Overhead Slides for

Chapter 3

of

Fundamentals of Atmospheric Modeling

by

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Wind Velocity and Speed

Velocity

Rate at which the position of a body changes with time

Velocity vector

$$\mathbf{v} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w \qquad (3.1)$$

Horizontal velocity vector

$$\mathbf{v}_h = \mathbf{i}u + \mathbf{j}v \qquad (3.1)$$

Scalar components of wind velocity

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt} \qquad w = \frac{dz}{dt} \qquad (3.2)$$

Wind speed

Magnitude of the velocity vector

$$|\mathbf{v}| = \sqrt{u^2 + v^2 + w^2} \qquad (3.3)$$

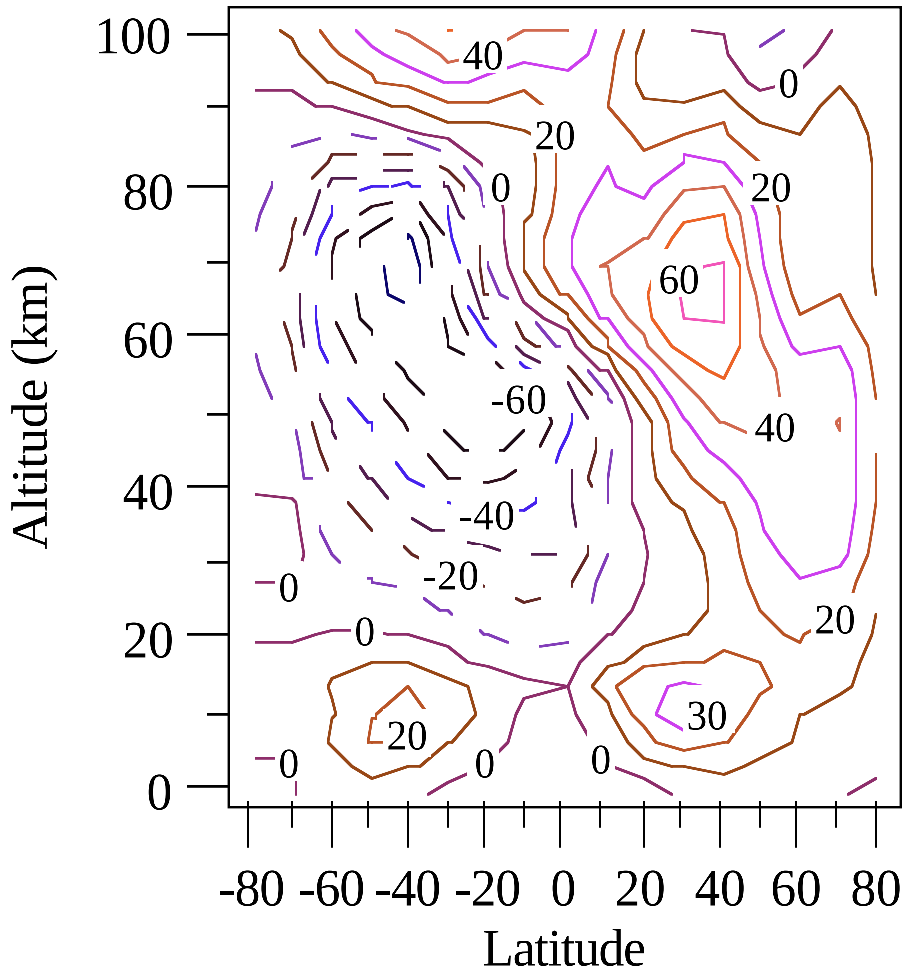
Horizontal wind speed

$$|\mathbf{v}_h| = \sqrt{u^2 + v^2} \qquad (3.3)$$

Zonally-/Monthly-Averaged Winds (m/s)

Figure 3.1 a

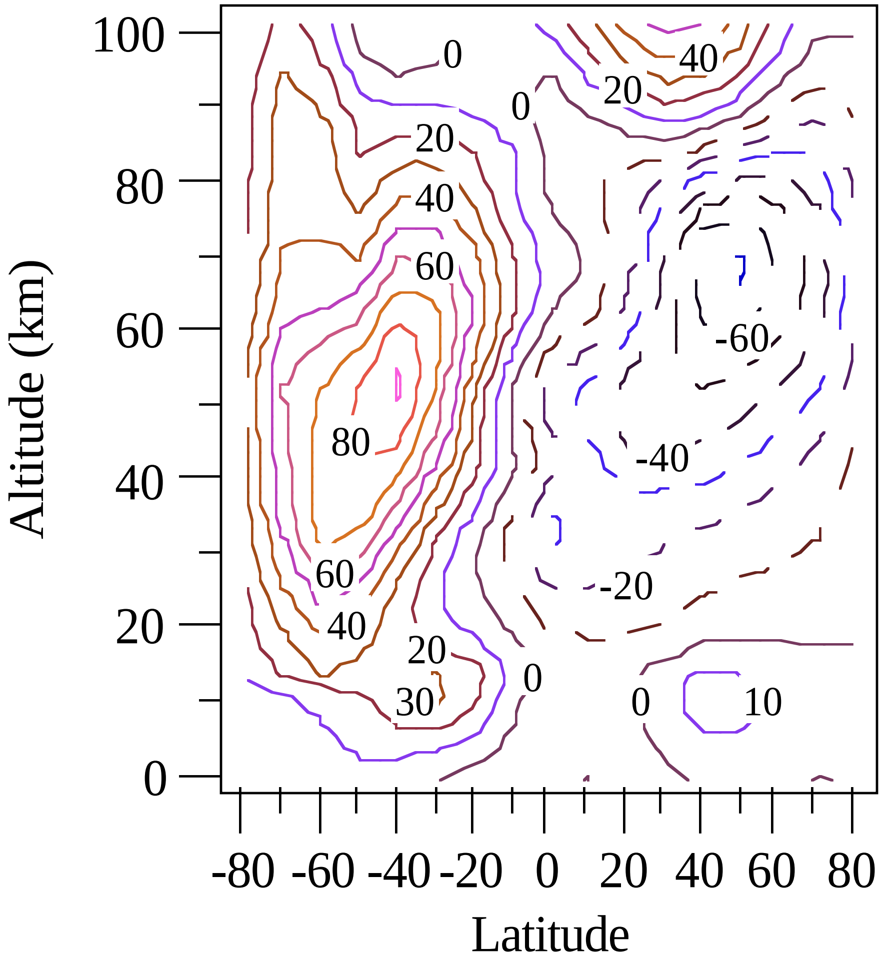
January



Zonally-/Monthly-Averaged Winds (m/s)

Figure 3.1 b

July



Local and Total Differentiation

Expansion of total derivative with chain rule

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\partial N}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial N}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial N}{\partial t} + u\frac{\partial N}{\partial x} \qquad (3.4)$$

Total derivative. Time rate of change along a trajectory

$$\frac{\mathrm{d}N}{\mathrm{d}t}$$

Local derivative. Time rate of change at a fixed point

$$\frac{\partial N}{\partial t}$$

Transport term. Time rate of change due to transport

$$u\frac{\partial N}{\partial x}$$

Eulerian frame of reference

Frame of reference of a fixed point in space

$$\frac{\partial N}{\partial t} + u\frac{\partial N}{\partial x}$$

Lagrangian frame of reference

Frame of reference of a moving parcel

$$\frac{\mathrm{d}N}{\mathrm{d}t}$$

Example

Example 3.1.

Balloon traveling with the wind from east to west

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 10^8 \text{ molec cm}^{-3} \text{ s}^{-1}$$

$$\frac{\partial N}{\partial x} = 10^{10} \text{ molec cm}^{-3} \text{ km}^{-1}$$

$$u = -10 \text{ m s}^{-1}$$

Find time rate of change in concentration at fixed point A

$$\frac{\partial N}{\partial t} = \frac{\mathrm{d}N}{\mathrm{d}t} - u\frac{\partial N}{\partial x}$$

$$= 10^8 \frac{\text{molec.}}{\text{cm}^3 \text{ s}} - \left(-10\frac{\text{m}}{\text{s}}\right) \times 10^{10} \frac{\text{molec.}}{\text{cm}^3 \text{ km}} \times 0.001\frac{\text{km}}{\text{m}} = 2 \times 10^8 \frac{\text{molec.}}{\text{cm}^3 \text{ s}}$$

Gradient Operator

Gradient operator in Cartesian / altitude coordinates

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \qquad (3.6)$$

Dot product of velocity vector with gradient operator

$$\mathbf{v} \bullet \nabla = (\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) \bullet \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad (3.7)$$

$\mathbf{i} \bullet \mathbf{i} = 1$ $\quad$ $\mathbf{j} \bullet \mathbf{j} = 1,$ $\quad$ $\mathbf{k} \bullet \mathbf{k} = 1$

$\mathbf{i} \bullet \mathbf{j} = 0$ $\quad$ $\mathbf{i} \bullet \mathbf{k} = 0$ $\quad$ $\mathbf{j} \bullet \mathbf{k} = 0$

Dot product of gradient operator with velocity vector

Divergence term

$$\nabla \bullet \mathbf{v} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet (\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (3.8)$$

Dot product of gradient operator with vector not symmetric

$$\nabla \bullet \mathbf{v} \neq \mathbf{v} \bullet \nabla$$

Dot product of two vectors is symmetric

$$\mathbf{a} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{a}$$

Gradient Operator

Scalar concentration divergence term

$$N(\nabla \bullet \mathbf{v}) = N\frac{\partial u}{\partial x} + N\frac{\partial v}{\partial y} + N\frac{\partial w}{\partial z} \qquad (3.9)$$

Gradient of a scalar variable is a vector

$$\nabla N = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)N = \mathbf{i}\frac{\partial N}{\partial x} + \mathbf{j}\frac{\partial N}{\partial y} + \mathbf{k}\frac{\partial N}{\partial z} \qquad (3.10)$$

Apply dot product of velocity and gradient operator to *N*

$$(\mathbf{v} \bullet \nabla)N = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)N = u\frac{\partial N}{\partial x} + v\frac{\partial N}{\partial y} + w\frac{\partial N}{\partial z}$$

(3.11)

Substitute into total derivative

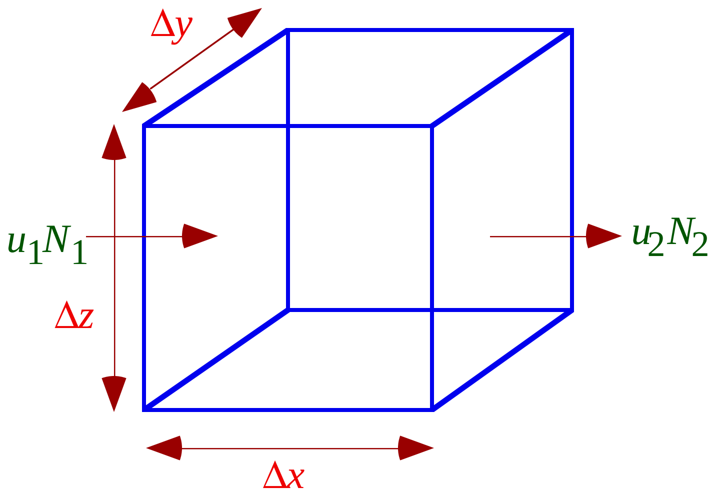
$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\partial N}{\partial t} + (\mathbf{v} \bullet \nabla)N \qquad (3.12)$$

Generalize and expand total derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \mathbf{v} \bullet \nabla \qquad (3.13)$$

Continuity Equation

Figure 3.2.



Accumulation = molecules entering - molecules leaving

$$\Delta N \Delta x \Delta y \Delta z = u_1 N_1 \Delta y \Delta z \Delta t - u_2 N_2 \Delta y \Delta z \Delta t \qquad (3.14)$$

Divide both sides by Δt and box volume ($\Delta x \Delta y \Delta z$)

$$\frac{\Delta N}{\Delta t} = -\left(\frac{u_2 N_2 - u_1 N_1}{\Delta x} \right) \qquad (3.15)$$

Δx --> 0 and Δt --> 0

--> Flux divergence form of continuity equation

$$\frac{\partial N}{\partial t} = -\frac{\partial (uN)}{\partial x} \qquad (3.16)$$

Continuity Equation

Flux divergence form of continuity equation in three dimensions

$$\frac{\partial N}{\partial t} = -\frac{\partial (uN)}{\partial x} - \frac{\partial (vN)}{\partial y} - \frac{\partial (wN)}{\partial z} = -\nabla \bullet (\mathbf{v}N) \qquad (3.17)$$

Take dot product of gradient operator with $\mathbf{v}N$

$$\nabla \bullet (\mathbf{v}N) = N(\nabla \bullet \mathbf{v}) + (\mathbf{v} \bullet \nabla)N \qquad (3.18)$$

Substitute into (3.17)

$$\frac{\partial N}{\partial t} = -N(\nabla \bullet \mathbf{v}) - (\mathbf{v} \bullet \nabla)N \qquad (3.19)$$

From definition of total derivative

$$(\mathbf{v} \bullet \nabla)N = \frac{\mathrm{d}N}{\mathrm{d}t} - \frac{\partial N}{\partial t} \qquad (3.21)$$

Substitute into (3.19) and similar equation for density

--> Velocity divergence forms of continuity equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -N(\nabla \bullet \mathbf{v}) \qquad (3.22)$$

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = -\rho_a(\nabla \bullet \mathbf{v}) \qquad (3.23)$$

Continuity Equation for Mass Mixing Ratio

Number concentration as a function of moist-air mass mixing ratio

$$N = \frac{A \rho_a q}{m} \tag{3.24}$$

Substitute (3.24) into (3.19)

$$q\left(\frac{\partial \rho_a}{\partial t} + \rho_a (\nabla \bullet \mathbf{v}) + (\mathbf{v} \bullet \nabla)\rho_a\right) + \rho_a \frac{\partial q}{\partial t} = -\rho_a (\mathbf{v} \bullet \nabla) q \tag{3.25}$$

Substitute continuity equation for air

--> Continuity equation for moist-air mass mixing ratio of a species with no external source or sink terms.

$$\frac{\partial q}{\partial t} = -(\mathbf{v} \bullet \nabla) q \tag{3.26}$$

Compressibility / Incompressibility

Compressible fluid (air)

Volume of an air parcel changes over time and density varies (thus air is inhomogeneous)

$$\nabla \bullet \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Incompressible fluid (water)

Volume of a water parcel stays constant, but density varies (thus water is also inhomogeneous)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (3.27)$$

Substitute water density (ρ_w) and $\nabla \bullet \mathbf{v} = 0$ into (3.23)

Density of incompressible fluid is constant along motion

$$\frac{d\rho_w}{dt} = 0 \qquad (3.28)$$

Substitute water density (ρ_w) and $\nabla \bullet \mathbf{v} = 0$ into (3.20). At a fixed point in the fluid, water density changes.

$$\frac{\partial \rho_w}{\partial t} = -(\mathbf{v} \bullet \nabla)\rho_w \qquad (3.29)$$

Expanded Continuity Equation

Expanded continuity equation

$$\frac{\partial N}{\partial t} = -\nabla \bullet (\mathbf{v}N) + D\nabla^2 N + \sum_{n=1}^{N_{e,t}} R_n \qquad (3.30)$$

Substitute

$$\nabla^2 N = (\nabla \bullet \nabla) N = \left[\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\right] N$$
$$= \frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} + \frac{\partial^2 N}{\partial z^2} \qquad (3.31)$$

into (3.30) to obtain

$$\frac{\partial N}{\partial t} + \frac{\partial (uN)}{\partial x} + \frac{\partial (vN)}{\partial y} + \frac{\partial (wN)}{\partial z} = D\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} + \frac{\partial^2 N}{\partial z^2}\right) + \sum_{n=1}^{N_{e,t}} R_n$$
$$(3.32)$$

Time and Grid Volume Averaging

Precise gas concentration

$$N = \overline{N} + N' \qquad (3.33)$$

Time and grid volume averaged concentration

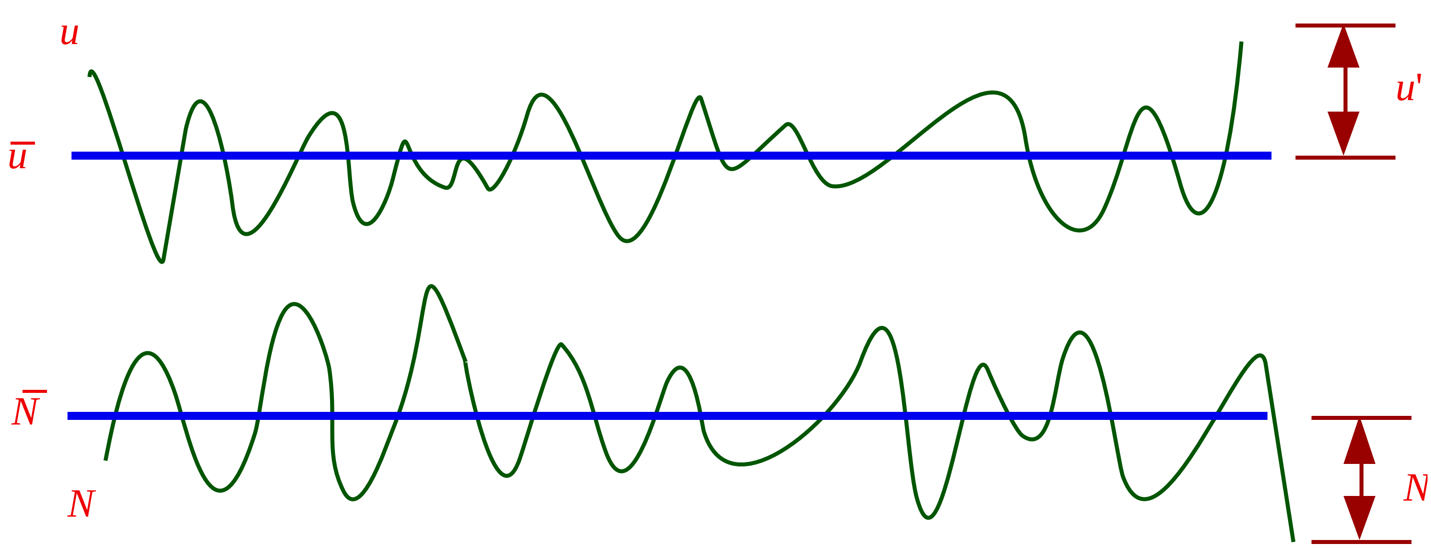
$$\overline{N} = \frac{1}{\Delta t \Delta x \Delta y \Delta z} \int_t^{t+\Delta t} \left\{ \int_x^{x+\Delta x} \left[\int_y^{y+\Delta y} \left(\int_z^{z+\Delta z} N \mathrm{d}z \right) \mathrm{d}y \right] \mathrm{d}x \right\} \mathrm{d}t \qquad (3.34)$$

Precise wind velocity vector

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}' \qquad (3.36)$$

$$u = \overline{u} + u' \qquad v = \overline{v} + v' \qquad w = \overline{w} + w' \qquad (3.35)$$

Figure 3.3. Precise, mean, and perturbation components of scalar velocity and gas concentration, respectively.



Continuity Equation

Substitute (3.33) and (3.35) into (3.32)

$$\overline{\left[\frac{\partial\left(\overline{N}+N'\right)}{\partial t}\right]}+\overline{\left[\frac{\partial\left(\overline{u}+u'\right)\left(\overline{N}+N'\right)}{\partial x}\right]}+\overline{\left[\frac{\partial\left(\overline{v}+v'\right)\left(\overline{N}+N'\right)}{\partial y}\right]}+\overline{\left[\frac{\partial\left(\overline{w}+w'\right)\left(\overline{N}+N'\right)}{\partial z}\right]}$$

$$=D\left\{\overline{\left[\frac{\partial^2\left(\overline{N}+N'\right)}{\partial x^2}\right]}+\overline{\left[\frac{\partial^2\left(\overline{N}+N'\right)}{\partial y^2}\right]}+\overline{\left[\frac{\partial^2\left(\overline{N}+N'\right)}{\partial z^2}\right]}\right\}+\overline{\sum_{n=1}^{N_{e,t}}R_n}$$

(3.37)

Simplify terms

$$\overline{\left[\frac{\partial\left(\overline{N}+N'\right)}{\partial t}\right]}=\frac{\partial\left(\overline{\overline{N}}+\overline{N'}\right)}{\partial t}=\frac{\partial\overline{N}}{\partial t} \qquad (3.38)$$

$$\overline{\partial\left(\overline{N}+N'\right)/\partial t}=\partial\left(\overline{\overline{N}+N'}\right)/\partial t \qquad \overline{\overline{N}+N'}=\overline{\overline{N}}+\overline{N'}$$

$$\overline{\overline{N}}=\overline{N} \qquad \overline{N'}=0$$

Simplify again

$$\overline{\left[\frac{\partial\left(\overline{u}+u'\right)\left(\overline{N}+N'\right)}{\partial x}\right]}=\frac{\partial\left(\overline{\overline{u}\,\overline{N}}+\overline{\overline{u}N'}+\overline{u'\overline{N}}+\overline{u'N'}\right)}{\partial x}=\frac{\partial\left(\overline{u}\,\overline{N}+\overline{u'N'}\right)}{\partial x}$$

(3.39)

$$\overline{u'\overline{N}}=0 \qquad \overline{\overline{u}N'}=0 \qquad \overline{\overline{u}\,\overline{N}}=\overline{u}\,\overline{N}$$

Continuity Equation

Substitute averaged terms into (3.50)

$$\frac{\partial \bar{N}}{\partial t}+\frac{\partial(\bar{u}\bar{N})}{\partial x}+\frac{\partial(\bar{v}\bar{N})}{\partial y}+\frac{\partial(\bar{w}\bar{N})}{\partial z}+\frac{\partial \overline{u'N'}}{\partial x}+\frac{\partial \overline{v'N'}}{\partial y}+\frac{\partial \overline{w'N'}}{\partial z}$$

$$= D\left(\frac{\partial^2 \bar{N}}{\partial x^2}+\frac{\partial^2 \bar{N}}{\partial y^2}+\frac{\partial^2 \bar{N}}{\partial z^2}\right)+\sum_{n=1}^{N_{e,t}} \bar{R}_n \qquad (3.40)$$

Turbulent diffusion is much larger than molecular diffusion -->

$$\frac{\partial \bar{N}}{\partial t}+\frac{\partial(\bar{u}\bar{N})}{\partial x}+\frac{\partial(\bar{v}\bar{N})}{\partial y}+\frac{\partial(\bar{w}\bar{N})}{\partial z}+\frac{\partial \overline{u'N'}}{\partial x}+\frac{\partial \overline{v'N'}}{\partial y}+\frac{\partial \overline{w'N'}}{\partial z}=\sum_{n=1}^{N_{e,t}} \bar{R}_n \qquad (3.41)$$

Continuity equation for gas in vector notation.

$$\frac{\partial \bar{N}}{\partial t}+\nabla \bullet (\bar{\mathbf{v}}\bar{N})+\nabla \bullet (\overline{\mathbf{v}'N'})=\sum_{n=1}^{N_{e,t}} \bar{R}_n \qquad (3.42)$$

Similar equation for air mass density

$$\frac{\partial \bar{\rho}_a}{\partial t}+\nabla \bullet (\bar{\mathbf{v}}\bar{\rho}_a)+\nabla \bullet (\overline{\mathbf{v}'\rho'_a})=0 \qquad (3.43)$$

Continuity Equation for Mass Mixing Ratio

Continuity equation for trace gas in mass mixing ratio units

$$\frac{\partial q}{\partial t} + \mathbf{v} \bullet \nabla q = \sum_{n=1}^{N_{e,t}} R_n \qquad (3.44)$$

Continuity equation for air

$$\frac{\partial \rho_a}{\partial t} = -\rho_a (\nabla \bullet \mathbf{v}) - (\mathbf{v} \bullet \nabla)\rho_a \qquad (3.20)$$

Sum continuity equations

$$\frac{\partial (\rho_a q)}{\partial t} + \nabla \bullet (\rho_a \mathbf{v} q) = \rho_a \sum_{n=1}^{N_{e,t}} R_n \qquad (3.45)$$

Take time and grid volume average of terms in equation

--> Species continuity equation

$$\frac{\partial \bar{q}}{\partial t} + (\bar{\mathbf{v}} \bullet \nabla)\bar{q} + \frac{1}{\bar{\rho}_a} \nabla \bullet \left(\bar{\rho}_a \overline{\mathbf{v}' q'}\right) = \sum_{n=1}^{N_{e,t}} \bar{R}_n \qquad (3.49)$$

Parameterization of Diffusion

Eddies

Swirling motions of air caused by wind shear and enhanced by buoyancy

Turbulence

Many eddies of different sizes acting together

Turbulent diffusion

Subgrid diffusion due to turbulence

K-theory

Relates turbulent flux of one parameter (*e.g.* concentration) to gradient of mean value of the parameter

Kinematic turbulent flux terms

$$\overline{u'N'} = -K_{h,xx}\frac{\partial \bar{N}}{\partial x} \qquad \overline{v'N'} = -K_{h,yy}\frac{\partial \bar{N}}{\partial y} \qquad \overline{w'N'} = -K_{h,zz}\frac{\partial \bar{N}}{\partial z} \tag{3.50}$$

Substitute (3.50) into (3.41)

$$\frac{\partial \bar{N}}{\partial t} + \frac{\partial(\bar{u}\bar{N})}{\partial x} + \frac{\partial(\bar{v}\bar{N})}{\partial y} + \frac{\partial(\bar{w}\bar{N})}{\partial z} = \frac{\partial}{\partial x}\left(K_{h,xx}\frac{\partial \bar{N}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{h,yy}\frac{\partial \bar{N}}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{h,zz}\frac{\partial \bar{N}}{\partial z}\right) + \sum_{n=1}^{N_{e,t}}\bar{R}_n \tag{3.51}$$

Continuity Equations for Trace Gases and Air

Continuity equation for trace gas number conc. (molec. cm^{-3})

$$\frac{\partial N}{\partial t} + \nabla \bullet (\mathbf{v}N) = (\nabla \bullet \mathbf{K}_h \nabla) N + \sum_{n=1}^{N_{e,t}} R_n \qquad (3.52)$$

Expanded continuity equation for a trace gas

$$\frac{\partial N_q}{\partial t} + \nabla \bullet (\mathbf{v}N_q) = (\nabla \bullet \mathbf{K}_h \nabla) N_q \qquad (3.56)$$

$$+R_{emisg} + R_{depg} + R_{washg} + R_{chemg} + R_{nucg} + R_{c/eg} + R_{dp/sg} + R_{ds/eg} + R_{hrg}$$

Continuity equation for air

$$\frac{\partial \rho_a}{\partial t} + \nabla \bullet (\mathbf{v}\rho_a) = 0 \qquad (3.55)$$

Continuity Equations for Particles

Continuity equation for particle number conc. (partic. cm^{-3})

$$\frac{\partial n_i}{\partial t} + \nabla \bullet (\mathbf{v} n_i) = (\nabla \bullet \mathbf{K}_h \nabla) n_i \qquad (3.58)$$

$$+R_{emisn} + R_{depn} + R_{sedn} + R_{washn} + R_{nucn} + R_{coagn}$$

Volume concentration versus number concentration and volume

$$v_{q,i} = n_i \upsilon_{q,i} \qquad (3.57)$$

Continuity equation for particle volume concentration (cm^3 cm^{-3})

$$\frac{\partial v_{q,i}}{\partial t} + \nabla \bullet (\mathbf{v} v_{q,i}) = (\nabla \bullet \mathbf{K}_h \nabla) v_{q,i} \qquad (3.59)$$

$$+R_{emisv} + R_{depv} + R_{sedv} + R_{washv} + R_{nucv} + R_{coagv}$$

$$+R_{c/ev} + R_{dp/sv} + R_{ds/ev} + R_{eqv} + R_{aqv} + R_{hrv}$$

Continuity Equations for Water

Water vapor

$$\frac{\partial q_v}{\partial t} + (\mathbf{v} \bullet \nabla) q_v = \frac{1}{\rho_a} (\nabla \rho_a \mathbf{K}_h \nabla) q_v \qquad (3.61)$$
$$+ R_{emisV} + R_{depV} + R_{chemV} + R_{nucV} + R_{c/eV} + R_{dp/sV}$$

Liquid water

$$\frac{\partial q_{L,i}}{\partial t} + (\mathbf{v} \bullet \nabla) q_{L,i} = \frac{1}{\rho_a} (\nabla \rho_a \mathbf{K}_h \nabla) q_{L,i} \qquad (3.62)$$
$$+ R_{emisL} + R_{depL} + R_{sedL} + R_{nucL} + R_{coagL} + R_{c/eL} + R_{f/mL}$$

Ice

$$\frac{\partial q_{I,i}}{\partial t} + (\mathbf{v} \bullet \nabla) q_{I,i} = \frac{1}{\rho_a} (\nabla \rho_a \mathbf{K}_h \nabla) q_{I,i} \qquad (3.63)$$
$$+ R_{depI} + R_{sedI} + R_{nucI} + R_{coagI} + R_{f/mI} + R_{dp/sI}$$

Bulk vs. Size-Resolved Process

Size resolved treatment of water

$$q_T = q_v + \sum_{i=1}^{N_B} \left(q_{L,i} + q_{I,i} \right) \qquad (3.60)$$

Bulk treatment of water

$$q_T = q_v + q_L + q_I$$

Thermodynamic Energy Equation

Energy transfer processes

- Conduction
- Advection
- Forced Convection
- Free Convection
- Turbulence
- Radiative transfer

Energy sources / sinks

- Latent heat release / absorption
- Solar / infrared emissions / absorption

First law of thermodynamics

$$\mathrm{d}Q = c_{p,d}\,\mathrm{d}T_v - \alpha_a \mathrm{d}p_a \qquad (2.76)$$

Differentiate, substitute $\alpha_a = 1/\rho_a$, rearrange -->

Thermodynamic energy equation in terms of temperature along the motion of an air parcel

$$\frac{\mathrm{d}T_v}{\mathrm{d}t} = \frac{1}{c_{p,d}}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{c_{p,d}\rho_a}\frac{\mathrm{d}p_a}{\mathrm{d}t} \qquad (3.64)$$

Thermodynamic Energy Equation

Differentiate $\theta_v = T_v (1000/p_a)^\kappa$ with respect to time

$$\frac{d\theta_v}{dt} = \frac{dT_v}{dt}\left(\frac{1000}{p_a}\right)^{\kappa} + T_v\kappa\left(\frac{1000}{p_a}\right)^{\kappa-1}\left(-\frac{1000}{p_a^2}\right)\frac{dp_a}{dt} = \frac{\theta_v}{T_v}\frac{dT_v}{dt} - \frac{\kappa\theta_v}{p_a}\frac{dp_a}{dt}$$

(3.65)

Substitute into (3.64) and expand total derivative

--> Thermodynamic energy equation

$$\frac{d\theta_v}{dt} = \frac{\partial\theta_v}{\partial t} + (\mathbf{v}\bullet\nabla)\theta_v = \frac{\theta_v}{c_{p,d}T_v}\frac{dQ}{dt} \qquad (3.66)$$

Multiply (3.66) by $c_{p,d}\rho_a$, continuity equation by $c_{p,d}\theta_v$ and sum

$$\frac{\partial\left(c_{p,d}\rho_a\theta_v\right)}{\partial t} + \nabla\bullet\left(\mathbf{v}c_{p,d}\rho_a\theta_v\right) = \rho_a\frac{\theta_v}{T_v}\frac{dQ}{dt} \qquad (3.67)$$

Define energy density (J m^{-3})

$$E = c_{p,d}\rho_a\theta_v$$

Substitute into (3.67)

$$\frac{\partial E}{\partial t} + \nabla\bullet(\mathbf{v}E) = \rho_a\frac{\theta_v}{T_v}\frac{dQ}{dt} \qquad (3.68)$$

Thermodynamic Energy Equation

Decompose terms (assume $\rho'_a << \bar{\rho}_a$)

$$\rho_a \approx \bar{\rho}_a \quad \theta_v = \bar{\theta}_v + \theta'_v \qquad\qquad \mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$$

Substitute into (3.67)

$$\frac{\partial\left(\bar{\rho}_a \bar{\theta}_v\right)}{\partial t} + \nabla \bullet \left(\bar{\rho}_a \bar{\mathbf{v}} \bar{\theta}_v\right) + \overline{\nabla \bullet \left(\bar{\rho}_a \mathbf{v}' \theta'_v\right)} = \frac{\bar{\rho}_a}{c_{p,d}} \overline{\frac{\theta_v}{T_v} \frac{\mathrm{dQ}}{\mathrm{d}t}} \qquad (3.70)$$

Expand and substitute continuity equation

$$\frac{\partial \bar{\theta}_v}{\partial t} + (\bar{\mathbf{v}} \bullet \nabla)\bar{\theta}_v + \frac{1}{\bar{\rho}_a} \nabla \bullet \left(\bar{\rho}_a \overline{\mathbf{v}' \theta'_v}\right) = \overline{\frac{\theta_v}{c_{p,d} T_v} \frac{\mathrm{dQ}}{\mathrm{d}t}} \qquad (3.72)$$

Subgrid scale turbulent flux divergence terms

$$\overline{u'\theta'_v} = -K_{h,xx} \frac{\partial \bar{\theta}_v}{\partial x} \qquad \overline{v'\theta'_v} = -K_{h,yy} \frac{\partial \bar{\theta}_v}{\partial y} \qquad \overline{w'\theta'_v} = -K_{h,zz} \frac{\partial \bar{\theta}_v}{\partial z}$$

(3.73)

Thermodynamic Energy Equation

Substitute $\overline{\mathbf{v}'\theta_v'} = -\mathbf{K}_h \nabla \bar{\theta}_v$ into (3.72), remove overbars

$$\frac{\partial \theta_v}{\partial t} + (\mathbf{v} \bullet \nabla)\theta_v = \frac{1}{\rho_a}(\nabla \bullet \rho_a \mathbf{K}_h \nabla)\theta_v + \frac{\theta_v}{c_{p,d} T_v} \frac{\mathrm{d}Q}{\mathrm{d}t} \qquad (3.74)$$

Rewrite diabatic energy source/sink term

$$\frac{\partial \theta_v}{\partial t} + (\mathbf{v} \bullet \nabla)\theta_v = \frac{1}{\rho_a}(\nabla \bullet \rho_a \mathbf{K}_h \nabla)\theta_v + \frac{\theta_v}{c_{p,d} T_v} \sum_{n=1}^{N_{e,h}} \frac{\mathrm{d}Q_n}{\mathrm{d}t} \qquad (3.76)$$

where

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \sum_{n=1}^{N_{e,h}} \frac{\mathrm{d}Q_n}{\mathrm{d}t} = \frac{\mathrm{d}Q_{c/e}}{\mathrm{d}t} + \frac{\mathrm{d}Q_{f/m}}{\mathrm{d}t} + \frac{\mathrm{d}Q_{dp/s}}{\mathrm{d}t} + \frac{\mathrm{d}Q_{solar}}{\mathrm{d}t} + \frac{\mathrm{d}Q_{ir}}{\mathrm{d}t}$$

(3.75)