qquad (2.87)$$

$p_{a,0}$ = 1000 mb --> T_0 = potential temperature of moist air ($\theta_{p,m}$)

$$\theta_{p,m} = T\left(\frac{1000 \text{ mb}}{p_a}\right)^{\kappa(1-0.251q_v)} \qquad (2.88)$$

Potential temperature of dry air

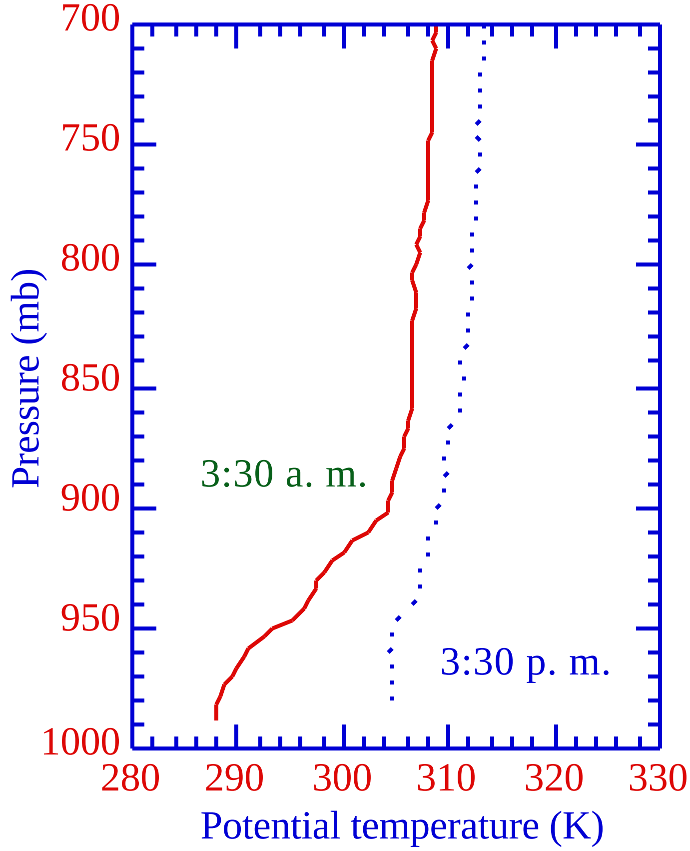
$$\theta_p = T\left(\frac{1000 \text{ mb}}{p_d}\right)^{\kappa} \qquad (2.89)$$

Example 2.20.

	p_d	= 800 mb	T	= 270 K
---->	θ_p	= 287.8 K		

Morning and Afternoon Potential Temperature Soundings at Riverside

Figure 2.12.



Potential Virtual Temperature

Potential virtual temperature

$$\theta_v = T\left(1 + 0.608 q_v\right)\left(\frac{1000\ \text{mb}}{p_a}\right)^{\kappa} = T_v\left(\frac{1000\ \text{mb}}{p_a}\right)^{\kappa} \qquad (2.90)$$

Virtual potential temperature

$$\theta_{p,v} = \theta_{p,m}\left(1 + 0.608 q_v\right) = T_v\left(\frac{1000\ \text{mb}}{p_a}\right)^{\kappa\left(1 - 0.251 q_v\right)} \qquad (2.91)$$

Exner function

$$c_{p,d}P$$

Potential temperature factor

$$P = \left(\frac{p_a}{1000\ \text{mb}}\right)^{\kappa} \qquad (2.92)$$

Temperature and virtual temperature

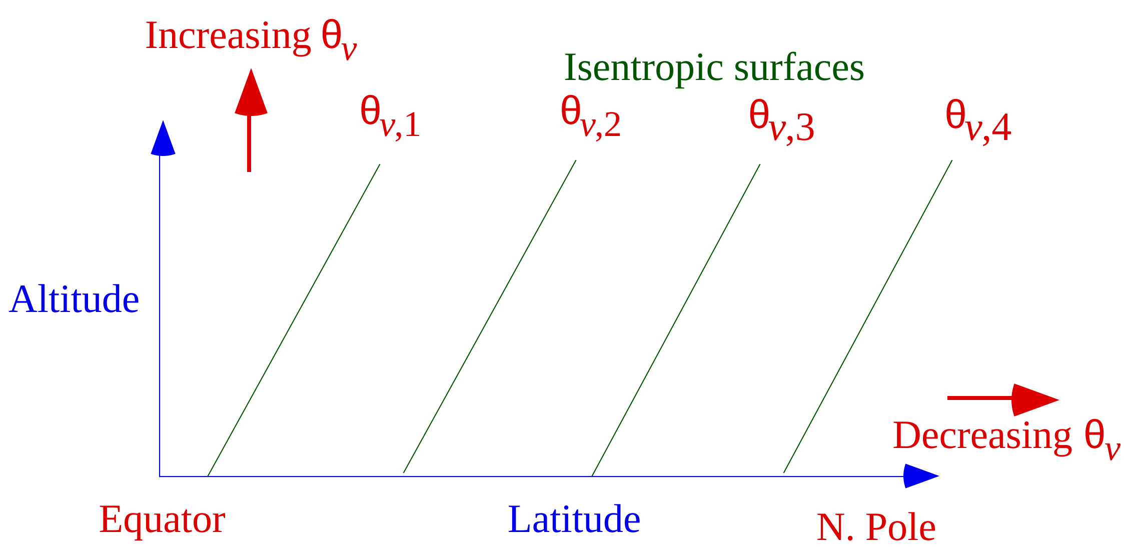
$$T = \theta_p P \qquad T_v = \theta_v P \qquad (2.93)$$

Isentropic Surfaces

Change in entropy

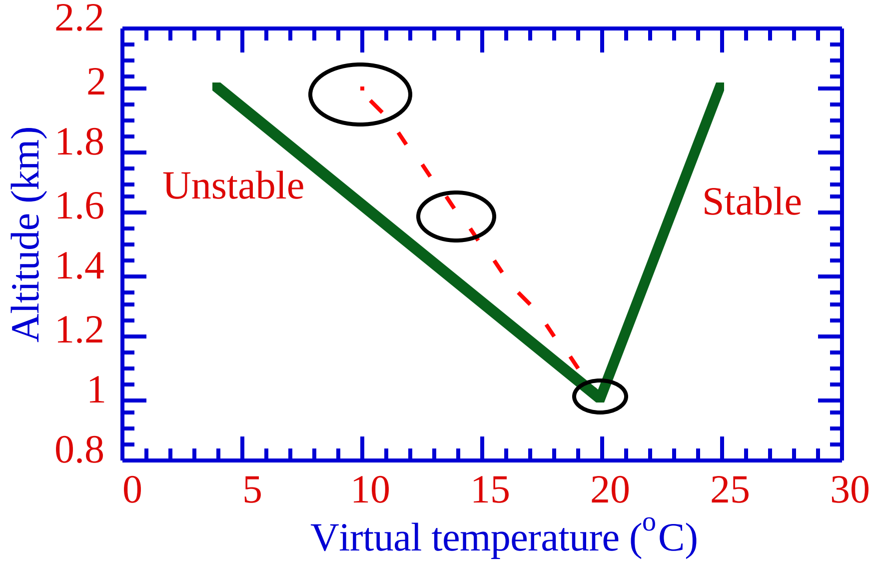
$$dS = \frac{dQ}{T}$$

Figure 2.13
Isentropic surfaces (surfaces of constant potential temperature) between the equator and the North Pole



Stability Criteria for Unsaturated Air

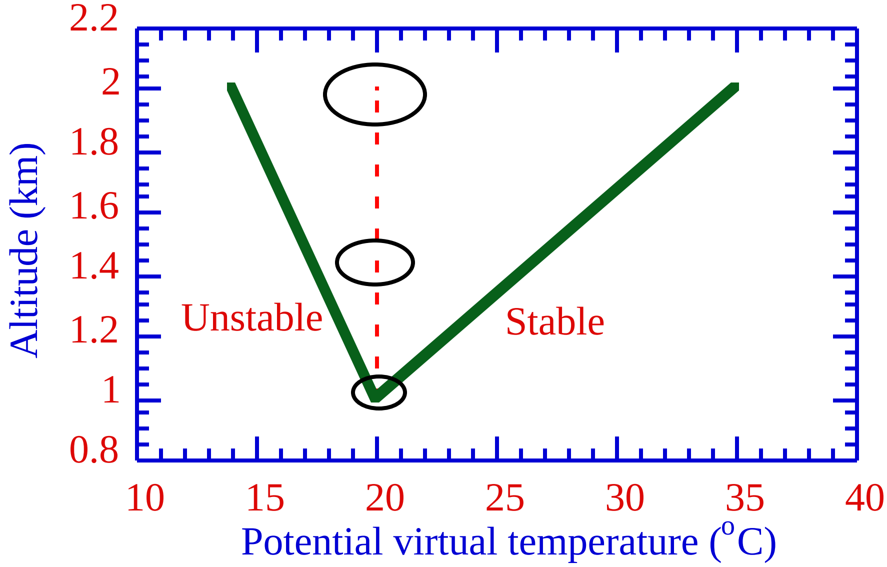
Figure 2.14.



$$\Gamma_v \begin{cases} > \Gamma_d & \text{unsaturated} \quad \text{unstable} \\ = \Gamma_d & \text{unsaturated} \quad \text{neutral} \\ < \Gamma_d & \text{unsaturated} \quad \text{stable} \end{cases} \qquad (2.94)$$

Stability Criteria from Potential Virtual Temperature

Figure 2.15.



$$\frac{\partial \theta_v}{\partial z} \begin{cases} < 0 & \text{unsaturated} \quad \text{unstable} \\ = 0 & \text{unsaturated} \quad \text{neutral} \\ > 0 & \text{unsaturated} \quad \text{stable} \end{cases} \qquad (2.95)$$

Stability Criteria from Potential Virtual Temperature

Differentiate (2.90)

$$d\theta_v = dT_v\left(\frac{1000}{p_a}\right)^{\kappa} + T_v\kappa\left(\frac{1000}{p_a}\right)^{\kappa-1}\left(-\frac{1000}{p_a^2}\right)dp_a = \frac{\theta_v}{T_v}dT_v - \kappa\frac{\theta_v}{p_a}dp_a \qquad (2.96)$$

Differentiate (2.96), substitute hydrostatic equation and Γ_v

$$\frac{\partial\theta_v}{\partial z} = \frac{\theta_v}{T_v}\frac{\partial T_v}{\partial z} - \kappa\frac{\theta_v}{p_a}\frac{\partial p_a}{\partial z} = -\frac{\theta_v}{T_v}\Gamma_v + \frac{R'}{c_{p,d}}\frac{\theta_v}{p_a}\rho_a g \qquad (2.97)$$

Substitute equation of state for air and definition of Γ_d

$$\frac{\partial\theta_v}{\partial z} = -\frac{\theta_v}{T_v}\Gamma_v + \frac{\theta_v g}{T_v c_{p,d}} = \frac{\theta_v}{T_v}\left(\Gamma_d - \Gamma_v\right) \qquad (2.98)$$

Example 2.23.

	p_a	= 925 mb	T_v	= 290 K
	Γ_v	= +7 K km^{-1}		
---->	θ_v	= 296.5 K	$\partial\theta_v/\partial z$	= 3.07 K km^{-1}

Brunt-Väisälä Frequency

Rewrite (2.98)

$$\frac{\partial \ln \theta_v}{\partial z} = \frac{1}{T_v}\left(\Gamma_d - \Gamma_v\right) \qquad (2.99)$$

Multiply by g --> Brunt-Väisälä (buoyancy) frequency

$$N_{bv}^2 = g\frac{\partial \ln \theta_v}{\partial z} = \frac{g}{T_v}\left(\Gamma_d - \Gamma_v\right) \qquad (2.100)$$

Period of oscillation

$$\tau_{bv} = \frac{2\pi}{N_{bv}}$$

Stability criteria

$$N_{bv}^2 \begin{cases} < 0 & \text{unsaturated unstable} \\ = 0 & \text{unsaturated neutral} \\ > 0 & \text{unsaturated stable} \end{cases} \qquad (2.101)$$

Example 2.24.

	T_v	= 288 K
	Γ_v	= +6.5 K km^{-1}
---->	N_{bv}	= 0.0106 s^{-1}
---->	τ_{bv}	= 593 s