**Overhead Slides for**

# Chapter 4

**of**

# Fundamentals of Atmospheric Modeling

by

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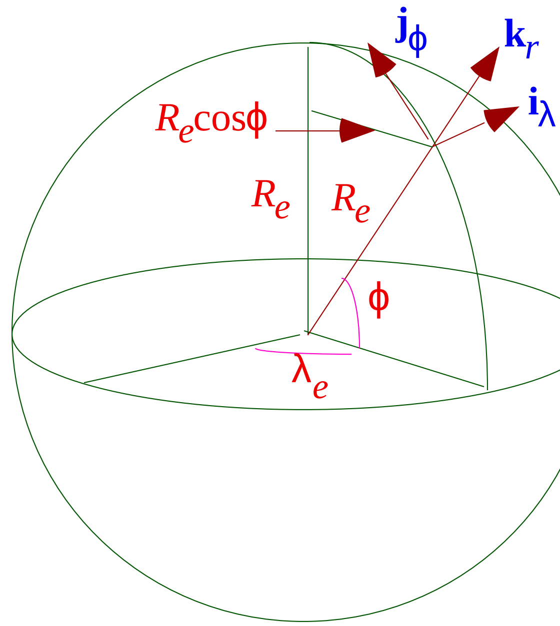
# Spherical Horizontal Coordinates

Fig. 4.1.



# Spherical Coordinate Conversions

West-east, south-north increments

$$dx = (R_e \cos\phi)\, d\lambda_e \qquad dy = R_e d\phi \qquad (4.1)$$

Example 4.1.

$d\lambda_e = 5^o$

$d\phi = 5^o$

$\phi = 30\ ^o$N latitude

----> $d\lambda_e = d\phi = 5^o \times \pi / 180^o = 0.0873$ radians

----> $dx = (6371)(0.866)(0.0873) = 482$ km

----> $dy = (6371)(0.0873) = 556$ km

Total / horizontal velocity vectors in spherical coordinates

$$\mathbf{v} = \mathbf{i}_\lambda u + \mathbf{j}_\phi v + \mathbf{k}_r w \qquad \mathbf{v}_h = \mathbf{i}_\lambda u + \mathbf{j}_\phi v \qquad (4.2)$$

Scalar velocities

$$u = \frac{dx}{dt} = R_e \cos\phi \frac{d\lambda_e}{dt} \qquad (4.3)$$

$$v = \frac{dy}{dt} = R_e \frac{d\phi}{dt}$$

$$w = \frac{dz}{dt}$$

# Spherical Coordinate Conversions

Gradient operator

$$= \mathbf{i}_{\lambda} \frac{1}{R_e \cos\phi} \frac{\partial}{\partial \lambda_e} + \mathbf{j}_{\phi} \frac{1}{R_e} \frac{\partial}{\partial \phi} + \mathbf{k}_r \frac{\partial}{\partial z} \qquad (4.4)$$

Dot product of gradient operator with velocity vector

$$\cdot\, \mathbf{v} = \left( \mathbf{i}_{\lambda} \frac{1}{R_e \cos\phi} \frac{\partial}{\partial \lambda_e} + \mathbf{j}_{\phi} \frac{1}{R_e} \frac{\partial}{\partial \phi} + \mathbf{k}_r \frac{\partial}{\partial z} \right) \cdot \left( \mathbf{i}_{\lambda} u + \mathbf{j}_{\phi} v + \mathbf{k}_r w \right)$$

$$= \left( \frac{1}{R_e \cos\phi} \frac{\partial u}{\partial \lambda_e} + \mathbf{i}_{\lambda} u \frac{1}{R_e \cos\phi} \frac{\partial \mathbf{i}_{\lambda}}{\partial \lambda_e} + \mathbf{i}_{\lambda} v \frac{1}{R_e \cos\phi} \frac{\partial \mathbf{j}_{\phi}}{\partial \lambda_e} + \mathbf{i}_{\lambda} w \frac{1}{R_e \cos\phi} \frac{\partial \mathbf{k}_r}{\partial \lambda_e} \right)$$

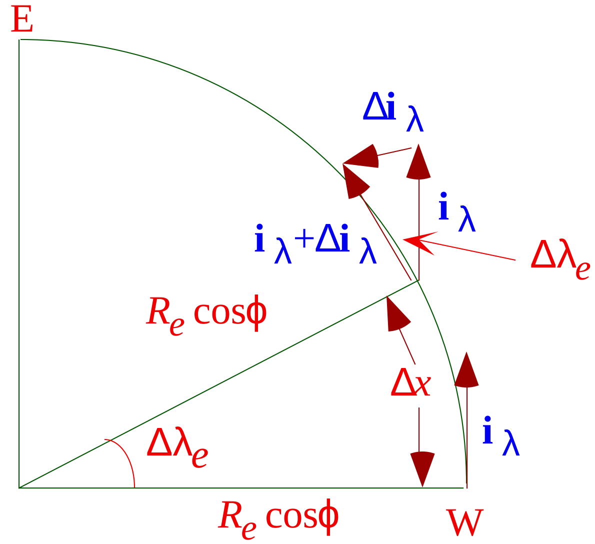
$$+ \left( \frac{1}{R_e} \frac{\partial v}{\partial \phi} + \mathbf{j}_{\phi} u \frac{1}{R_e} \frac{\partial \mathbf{i}_{\lambda}}{\partial \phi} + \mathbf{j}_{\phi} v \frac{1}{R_e} \frac{\partial \mathbf{j}_{\phi}}{\partial \phi} + \mathbf{j}_{\phi} w \frac{1}{R_e} \frac{\partial \mathbf{k}_r}{\partial \phi} \right)$$

$$+ \left( \frac{\partial w}{\partial z} + \mathbf{k}_r u \frac{\partial \mathbf{i}_{\lambda}}{\partial z} + \mathbf{k}_r v \frac{\partial \mathbf{j}_{\phi}}{\partial z} + \mathbf{k}_r w \frac{\partial \mathbf{k}_r}{\partial z} \right) \qquad (4.5)$$

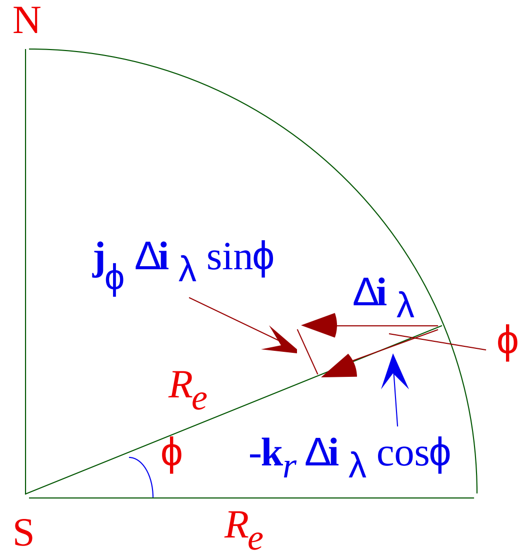
# Spherical Coordinate Conversions

Figs. 4.2 a and b.
Angles used to determine $\partial \mathbf{i}_\lambda / \partial \lambda_e$ in spherical coordinates.

Top view




Side view



# Spherical Coordinate Conversions

Magnitude of an east-west unit vector increment

$$\left|\Delta \mathbf{i}_\lambda\right| = \left|\mathbf{i}_\lambda\right| \Delta\lambda_e = \Delta\lambda_e \qquad (4.6)$$

From Fig. 4.2 (b),

$$\Delta \mathbf{i}_\lambda = \mathbf{j}_\phi \left|\Delta \mathbf{i}_\lambda\right| \sin\phi - \mathbf{k}_r \left|\Delta \mathbf{i}_\lambda\right| \cos\phi \qquad (4.7)$$

Substitute (4.6) into (4.7), divide by $\Delta\lambda_e$

--> let $\Delta \mathbf{i}_\lambda \to 0$, $\Delta\lambda_e \to 0$

$$\frac{\partial \mathbf{i}_\lambda}{\partial \lambda_e} \approx \frac{\mathbf{j}_\phi \Delta\lambda_e \sin\phi - \mathbf{k}_r \Delta\lambda_e \cos\phi}{\Delta\lambda_e} \approx \mathbf{j}_\phi \sin\phi - \mathbf{k}_r \cos\phi \qquad (4.8)$$

Substitute (4.8) and other terms into (4.5)

$$\cdot\, \mathbf{v} = \frac{1}{R_e \cos\phi} \frac{\partial u}{\partial \lambda_e} + \frac{1}{R_e \cos\phi} \frac{\partial}{\partial \phi}\left(v \cos\phi\right) + \frac{1}{R_e^2} \frac{\partial}{\partial z}\left(w R_e^2\right) \qquad (4.10)$$

Assume $R_e$ is constant

$$\cdot\, \mathbf{v} = \frac{1}{R_e \cos\phi} \frac{\partial u}{\partial \lambda_e} + \frac{1}{R_e \cos\phi} \frac{\partial}{\partial \phi}\left(v \cos\phi\right) + \frac{\partial w}{\partial z} \qquad (4.11)$$

# Newton's Second Law of Motion

Newton's second law of motion

$$F = Ma$$

Inertial acceleration (momentum equation in inertial frame)

$$\mathbf{a}_i = \frac{1}{M_a} \sum \mathbf{F} \qquad (4.12)$$

Inertial acceleration in terms of absolute velocity

$$\mathbf{a}_i = \frac{\mathrm{d}\mathbf{v}_A}{\mathrm{d}t} + \Omega \times \mathbf{v}_A \qquad (4.15)$$

Absolute velocity of a body in motion

$$\mathbf{v}_A = \mathbf{v} + \Omega \times \mathbf{R}_e \qquad (4.13)$$

Expand inertial acceleration

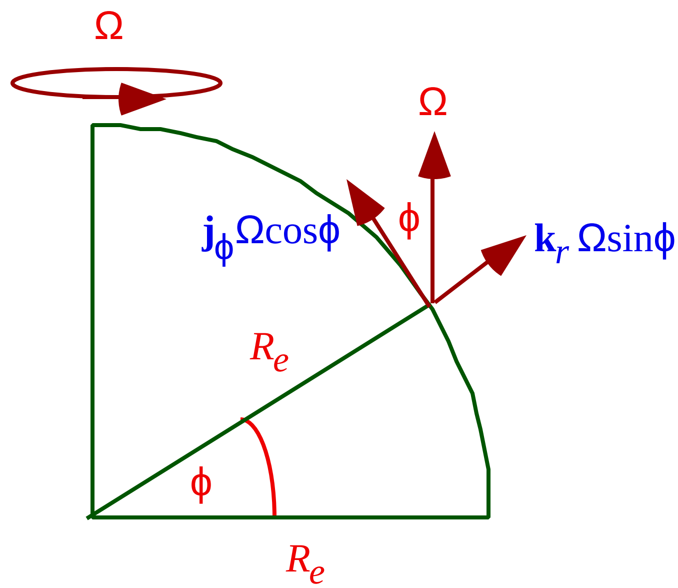
$$\mathbf{a}_i = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \Omega \times \frac{\mathrm{d}\mathbf{R}_e}{\mathrm{d}t} + \Omega \times \mathbf{v} + \Omega \times \left(\Omega \times \mathbf{R}_e\right) \qquad (4.16)$$

# Angular Velocity of Earth

Vector giving angular velocity of earth

$$\Omega = \mathbf{j}_\phi \Omega \cos\phi + \mathbf{k}_r \Omega \sin\phi \qquad (4.14)$$

Figs. 4.3. Components of earth's angular velocity vector.



# Inertial Acceleration

Vector giving radius of earth

$$\mathbf{R}_e = \mathbf{k}_r R_e \tag{4.14}$$

Total derivative of radius of earth vector

$$\frac{\mathrm{d}\mathbf{R}_e}{\mathrm{d}t} = R_e \frac{\mathrm{d}\mathbf{k}_r}{\mathrm{d}t} = \mathbf{i}_\lambda u + \mathbf{j}_\phi v \approx \mathbf{v} \tag{4.17}$$

$$\frac{\mathrm{d}\mathbf{k}_r}{\mathrm{d}t} = \mathbf{i}_\lambda \frac{u}{R_e} + \mathbf{j}_\phi \frac{v}{R_e} \tag{4.28}$$

Substitute radius of earth term into inertial acceleration

$$\mathbf{a}_i = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + 2\Omega \times \mathbf{v} + \Omega \times \left(\Omega \times \mathbf{R}_e\right) = \mathbf{a}_l + \mathbf{a}_c + \mathbf{a}_r \tag{4.18}$$

Local, Coriolis, and earth's centripetal acceleration

$$\mathbf{a}_l = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{4.19}$$

$$\mathbf{a}_c = 2\Omega \times \mathbf{v} \tag{4.19}$$

$$\mathbf{a}_r = \Omega \times \left(\Omega \times \mathbf{R}_e\right) \tag{4.19}$$

# Momentum Equation

Expand both sides of (4.12)

$$\mathbf{a}_l + \mathbf{a}_c + \mathbf{a}_r = \frac{1}{M_a}\left(\mathbf{F}_g^* + \mathbf{F}_p + \mathbf{F}_v\right) \qquad (4.20)$$

Treat Coriolis and earth's centripetal terms as apparent forces

$$\mathbf{a}_c = \mathbf{F}_c / M_a \qquad\qquad \mathbf{a}_r = -\mathbf{F}_r / M_a$$

Momentum equation from reference frame fixed on earth

$$\mathbf{a}_l = \frac{1}{M_a}\left(\mathbf{F}_r - \mathbf{F}_c + \mathbf{F}_g^* + \mathbf{F}_p + \mathbf{F}_v\right) \qquad (4.21)$$

# Local Acceleration

Expand local acceleration term

$$\mathbf{a}_l = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot \quad)\mathbf{v} \qquad (4.22)$$

Expand left side in Cartesian / altitude coordinates

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w)}{\mathrm{d}t} = \mathbf{i}\frac{\mathrm{d}u}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}v}{\mathrm{d}t} + \mathbf{k}\frac{\mathrm{d}w}{\mathrm{d}t} \qquad (4.23)$$

Expand right side in Cartesian / altitude coordinates

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot \quad)\mathbf{v} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w)$$

$$\mathbf{i}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \mathbf{j}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$+\mathbf{k}\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \qquad (4.24)$$

Horizontal magnitude --> $10^{-4}$ m $s^{-2}$
Vertical magnitude over large scales --> $10^{-7}$ m $s^{-2}$

# Local Acceleration

Expand left side of (4.22) in spherical / altitude coordinates

$$\frac{d\mathbf{v}}{dt} = \frac{d\left(\mathbf{i}_\lambda u + \mathbf{j}_\phi v + \mathbf{k}_r w\right)}{dt} \qquad (4.25)$$

$$= \left(\mathbf{i}_\lambda \frac{du}{dt} + u\frac{d\mathbf{i}_\lambda}{dt}\right) + \left(\mathbf{j}_\phi \frac{dv}{dt} + v\frac{d\mathbf{j}_\phi}{dt}\right) + \left(\mathbf{k}_r \frac{dw}{dt} + w\frac{d\mathbf{k}_r}{dt}\right)$$

Total derivative in spherical / altitude coordinates

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{1}{R_e \cos\phi}\frac{\partial}{\partial \lambda_e} + v\frac{1}{R_e}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial z} \qquad (4.26)$$

Total derivatives of unit vectors

$$\frac{d\mathbf{i}_\lambda}{dt} = \mathbf{j}_\phi \frac{u\tan\phi}{R_e} - \mathbf{k}_r \frac{u}{R_e} \qquad (4.28)$$

$$\frac{d\mathbf{j}_\phi}{dt} = -\mathbf{i}_\lambda \frac{u\tan\phi}{R_e} - \mathbf{k}_r \frac{v}{R_e} \qquad (4.28)$$

$$\frac{d\mathbf{k}_r}{dt} = \mathbf{i}_\lambda \frac{u}{R_e} + \mathbf{j}_\phi \frac{v}{R_e} \qquad (4.28)$$

Substitute (4.28) into (4.25)

$$\frac{d\mathbf{v}}{dt} = \mathbf{i}_\lambda\left(\frac{du}{dt} - \frac{uv\tan\phi}{R_e} + \frac{uw}{R_e}\right) + \mathbf{j}_\phi\left(\frac{dv}{dt} + \frac{u^2\tan\phi}{R_e} + \frac{vw}{R_e}\right) + \mathbf{k}_r\left(\frac{dw}{dt} - \frac{u^2}{R_e} - \frac{v^2}{R_e}\right)$$

(4.29)

# Total Derivative of Velocity

Example 4.2.

| | | | |
|---|---|---|---|
| $u$ | $= 20\ \mathrm{m\ s^{-1}}$ | $\Delta x$ | $= 500$ km |
| $v$ | $= 10\ \mathrm{m\ s^{-1}}$ | $\Delta y$ | $= 500$ km |
| $w$ | $= 0.01\ \mathrm{m\ s^{-1}}$ | $\Delta z$ | $= 10$ km |
| $\phi$ | $= 45\ ^{o}\mathrm{N}$ | $R_e$ | $= 6371$ km |

$\frac{du}{dt} = 8 \times 10^{-4}$ $\quad \frac{uv\tan\phi}{R_e} = 3.1 \times 10^{-5}$ $\quad \frac{uw}{R_e} \approx = 3.1 \times 10^{-8}$

$\frac{dv}{dt} = 2 \times 10^{-4}$ $\quad \frac{u^2\tan\phi}{R_e} = 6.3 \times 10^{-5}$ $\quad \frac{vw}{R_e} = 1.6 \times 10^{-8}$

$\frac{dw}{dt} = 1 \times 10^{-8}$ $\quad \frac{u^2}{R_e} = 6.3 \times 10^{-5}$ $\quad \frac{v^2}{R_e} = 1.6 \times 10^{-5}$

All in units of m s$^{-2}$

Simplify local acceleration

$$\frac{d\mathbf{v}}{dt} = \mathbf{i}_\lambda \left( \frac{du}{dt} - \frac{uv\tan\phi}{R_e} \right) + \mathbf{j}_\phi \left( \frac{dv}{dt} + \frac{u^2\tan\phi}{R_e} \right) + \mathbf{k}_r \frac{dw}{dt} \qquad (4.30)$$

# Local Acceleration

Expand right side of (4.22) in spherical / altitude coordinates

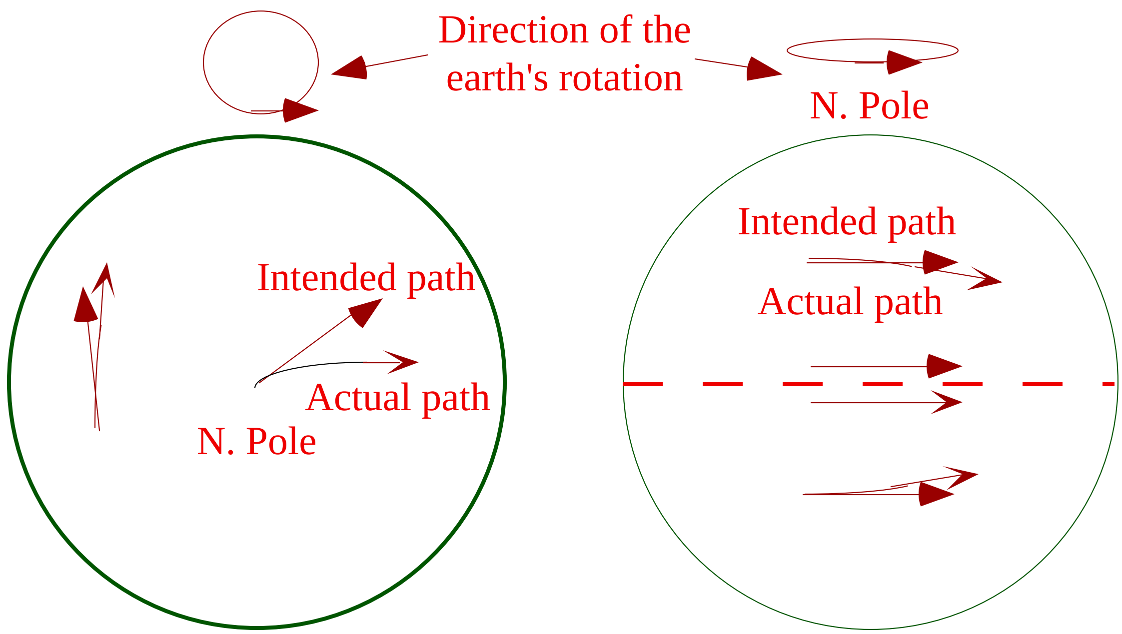
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{i}_\lambda\left(\frac{\partial u}{\partial t} + \frac{u}{R_e\cos\phi}\frac{\partial u}{\partial \lambda_e} + \frac{v}{R_e}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} - \frac{uv\tan\phi}{R_e}\right)$$

$$+\mathbf{j}_\phi\left(\frac{\partial v}{\partial t} + \frac{u}{R_e\cos\phi}\frac{\partial v}{\partial \lambda_e} + \frac{v}{R_e}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + \frac{u^2\tan\phi}{R_e}\right)$$

$$+\mathbf{k}_r\left(\frac{\partial w}{\partial t} + \frac{u}{R_e\cos\phi}\frac{\partial w}{\partial \lambda_e} + \frac{v}{R_e}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial z}\right) \qquad (4.31)$$

# Coriolis Force

Fig. 4.4. Example of Coriolis force deflections.



# Coriolis Force

Apparent Coriolis force per unit mass

$$\frac{\mathbf{F}_c}{M_a} = 2\Omega \times \mathbf{v} = 2\Omega \begin{vmatrix} \mathbf{i}_\lambda & \mathbf{j}_\phi & \mathbf{k}_r \\ 0 & \cos\phi & \sin\phi \\ u & v & w \end{vmatrix} \qquad (4.32)$$

$$= \mathbf{i}_\lambda 2\Omega\left(w\cos\phi - v\sin\phi\right) + \mathbf{j}_\phi 2\Omega u \sin\phi - \mathbf{k}_r 2\Omega u \cos\phi$$

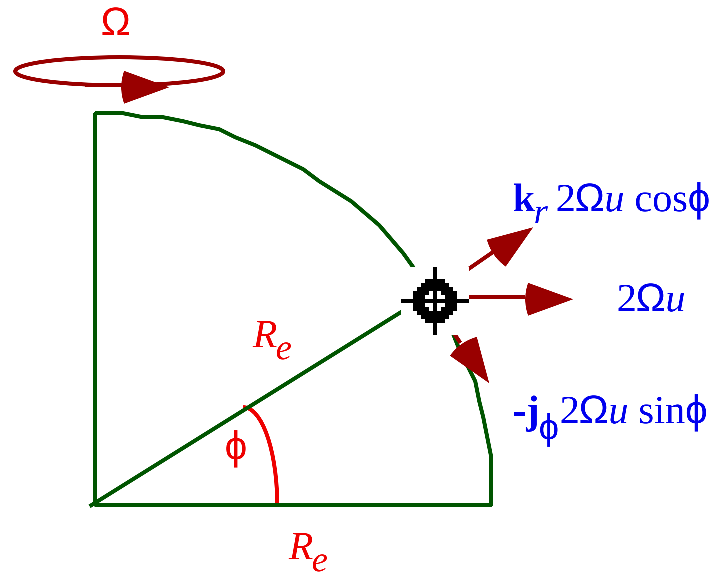
Consider only zonal (west-east) velocity

$$\frac{\mathbf{F}_c}{M_a} = 2\Omega \times \mathbf{v} = \mathbf{j}_\phi 2\Omega u \sin\phi - \mathbf{k}_r 2\Omega u \cos\phi \qquad (4.33)$$

Equate local acceleration to Coriolis force ($\mathbf{a}_l = \mathrm{d}\mathbf{v}/\mathrm{d}t$)

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\mathbf{j}_\phi 2\Omega u \sin\phi + \mathbf{k}_r 2\Omega u \cos\phi$$

Figs. 4.5. Coriolis acceleration components when the Coriolis force acts on a zonal wind



# Coriolis Force

Eliminate vertical velocity term
Eliminate **k** term

Coriolis force per unit mass

$$\frac{\mathbf{F}_c}{M_a} = 2\Omega \times \mathbf{v} \approx -\mathbf{i}_\lambda 2\Omega v \sin\phi + \mathbf{j}_\phi 2\Omega u \sin\phi \qquad (4.34)$$

Coriolis parameter

$$f = 2\Omega \sin\phi \qquad (4.35)$$

Rewrite Coriolis force

$$\frac{\mathbf{F}_c}{M_a} \approx -\mathbf{i}_\lambda f v + \mathbf{j}_\phi f u = f \begin{vmatrix} \mathbf{i}_\lambda & \mathbf{j}_\phi & \mathbf{k}_r \\ 0 & 0 & 1 \\ u & v & 0 \end{vmatrix} = f\mathbf{k}_r \times \mathbf{v}_h \qquad (4.36)$$

Magnitude of the Coriolis force per unit mass

$$\frac{|\mathbf{F}_c|}{M_a} = f|\mathbf{v}_h| = f\sqrt{u^2 + v^2}$$

Example 4.4.

| | | |
|---|---|---|
| | $|\mathbf{v}_h|$ | $= 10$ m s$^{-1}$ at the North Pole |
| ----> | $|\mathbf{F}_c| / M_a$ | $= 0.001454$ m s$^{-2}$ |

# Gravitational Acceleration

True gravitational force vector

$$\frac{\mathbf{F}_g^*}{M_a} = -\mathbf{k}_r^* g^* \quad (4.37)$$

Newton's law of universal gravitation

$$\mathbf{F}_{12,g} = -\mathbf{r}_{21} \frac{G M_1 M_2}{r_{21}^3} \quad (4.38)$$

Magnitude of true gravitational force

$$F_g = \frac{G M_1 M_2}{r_{21}^2}$$

Gravitational force vector for earth

$$\frac{\mathbf{F}_g^*}{M_a} = -\mathbf{k}_r^* \frac{G M_e}{R_e^2} \quad (4.39)$$

Equate (4.37) and (4.39)

$$\frac{\left|\mathbf{F}_g^*\right|}{M_a} = g^* = \frac{G M_e}{R_e^2} \quad (4.40)$$

Example 4.5

| | | |
|---|---|---|
| | $M_e$ | $= 5.98 \times 10^{24}$ kg |
| | $R_e$ | $= 6.37 \times 10^{6}$ m. |
| ----> | $g^*$ | $= 9.833$ m s$^{-2}$ |

# Apparent Centrifugal Force

Angular velocity and radius vectors for a spherical earth

$$\boldsymbol{\Omega} = \mathbf{j}^*_\phi \Omega \cos\phi + \mathbf{k}^*_r \Omega \sin\phi \qquad \mathbf{R}_e = \mathbf{k}^*_r R_e \tag{4.42}$$

Cross product of angular velocity vector with radius vector

$$\boldsymbol{\Omega} \times \mathbf{R}_e = \Omega \begin{vmatrix} \mathbf{i}^* & \mathbf{j}^* & \mathbf{k}^* \\ 0 & \cos\phi & \sin\phi \\ 0 & 0 & R_e \end{vmatrix} = \mathbf{i}^* R_e \Omega \cos\phi \tag{4.43}$$

Apparent centrifugal force per unit mass

$$\frac{\mathbf{F}_r}{M_a} = -\mathbf{a}_r = -\boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \mathbf{R}_e\right) = -\Omega \begin{vmatrix} \mathbf{i}^*_\lambda & \mathbf{j}^*_\phi & \mathbf{k}^*_r \\ 0 & \cos\phi & \sin\phi \\ R_e \cos\phi & 0 & 0 \end{vmatrix}$$

$$= -\mathbf{j}^*_\phi R_e \Omega^2 \cos\phi \sin\phi + \mathbf{k}^*_r R_e \Omega^2 \cos^2\phi \tag{4.41}$$

# Effective Gravity

Add apparent centrifugal and gravitational force vectors

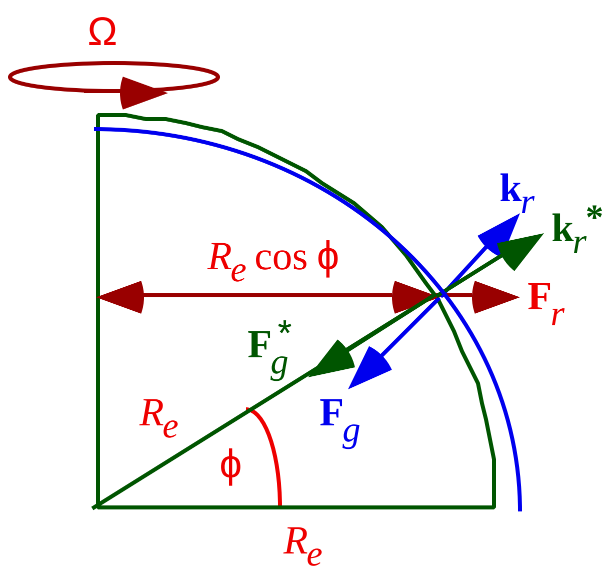
$$\frac{\mathbf{F}_g}{M_a} = \frac{\mathbf{F}_g^*}{M_a} + \frac{\mathbf{F}_r}{M_a} = -\mathbf{j}_\phi^* R_e \Omega^2 \cos\phi \sin\phi + \mathbf{k}_r^* \left( R_e \Omega^2 \cos^2\phi - g^* \right) = -\mathbf{k}_r g$$

(4.44)

Effective gravitational acceleration

$$g = \left[ \left( R_e \Omega^2 \cos\phi \sin\phi \right)^2 + \left( g^* - R_e \Omega^2 \cos^2\phi \right)^2 \right]^{1/2} \qquad (4.45)$$

Fig. 4.6. Gravitational acceleration components for earth.



# Effective Gravity

Example

$g$ = 9.799 m $s^{-2}$ at the equator

$g$ = 9.833 m $s^{-2}$ at the poles

----> $g_0$ = 981 cm $s^{-2}$ = average surface effective gravity

Difference between $g$ at the equator and pole = 0.34%

Difference between $R_e$ at the equator and pole = 0.33%

Example 4.6.

$z$ = 100 km

----> $g^*$ = 9.531 m $s^{-2}$ (3.1% lower than surface value)

----> $g$ = 9.497 m $s^{-2}$ (3.1% lower than surface value)

Gravity varies more with altitude than with latitude

# Geopotential

Work done against gravity to raise a unit mass of air from sea level to a given altitude = gravitational potential energy of air per unit mass.

Geopotential vector ($m^2\ s^{-2}$)

$$\boldsymbol{\Phi}(z) = \mathbf{k}_r \Phi(z) \qquad \Phi(z) = \int_0^z g(z)\mathrm{d}z \qquad (4.46)$$

Geopotential height

$$Z = \frac{\Phi(z)}{g_0} \qquad (4.47)$$

Geopotential vector for lower atmosphere ($g \approx g_0$)

$$\Phi(z) \approx gz \qquad (4.48)$$

Gradient of geopotential

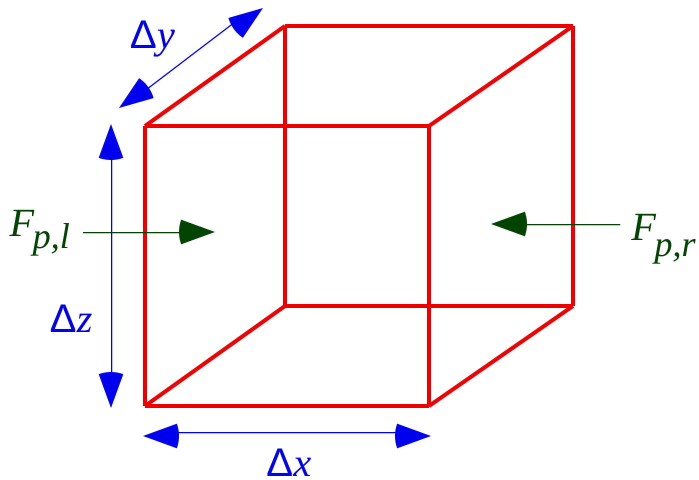
$$\Phi = \mathbf{k}_r \frac{\partial \Phi}{\partial z} \approx \mathbf{k}_r g \qquad (4.48)$$

Substitute (4.48) into (4.44)

$$\frac{\mathbf{F}_g}{M_a} = -\mathbf{k}_r g = - \quad \Phi \qquad (4.49)$$

# Pressure Gradient Force

Fig. 4.7. Example of pressure gradient force.




Forces acting on right and left sides of box

$$F_{p,r} = -\left(p_c + \frac{\partial p}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z \qquad F_{p,l} = \left(p_c - \frac{\partial p}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z \tag{4.50}$$

Sum forces

$$F_{p,x} = -\frac{\partial p_a}{\partial x}\Delta x \Delta y \Delta z$$

Mass of air parcel

$$M_a = \rho_a \Delta x \Delta y \Delta z$$

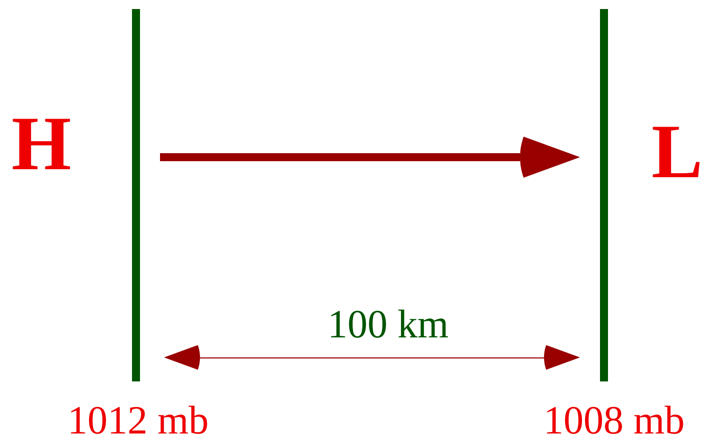
Pressure gradient force per unit mass in *x*-direction

$$\frac{F_{p,x}}{M_a} = -\frac{1}{\rho_a}\frac{\partial p_a}{\partial x} \tag{4.51}$$

# Pressure Gradient Force

Example 4.7.

Fig. 4.8. Example of a pressure gradient.




d$x$ = 100 km

$\rho_a$ = 1.2 kg m$^{-3}$

$$\frac{1}{\rho_a}\frac{\partial p_a}{\partial x} \approx \frac{1}{1.2\ \text{kg m}^{-3}}\left(\frac{1012-1008\ \text{mb}}{10^5\ \text{m}}\right)\frac{100\ \text{kg m}^{-1}\ \text{s}^{-2}}{\text{mb}} = 0.0033\ \text{m s}^{-2}$$

# Pressure Gradient Force per Unit Mass

Cartesian / altitude coordinates

$$\frac{\mathbf{F}_p}{M_a} = -\frac{1}{\rho_a}\nabla p_a = -\frac{1}{\rho_a}\left(\mathbf{i}\frac{\partial p_a}{\partial x} + \mathbf{j}\frac{\partial p_a}{\partial y} + \mathbf{k}\frac{\partial p_a}{\partial z}\right). \qquad (4.52)$$

Spherical / altitude coordinates

$$\frac{\mathbf{F}_p}{M_a} = -\frac{1}{\rho_a}\nabla p_a = -\frac{1}{\rho_a}\left(\mathbf{i}_\lambda \frac{1}{R_e \cos\phi}\frac{\partial p_a}{\partial \lambda_e} + \mathbf{j}_\phi \frac{1}{R_e}\frac{\partial p_a}{\partial \phi} + \mathbf{k}_r \frac{\partial p_a}{\partial z}\right) \qquad (4.53)$$

Example 4.8.

| | | | |
|---|---|---|---|
| $z$ | = 0 | $p_a$ | = 1013 mb |
| $z$ | = 100 m | $p_a$ | = 1000 mb |
| $\rho_a$ | = 1.2 kg m$^{-3}$ | | |

Pressure gradient force per unit mass in the vertical

$$\frac{1}{\rho_a}\frac{\partial p_a}{\partial z} \approx \frac{1}{1.2\ \text{kg m}^{-3}}\left(\frac{1013-1000\ \text{mb}}{100\ \text{m}}\right)\frac{100\ \text{kg m}^{-1}\ \text{s}^{-2}}{\text{mb}} = 10.80\ \text{m s}^{-2}$$

3000 times the horizontal pressure gradient acceleration

# Viscous Force

Shear

Change of wind velocity with height

Shearing stress

Viscous force per unit area resulting from shear

Shearing stress in the $x$-$z$ plane (N $m^{-2}$)

Force per unit area in the $x$-direction acting on the $x$-$y$ plane (normal to the $z$-direction).

$$\tau_{zx} = \eta_a \frac{\partial u}{\partial z} \qquad (4.54)$$

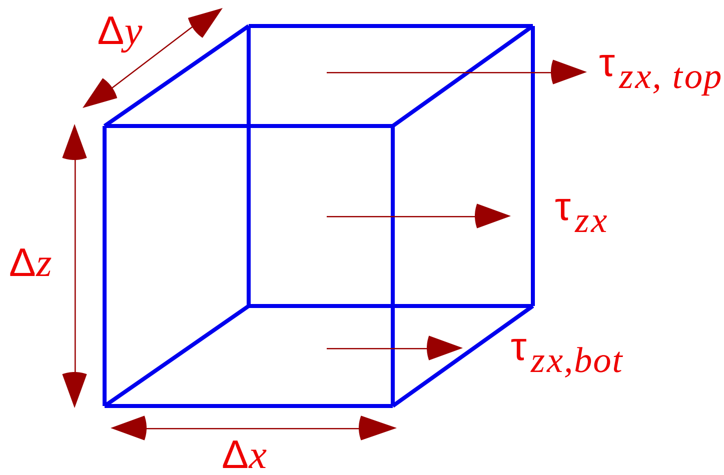
The dynamic viscosity of air (kg $m^{-1}$ $s^{-1}$)

Ratio of shearing stress to shear

$$\eta_a \approx 1.8325 \times 10^{-5} \left(\frac{416.16}{T + 120}\right)\left(\frac{T}{296.16}\right)^{1.5} \qquad (4.55)$$

# Viscous Force

Fig. 4.9. Example of shear stress in the *x*-direction




Shearing stress at the top and bottom

$$\tau_{zx,top} = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2} \qquad \tau_{zx,bot} = \tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2} \qquad (4.56)$$

Net viscous force on the parcel in the *x*-direction

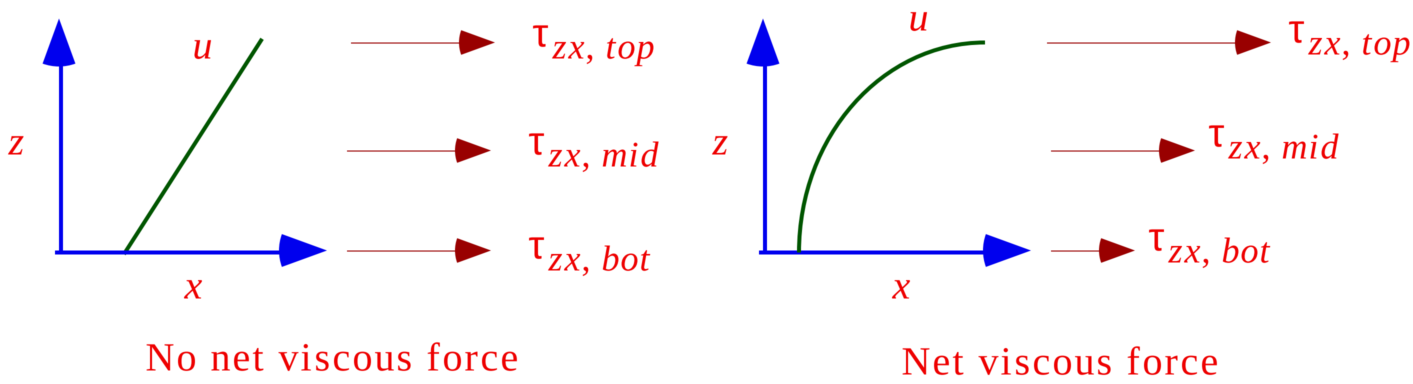
$$\frac{F_{v,zx}}{M_a} = \left(\tau_{zx,top} - \tau_{zx,bot}\right)\frac{\Delta x \Delta y}{\rho_a \Delta x \Delta y \Delta z} = \frac{1}{\rho_a}\frac{\partial \tau_{zx}}{\partial z} \qquad (4.57)$$

Substitute shearing stress

$$\frac{F_{v,zx}}{M_a} = \frac{1}{\rho_a}\frac{\partial}{\partial z}\left(\eta_a \frac{\partial u}{\partial z}\right) \approx \frac{\eta_a}{\rho_a}\frac{\partial^2 u}{\partial z^2} \qquad (4.58)$$

# Viscous Force

Fig. 4.10. Linear versus nonlinear wind shear




Expand (4.58)

$$\frac{\mathbf{F}_v}{M_a} = \frac{\eta_a}{\rho_a} \nabla^2 \mathbf{v} = \nu_a \nabla^2 \mathbf{v} \qquad (4.59)$$

Kinematic viscosity of air

$$\nu_a = \eta_a / \rho_a$$

Velocity term

$$\nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v} \qquad (4.60)$$

$$= \mathbf{i}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mathbf{j}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \mathbf{k}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

# Viscous Force

Example 4.9

| | | | |
|---|---|---|---|
| $z_1$ | = 1 km | $u_1$ | = 10 m s$^{-1}$ |
| $z_2$ | = 1.25 km | $u_2$ | = 14 m s$^{-1}$ |
| $z_3$ | = 1.5 km | $u_3$ | = 20 m s$^{-1}$ |
| $T$ | = 280 K | $\rho_a$ | = 1.085 kg m$^{-3}$ |

----> $\eta_a$ = 1.753 x 10$^{-5}$ kg m$^{-1}$ s$^{-2}$

---->

$$\frac{F_{v,zx}}{M_a} \approx \frac{\eta_a}{\rho_a}\frac{1}{(z_3 - z_1)/2}\left(\frac{u_3 - u_2}{z_3 - z_2} - \frac{u_2 - u_1}{z_2 - z_1}\right) = 5.17 \text{ x } 10^{-10} \text{ m s}^{-2}$$

Viscous forces aloft are small

Example 4.10

| | | | |
|---|---|---|---|
| $z_1$ | = 0 m | $u_1$ | = 0 m s$^{-1}$ |
| $z_2$ | = 0.05 m | $u_2$ | = 0.4 m s$^{-1}$ |
| $z_3$ | = 0.1 m | $u_3$ | = 1 m s$^{-1}$ |
| $T$ | = 288 K | $\rho_a$ | = 1.225 kg m$^{-3}$ |

----> $\eta_a$ = 1.753 x 10$^{-5}$ kg m$^{-1}$ s$^{-2}$

---->

$$\frac{F_{v,zx}}{M_a} \approx \frac{\eta_a}{\rho_a}\frac{1}{(z_3 - z_1)/2}\left(\frac{u_3 - u_2}{z_3 - z_2} - \frac{u_2 - u_1}{z_2 - z_1}\right) = 1.17 \text{ x } 10^{-3} \text{ m s}^{-2}$$

Viscous forces at the surface are comparable with horizontal pressure gradient accelerations

# Turbulent Flux Divergence

Local acceleration

$$\mathbf{a}_l = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} \qquad (4.22)$$

Continuity equation for air

$$\frac{\partial \rho_a}{\partial t} + \rho_a(\nabla\cdot\mathbf{v}) + (\mathbf{v}\cdot\nabla)\rho_a = 0 \qquad (3.20)$$

Combine

$$\rho_a\mathbf{a}_l = \rho_a\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] + \mathbf{v}\left[\frac{\partial \rho_a}{\partial t} + \nabla\cdot(\mathbf{v}\rho_a)\right] \qquad (4.61)$$

Decompose variables

$$\rho_a \approx \bar{\rho}_a \quad \mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$$

Take time- and grid volume-average of (4.61)

Local acceleration term

$$\bar{\mathbf{a}}_l = \frac{\partial \bar{\mathbf{v}}}{\partial t} + (\bar{\mathbf{v}}\cdot\nabla)\bar{\mathbf{v}} \qquad (4.64)$$

Turbulent flux divergence term

$$\mathbf{a}'_l = \frac{\mathbf{F}_t}{M_a} = \frac{1}{\bar{\rho}_a}\left[\overline{\bar{\rho}_a(\mathbf{v}'\cdot\nabla)\mathbf{v}' + \mathbf{v}'\nabla\cdot(\mathbf{v}'\bar{\rho}_a)}\right] \qquad (4.64)$$

# Turbulent Flux Divergence

Expand

$$\frac{\mathbf{F}_t}{M_a} = \mathbf{i}\frac{1}{\rho_a}\left[\frac{\partial\left(\rho_a\overline{u'u'}\right)}{\partial x} + \frac{\partial\left(\rho_a\overline{v'u'}\right)}{\partial y} + \frac{\partial\left(\rho_a\overline{w'u'}\right)}{\partial z}\right]$$

$$+\mathbf{j}\frac{1}{\rho_a}\left[\frac{\partial\left(\rho_a\overline{u'v'}\right)}{\partial x} + \frac{\partial\left(\rho_a\overline{v'v'}\right)}{\partial y} + \frac{\partial\left(\rho_a\overline{w'v'}\right)}{\partial z}\right]$$

$$+\mathbf{k}\frac{1}{\rho_a}\left[\frac{\partial\left(\rho_a\overline{u'w'}\right)}{\partial x} + \frac{\partial\left(\rho_a\overline{v'w'}\right)}{\partial y} + \frac{\partial\left(\rho_a\overline{w'w'}\right)}{\partial z}\right] \qquad (4.65)$$

# Diffusion Coefficients

Vertical kinematic turbulent fluxes from *K*-theory

$$\overline{w'u'} = -K_{m,zx}\frac{\partial \bar{u}}{\partial z} \qquad \overline{w'v'} = -K_{m,zy}\frac{\partial \bar{v}}{\partial z} \qquad (4.66)$$

Substitute fluxes into turbulent flux divergence equation

$$\frac{\mathbf{F}_t}{M_a} = -\mathbf{i}\frac{1}{\rho_a}\left[\frac{\partial}{\partial x}\left(\rho_a K_{m,xx}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho_a K_{m,yx}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho_a K_{m,zx}\frac{\partial u}{\partial z}\right)\right]$$

$$-\mathbf{j}\frac{1}{\rho_a}\left[\frac{\partial}{\partial x}\left(\rho_a K_{m,xy}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho_a K_{m,yy}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho_a K_{m,zy}\frac{\partial v}{\partial z}\right)\right]$$

$$-\mathbf{k}\frac{1}{\rho_a}\left[\frac{\partial}{\partial x}\left(\rho_a K_{m,xz}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho_a K_{m,yz}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho_a K_{m,zz}\frac{\partial w}{\partial z}\right)\right]$$

(4.67)

Vector / tensor notation

$$\frac{\mathbf{F}_t}{M_a} = -\frac{1}{\rho_a}\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)\mathbf{v} \qquad (4.69)$$

$$\mathbf{K}_m = \begin{bmatrix} K_{m,xx} & 0 & 0 \\ 0 & K_{m,yx} & 0 \\ 0 & 0 & K_{m,zx} \end{bmatrix} \text{ for } u \qquad (4.68)$$

# Diffusion Coefficients

Example 4.11. (vertical diffusion in middle of boundary layer)

| | | | |
|---|---|---|---|
| $K_m$ | $= 50\ \text{m}^2\ \text{s}^{-1}$ | | |
| $z_1$ | $= 300$ m | $u_1$ | $= 10\ \text{m s}^{-1}$ |
| $z_2$ | $= 350$ m | $u_2$ | $= 12\ \text{m s}^{-1}$ |
| $z_3$ | $= 400$ m | $u_3$ | $= 15\ \text{m s}^{-1}$ |

--->

$$\frac{F_{t,zx}}{M_a} = \frac{1}{\rho_a}\frac{\partial}{\partial z}\left(\rho_a K_{m,zx}\frac{\partial u}{\partial z}\right) \approx \frac{K_{m,zx}}{(z_3 - z_1)/2}\left(\frac{u_3 - u_2}{z_3 - z_2} - \frac{u_2 - u_1}{z_2 - z_1}\right) = 0.02\ \text{m s}^{-2}$$

Example 4.12. (horizontal diffusion)

| | | | |
|---|---|---|---|
| $K_m$ | $= 100\ \text{m}^2\ \text{s}^{-1}$ | | |
| $y_1$ | $= 0$ m | $u_1$ | $= 10\ \text{m s}^{-1}$ |
| $y_2$ | $= 500$ m | $u_2$ | $= 9\ \text{m s}^{-1}$ |
| $y_3$ | $= 1000$ m | $u_3$ | $= 7\ \text{m s}^{-1}$ |

--->

$$\frac{F_{t,yx}}{M_a} = \frac{1}{\rho_a}\frac{\partial}{\partial y}\left(\rho_a K_{m,yx}\frac{\partial u}{\partial y}\right) \approx \frac{K_{m,yx}}{(y_3 - y_1)/2}\left(\frac{u_3 - u_2}{y_3 - y_2} - \frac{u_2 - u_1}{y_2 - y_1}\right) = -0.0004\ \text{m}$$

# Complete Momentum Equation

Table 4.1. Magnitude of terms in momentum equation

| Term | Acceleration or Force / Mass Expression | Horizontal Accel. ($m\ s^{-2}$) | Vertical Accel. ($m\ s^{-2}$) |
|---|---|---|---|
| Local acceleration | $\bar{\mathbf{a}}_l = \frac{d\bar{\mathbf{v}}}{dt} = \frac{\partial \bar{\mathbf{v}}}{\partial t} + (\bar{\mathbf{v}} \cdot \ \ )\bar{\mathbf{v}}$ | $10^{-4}$ | $10^{-7}$-1 |
| Coriolis force per unit mass | $\frac{\mathbf{F}_c}{M_a} = f\mathbf{k} \times \mathbf{v}$ | $10^{-3}$ | 0 |
| Effective gravitational force per unit mass | $\frac{\mathbf{F}_g}{M_a} = \frac{\mathbf{F}_g^*}{M_a} + \frac{\mathbf{F}_r}{M_a} = -\ \ \Phi$ | 0 | 10 |
| Pressure gradient force per unit mass | $\frac{\mathbf{F}_p}{M_a} = -\frac{1}{\rho_a}\ \ p_a$ | $10^{-3}$ | 10 |
| Viscous force per unit mass | $\frac{\mathbf{F}_v}{M_a} = \frac{\eta_a}{\rho_a}\ \ ^2\mathbf{v}$ | $10^{-12}$-$10^{-3}$ | $10^{-15}$-$10^{-5}$ |
| Turbulent flux divergence of momentum | $\frac{\mathbf{F}_t}{M_a} = -\frac{1}{\rho_a}(\ \ \cdot \rho_a \mathbf{K}_m\ \ )\mathbf{v}$ | 0-0.005 | 0-1 |

Momentum equation

$$\frac{d\mathbf{v}}{dt} = -f\mathbf{k} \times \mathbf{v} - \ \ \Phi - \frac{1}{\rho_a}\ \ p_a + \frac{\eta_a}{\rho_a}\ \ ^2\mathbf{v} + \frac{1}{\rho_a}(\ \ \cdot \rho_a \mathbf{K}_m\ \ )\mathbf{v} \quad (4.70)$$

# Momentum Equations in Cartesian / Altitude Coordinates

*u*-direction

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = fv - \frac{1}{\rho_a}\frac{\partial p_a}{\partial x} \qquad (4.72)$$

$$+\frac{1}{\rho_a}\left[\frac{\partial}{\partial x}\left(\rho_a K_{m,xx}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho_a K_{m,yx}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho_a K_{m,zx}\frac{\partial u}{\partial z}\right)\right]$$

*v*-direction

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -fu - \frac{1}{\rho_a}\frac{\partial p_a}{\partial y} \qquad (4.73)$$

$$+\frac{1}{\rho_a}\left[\frac{\partial}{\partial x}\left(\rho_a K_{m,xy}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho_a K_{m,yy}\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho_a K_{m,zy}\frac{\partial v}{\partial z}\right)\right]$$

*w* direction

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -g - \frac{1}{\rho_a}\frac{\partial p_a}{\partial z} \qquad (4.74)$$

$$+\frac{1}{\rho_a}\left[\frac{\partial}{\partial x}\left(\rho_a K_{m,xz}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho_a K_{m,yz}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho_a K_{m,zz}\frac{\partial w}{\partial z}\right)\right]$$

# Momentum Equations in Spherical / Altitude Coordinates

*u*-direction

$$\frac{\partial u}{\partial t}+\frac{u}{R_e\cos\phi}\frac{\partial u}{\partial\lambda_e}+\frac{v}{R_e}\frac{\partial u}{\partial\phi}+w\frac{\partial u}{\partial z}=\frac{uv\tan\phi}{R_e}+fv-\frac{1}{\rho_a R_e\cos\phi}\frac{\partial p_a}{\partial\lambda_e}$$

$$+\frac{1}{\rho_a}\left[\frac{1}{R_e^2\cos\phi}\frac{\partial}{\partial\lambda_e}\left(\frac{\rho_a K_{m,xx}}{\cos\phi}\frac{\partial u}{\partial\lambda_e}\right)+\frac{1}{R_e^2}\frac{\partial}{\partial\phi}\left(\rho_a K_{m,yx}\frac{\partial u}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\rho_a K_{m,zx}\frac{\partial u}{\partial z}\right)\right] \qquad (4.75)$$

*v*-direction

$$\frac{\partial v}{\partial t}+\frac{u}{R_e\cos\phi}\frac{\partial v}{\partial\lambda_e}+\frac{v}{R_e}\frac{\partial v}{\partial\phi}+w\frac{\partial v}{\partial z}=-\frac{u^2\tan\phi}{R_e}-fu-\frac{1}{\rho_a R_e}\frac{\partial p_a}{\partial\phi}$$

$$+\frac{1}{\rho_a}\left[\frac{1}{R_e^2\cos\phi}\frac{\partial}{\partial\lambda_e}\left(\frac{\rho_a K_{m,xy}}{\cos\phi}\frac{\partial v}{\partial\lambda_e}\right)+\frac{1}{R_e^2}\frac{\partial}{\partial\phi}\left(\rho_a K_{m,yy}\frac{\partial v}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\rho_a K_{m,zy}\frac{\partial v}{\partial z}\right)\right] \qquad (4.76)$$

*w* direction

$$\frac{\partial w}{\partial t}+\frac{u}{R_e\cos\phi}\frac{\partial w}{\partial\lambda_e}+\frac{v}{R_e}\frac{\partial w}{\partial\phi}+w\frac{\partial w}{\partial z}=-g-\frac{1}{\rho_a}\frac{\partial p_a}{\partial z} \qquad (4.77)$$

$$+\frac{1}{\rho_a}\left[\frac{1}{R_e^2\cos\phi}\frac{\partial}{\partial\lambda_e}\left(\frac{\rho_a K_{m,xz}}{\cos\phi}\frac{\partial w}{\partial\lambda_e}\right)+\frac{1}{R_e^2}\frac{\partial}{\partial\phi}\left(\rho_a K_{m,yz}\frac{\partial w}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\rho_a K_{m,zz}\frac{\partial w}{\partial z}\right)\right]$$

# Scaling the Equation of Motion

Ekman Number

$$\text{Ek} = \frac{\nu_a u / x^2}{uf} \qquad (4.71)$$

Rossby Number

$$\text{Ro} = \frac{u^2 / x}{uf} \qquad (4.71)$$

Froude number

$$\text{Fr}^2 = \frac{w^2 / z}{g} \qquad (4.71)$$

Example 4.13

| | | |
|---|---|---|
| | $\nu_a$ | $= 10^{-6}$ m$^2$ s$^{-1}$ |
| | $u$ | = 10 m s$^{-1}$ |
| | $w$ | = 0.01 m s$^{-1}$ |
| | $x$ | $= 10^6$ m |
| | $z$ | $= 10^4$ m |
| | $f$ | $= 10^{-4}$ s$^{-1}$ |
| ---> | Ek | $= 10^{-14}$ |
| ---> | Ro | = 0.1 |
| ---> | Fr | $= 3 \times 10^{-5}$ |

Viscous accelerations negligible over large scales
Coriolis more important than local horizontal acceleration
Gravity more important than vertical inertial acceleration

# Geostrophic Wind

Geostrophic wind

$$v_g = \frac{1}{f\rho_a}\frac{\partial p_a}{\partial x} \qquad u_g = -\frac{1}{f\rho_a}\frac{\partial p_a}{\partial y} \qquad (4.78)$$

Example 4.14

$\phi$ = 30 °N

$p_a$ = 568 mb

$\rho_a$ = 0.00076 g cm$^{-3}$

$$f = 7.292\times 10^{-5}\,\text{rad s}^{-1}$$

$$u_g = 48.1\ \text{m s}^{-1}$$

Geostrophic wind in vector and cross product notation

$$\mathbf{v}_g = \mathbf{i}u_g + \mathbf{j}v_g = \frac{1}{f\rho_a}\left(-\mathbf{i}\frac{\partial p_a}{\partial y} + \mathbf{j}\frac{\partial p_a}{\partial x}\right) = \frac{1}{f\rho_a}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \frac{\partial p_a}{\partial x} & \frac{\partial p_a}{\partial y} & 0 \end{vmatrix} = \frac{1}{f\rho_a}\mathbf{k}\times \quad {}_z p_a$$

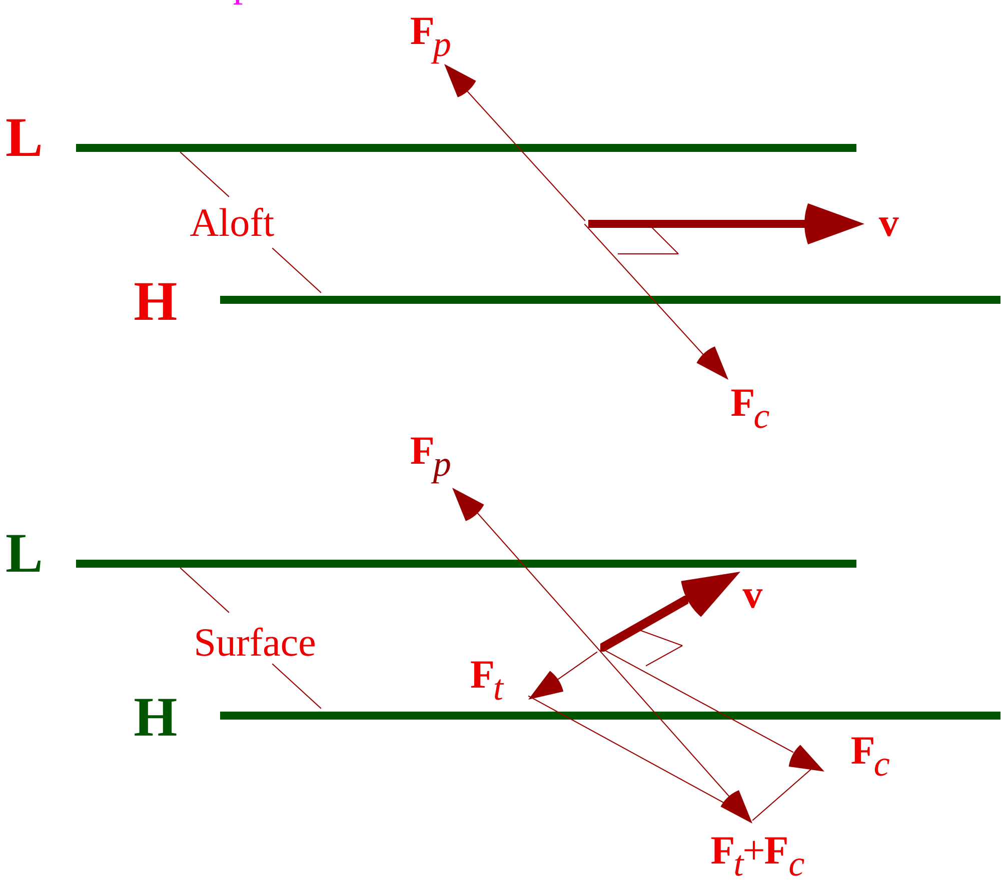
(4.79)

Horizontal gradient operator in altitude coordinate

$$ {}_z = \left(\mathbf{i}\frac{\partial}{\partial x}\right)_z + \left(\mathbf{j}\frac{\partial}{\partial y}\right)_z = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \qquad (4.80)$$

# Surface Winds

Fig. 4.11. Force and wind vectors aloft and at the surface in the Northern Hemisphere.



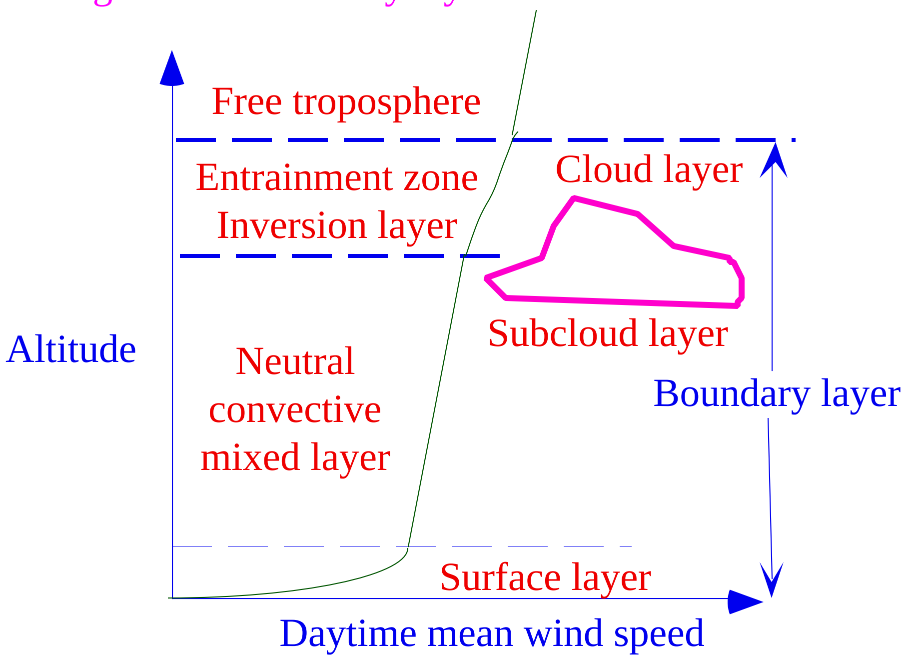

Horizontal equations of motion near the surface

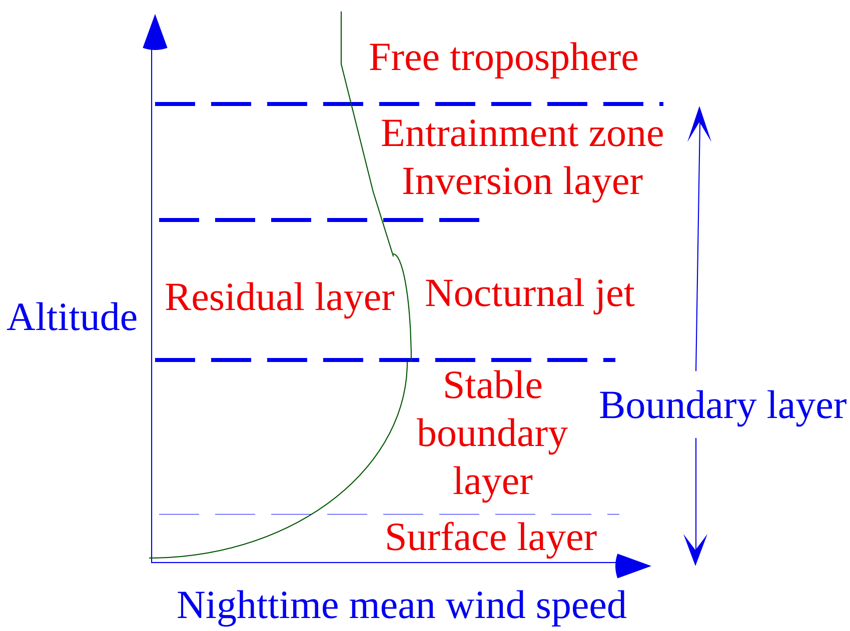
$$-fv = -\frac{1}{\rho_a}\frac{\partial p_a}{\partial x} + \frac{1}{\rho_a}\frac{\partial}{\partial z}\left(\rho_a K_{m,zx}\frac{\partial u}{\partial z}\right) \qquad (4.81)$$

$$fu = -\frac{1}{\rho_a}\frac{\partial p_a}{\partial y} + \frac{1}{\rho_a}\frac{\partial}{\partial z}\left(\rho_a K_{m,zy}\frac{\partial v}{\partial z}\right) \qquad (4.81)$$

# Surface Winds

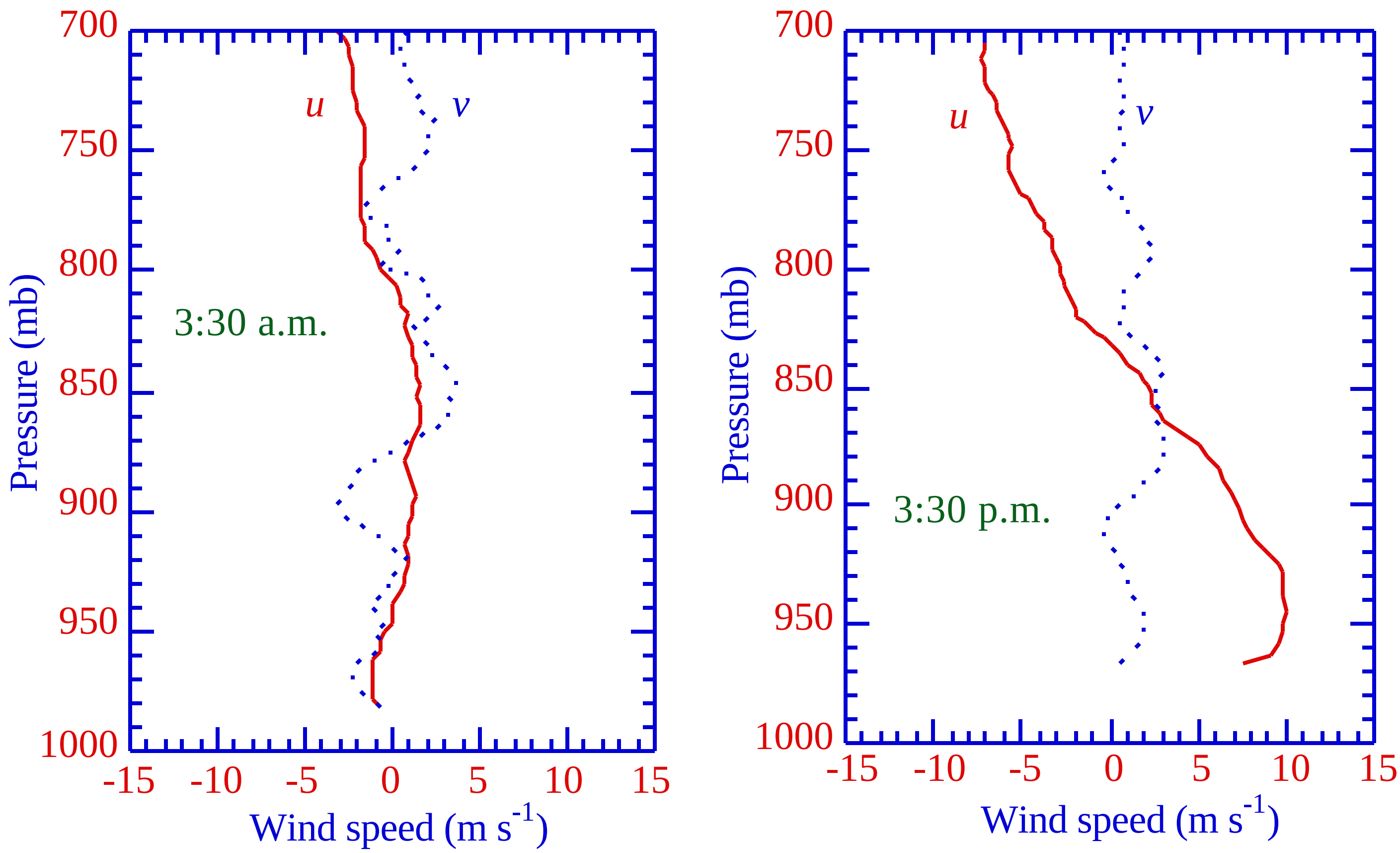
Figs. 4.12 a and b. Variation of wind speed with height during the day and night in the boundary layer.



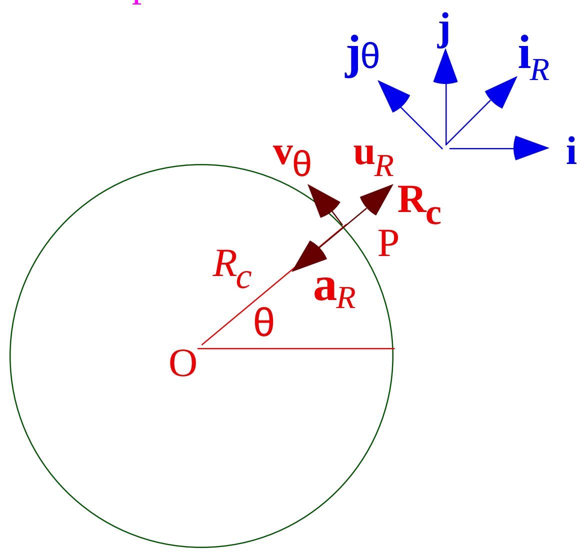
# Morning / Afternoon Observed Winds at Riverside

Fig. 4.13 a and b



# Gradient Wind

Fig. 4.14. Diagram of directional components used for calculating local centripetal acceleration.




Cartesian and cylindrical coordinates

$x = R_c \cos\theta$ $\quad$ $y = R_c \sin\theta$

$R_c^2 = x^2 + y^2$ $\quad$ $\theta = \tan^{-1}(y/x)$

Radial vector

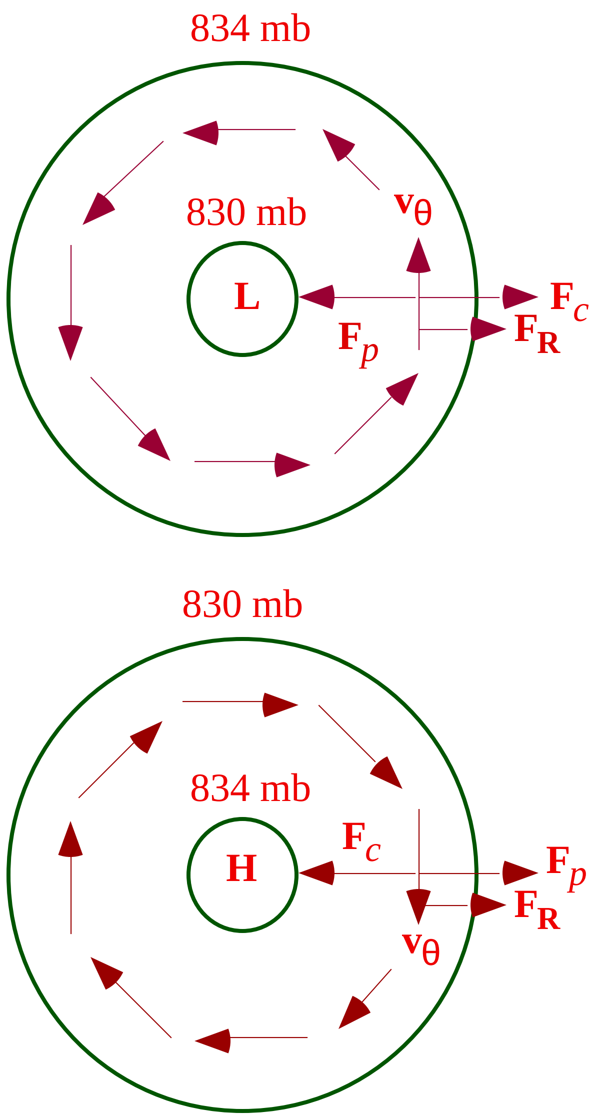
$\mathbf{R}_c = \mathbf{i}_R R_c$

Radial and tangential scalar velocities

$u_\mathrm{R} = \mathrm{d}R_c/\mathrm{d}t$ $\quad$ $v_\theta = R_c\, \mathrm{d}\theta/\mathrm{d}t$

# Gradient Wind

Figs. 4.15 a and b. Gradient winds in N. Hemisphere.



# Gradient Wind

Angular scalar velocity

$$\omega_z = \frac{d\theta}{dt} = \frac{v_\theta}{R_c}$$

Angular velocity vector

$$\boldsymbol{\omega}_z = \mathbf{k}\omega_z$$

Tangential velocity vector

$$\mathbf{v}_\theta = \boldsymbol{\omega}_z \times \mathbf{R}_c = \begin{vmatrix} \mathbf{i}_R & \mathbf{j}_\theta & \mathbf{k} \\ 0 & 0 & v_\theta / R_c \\ R_c & 0 & 0 \end{vmatrix} = \mathbf{j}_\theta v_\theta \qquad (4.82)$$

Centripetal acceleration

$$\mathbf{a}_R = \boldsymbol{\omega}_z \times \mathbf{v}_\theta = \begin{vmatrix} \mathbf{i}_R & \mathbf{j}_\theta & \mathbf{k} \\ 0 & 0 & v_\theta / R_c \\ 0 & v_\theta & 0 \end{vmatrix} = -\mathbf{i}_R \frac{v_\theta^2}{R_c^2} = -\mathbf{i}_R R_c a_R = \mathbf{R}_c a_R$$

(4.83)

# Gradient Wind

Horizontal momentum equations in cylindrical coordinates

$$\frac{du_R}{dt} = fv_\theta - \frac{1}{\rho_a}\frac{\partial p_a}{\partial R_c} + \frac{v_\theta^2}{R_c} \qquad (4.84)$$

$$\frac{dv_\theta}{dt} = -fu_R - \frac{u_R v_\theta}{R_c} \qquad (4.85)$$

Local acceleration in (4.84) is small ( $du_r/dt = 0$ )

- -> Solve for gradient wind

$$v_\theta = -\frac{R_c f}{2} \pm \frac{R_c}{2}\sqrt{f^2 + 4\frac{1}{R_c\rho_a}\frac{\partial p_a}{\partial R_c}} \qquad (4.86)$$

The correct root is the positive root

# Gradient Wind

Example 4.15

Near the center of a hurricane (low pressure)

| | | |
|---|---|---|
| | $\partial p_a / \partial R_c$ | = 45 mb per 100 km |
| | $R_c$ | = 70 km |
| | $\phi$ | = 15$^o$ N latitude |
| | $p_a$ | = 850 mb |
| | $\rho_a$ | = 1.06 kg m$^{-3}$) |
| ----> | $v$ | = 52 m s$^{-1}$ |
| ----> | $v_g$ | = 1123 m s$^{-1}$ |

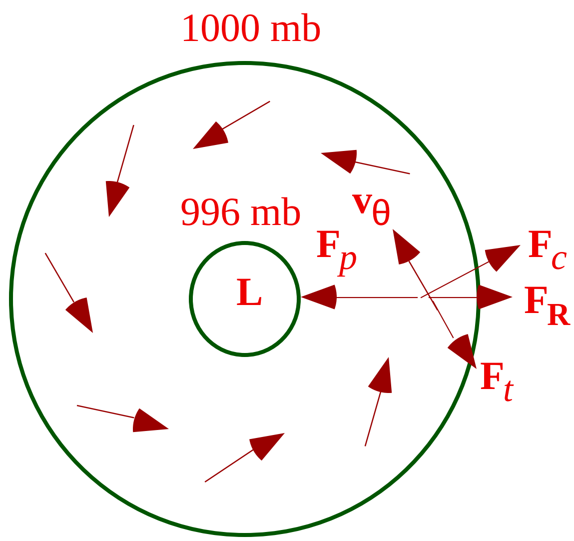
Near the center of high pressure

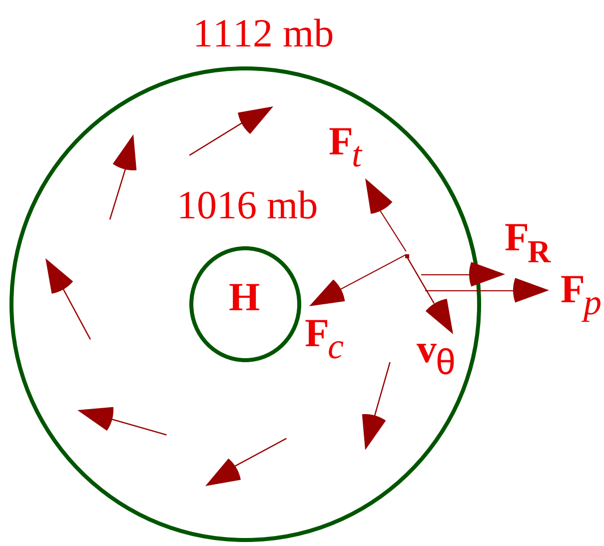
| | | |
|---|---|---|
| | $\partial p_a / \partial R_c$ | = -0.1 mb per 100 km |
| | $R_c$ | = 70 km |
| | $\phi$ | = 15$^o$ N latitude |
| | $p_a$ | = 850 mb |
| | $\rho_a$ | = 1.06 kg m$^{-3}$) |
| ----> | $v$ | = -1.7 m s$^{-1}$ |
| ----> | $v_g$ | = 1123 m s$^{-1}$ |

Around a high pressure center, $\partial p_a / \partial x$ and gradient winds must be smaller than around a low pressure center

# Surface Winds Around Lows and Highs

Figs. 4.16 a and b




$$\frac{du_R}{dt} = fv_\theta - \frac{1}{\rho_a}\frac{\partial p_a}{\partial R_c} + \frac{v_\theta^2}{R_c} + \frac{1}{\rho_a}\frac{\partial\left(\rho_a \overline{w'u'_R}\right)}{\partial z} \qquad (4.87)$$

$$\frac{dv_\theta}{dt} = -fu_R - \frac{u_R v_\theta}{R_c} + \frac{1}{\rho_a}\frac{\partial\left(\rho_a \overline{w'v'_\theta}\right)}{\partial z} \qquad (4.87)$$

# Atmospheric Waves

Wavenumber and wavelength

$$\tilde{k} = \frac{2\pi}{\lambda_{\alpha,x}} \qquad \tilde{l} = \frac{2\pi}{\lambda_{\alpha,y}} \qquad \tilde{m} = \frac{2\pi}{\lambda_{\alpha,z}} \tag{4.88}$$

Frequency of oscillation (dispersion relationship)

$$\nu_\alpha = c_\alpha \sqrt{\tilde{k}^2 + \tilde{l}^2 + \tilde{m}^2} = c_\alpha \left|\tilde{\mathbf{K}}\right| \tag{4.89}$$

Wavenumber vector

$$\left|\tilde{\mathbf{K}}\right| = \sqrt{\tilde{k}^2 + \tilde{l}^2 + \tilde{m}^2}$$

Group velocity

$$\mathbf{c_g} = \mathbf{i}c_{g,x} + \mathbf{j}c_{g,y} + \mathbf{k}c_{g,z}$$

Group speeds

$$c_{g,x} = \frac{\partial \nu_\alpha}{\partial \tilde{k}} \qquad c_{g,y} = \frac{\partial \nu_\alpha}{\partial \tilde{l}} \qquad c_{g,z} = \frac{\partial \nu_\alpha}{\partial \tilde{m}} \tag{4.90}$$

Group speed as a function of wavenumber

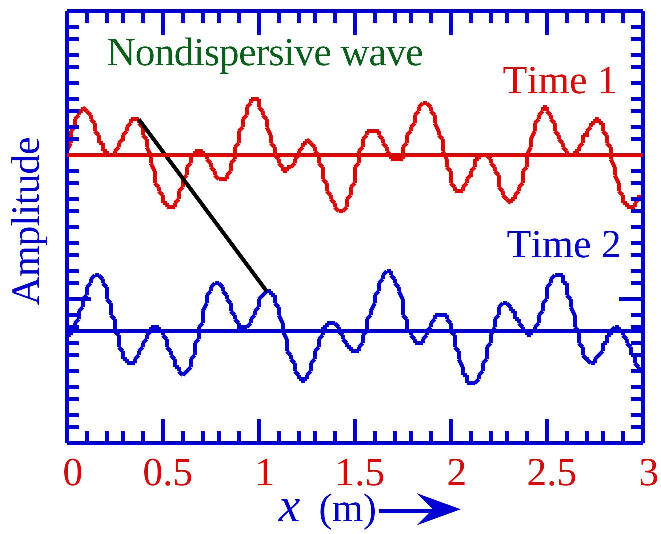
$$c_{g,x} = c_\alpha \frac{\tilde{k}}{\left|\tilde{\mathbf{K}}\right|} + \left|\tilde{\mathbf{K}}\right| \frac{\partial c_\alpha}{\partial \tilde{k}} \tag{4.91}$$

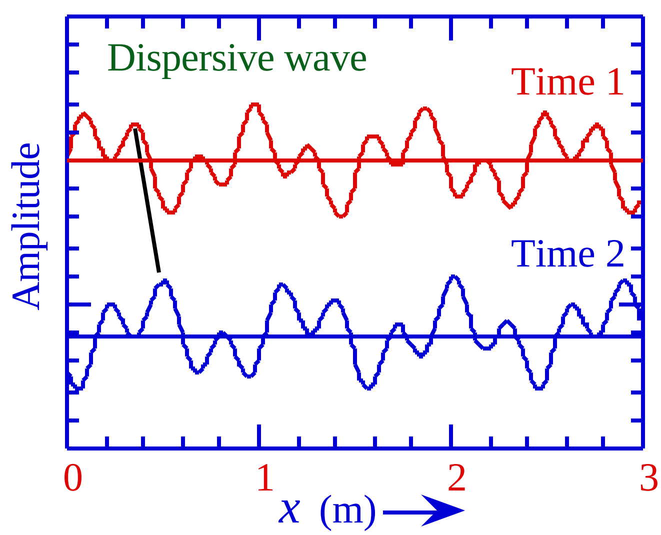
Nondispersive medium $\qquad c_\alpha$ is independent of $\tilde{k}$, $\tilde{l}$, $\tilde{m}$

# Atmospheric Waves

Fig. 4.17 a and b.

Wave pulses in nondispersive and dispersive media, respectively.



# Acoustic Waves

*U*-momentum equation

$$\frac{du}{dt} = -\frac{1}{\rho_a}\frac{\partial p_a}{\partial x} \qquad (4.92)$$

Continuity equation

$$\frac{d\rho_a}{dt} = -\rho_a \frac{\partial u}{\partial x} \qquad (4.92)$$

Thermodynamic energy equation

$$\frac{1}{\theta_v}\frac{d\theta_v}{dt} = \frac{d\ln\theta_v}{dt} = 0 \qquad (4.92)$$

Substitute $\theta_v = T_v\left(1000/p_a\right)^{\kappa}$ and $p_a = \rho_a R' T_v$ into (4.92)

$$\frac{d\rho_a}{dt} = \frac{\rho_a}{\gamma}\frac{d\ln p_a}{dt} \qquad \gamma = \frac{1}{1-\kappa} = \frac{c_{p,d}}{c_{v,d}} \approx 1.4 \qquad (4.93)$$

# Acoustic Waves

Substitute (4.93) into continuity equation

$$\frac{\mathrm{d}\ln p_a}{\mathrm{d}t} = \frac{1}{p_a}\frac{\mathrm{d}p_a}{\mathrm{d}t} = -\gamma\frac{\partial u}{\partial x} \qquad (4.94)$$

Combine with momentum equation --> wave equation

$$\frac{\mathrm{d}^2 p'_a}{\mathrm{d}t^2} = \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 p'_a = c_s^2\frac{\partial^2 p'_a}{\partial x^2} \qquad (4.95)$$

Speed of sound under adiabatic conditions

$$c_s = \pm\sqrt{\gamma R' \bar{T}_v}$$

Solution to wave equation

$$p'_a = p'_{a,0}\sin\left(\tilde{k}x - \nu_\alpha t\right)$$

Dispersion relationship for acoustic waves

$$\nu_\alpha = \left(\bar{u} \pm c_s\right)\tilde{k} \qquad (4.96)$$

Scalar group velocity

$$c_{g,x} = \bar{u} \pm c_s$$

# Acoustic-Gravity Waves

*U*-momentum equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\rho_a}\frac{\partial p_a}{\partial x} \quad (4.97)$$

*W*-momentum equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{1}{\rho_a}\frac{\partial p_a}{\partial z} - g \quad (4.97)$$

Continuity equation

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = -\rho_a\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \quad (4.97)$$

Thermodynamic energy equation

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \frac{\rho_a}{\gamma}\frac{\mathrm{d}\ln p_a}{\mathrm{d}t} \quad (4.97)$$

Dispersion relationship for acoustic-gravity waves

$$\frac{N_{bv}^2}{\left(\nu_\alpha - \bar{u}\tilde{k}\right)^2}\tilde{k}^2 + \frac{\left(\nu_\alpha - \bar{u}\tilde{k}\right)^2}{c_s^2} = \tilde{m}^2 + \tilde{k}^2 + \frac{\nu_c^2}{c_s^2} \quad (4.98)$$

Acoustic cutoff frequency

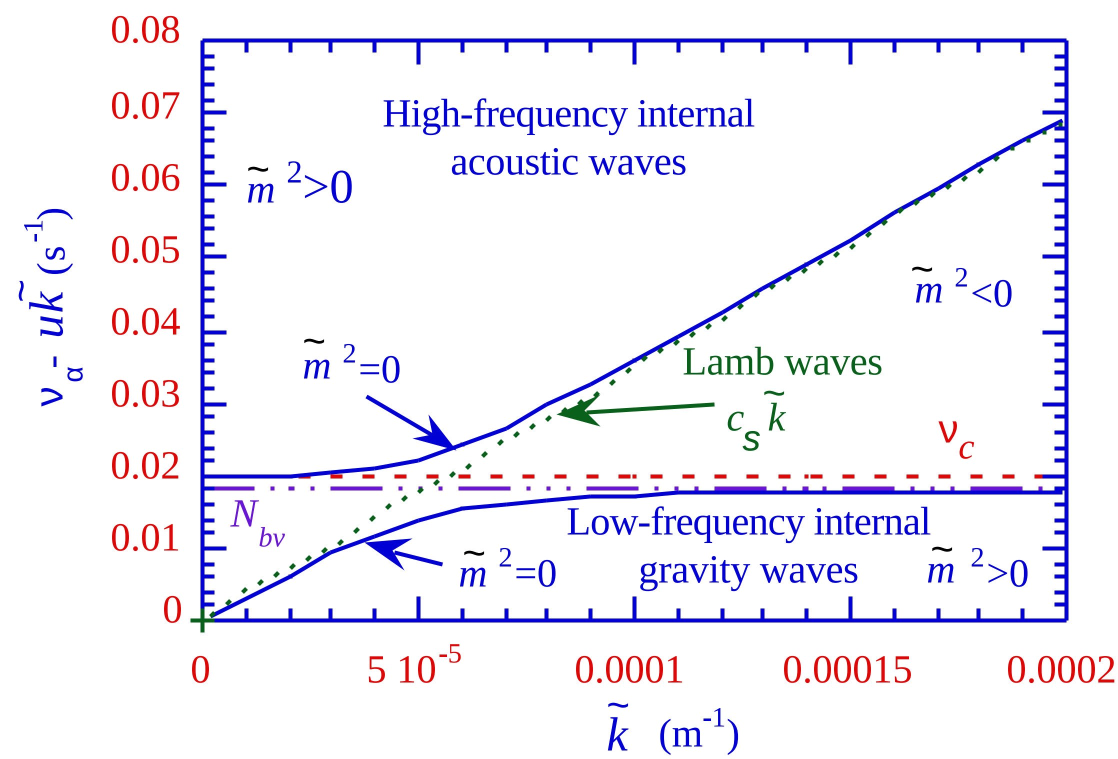
$$\nu_c = \frac{c_s}{2H}$$

# Acoustic-Gravity Waves

$$\nu_\alpha = \begin{cases} \bar{u}\tilde{k} + \dfrac{N_{bv}\tilde{k}}{\left(\tilde{k}^2 + \tilde{m}^2 + \nu_c^2/c_s^2\right)^{1/2}} & \nu_\alpha^2 << \nu_c^2 & \left\{\begin{array}{l}\text{low}-\text{frequency}\\ \text{gravity waves}\end{array}\right. \\ \bar{u}\tilde{k} + \left(c_s^2\tilde{k}^2 + c_s^2\tilde{m}^2 + \nu_c^2\right)^{1/2} & \nu_\alpha^2 >> N_{bv}^2 & \left\{\begin{array}{l}\text{high}-\text{frequency}\\ \text{acoustic waves}\end{array}\right. \\ \bar{u}\tilde{k} + \dfrac{N_{bv}\tilde{k}}{\left(\tilde{k}^2 + \tilde{m}^2\right)^{1/2}} & \tilde{k} \to \infty & \left\{\text{mountain waves}\right. \\ \left(\bar{u} + c_s\right)\tilde{k} & \left\{\begin{array}{l}\nu_\alpha^2 << \nu_c^2,\ \tilde{m}^2 = 0,\ \tilde{k} \to 0 \ \ or \\ \nu_\alpha^2 >> N_{bv}^2,\ \tilde{m}^2 = 0,\ \tilde{k} \to \infty\end{array}\right. & \left\{\begin{array}{l}\text{Lamb}\\ \text{waves}\end{array}\right. \end{cases}$$

(4.99)

Fig. 4.18.



# Inertial Oscillations

Horizontal momentum equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = fv = f\,\frac{\mathrm{d}y}{\mathrm{d}t} \qquad\qquad \frac{\mathrm{d}v}{\mathrm{d}t} = f\left(u_g - u\right) \qquad (4.101)$$

Integrate $u$-momentum equation between $y_0$ and $y_0 + \Delta y$

$$u\left(y_0 + \Delta y\right) - u_g\left(y_0\right) \approx f\Delta y \qquad (4.102)$$

Taylor series expansion of geostrophic wind at $y_0 + \Delta y$

$$u_g\left(y_0 + \Delta y\right) \approx u_g\left(y_0\right) + \frac{\partial u_g}{\partial y}\Delta y \qquad (4.103)$$

Substitute (4.103) into (4.102)

$$u_g\left(y_0 + \Delta y\right) - u\left(y_0 + \Delta y\right) \approx -\left(f - \frac{\partial u_g}{\partial y}\right)\Delta y \qquad (4.104)$$

Substitute (4.104) into $v$-momentum equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -f\left(f - \frac{\partial u_g}{\partial y}\right)\Delta y \qquad (4.105)$$

Inertial stability criteria in Northern hemisphere

$$f - \frac{\partial u_g}{\partial y}\begin{cases} < 0 & \text{inertially unstable} \\ = 0 & \text{inertially neutral} \\ > 0 & \text{inertially stable} \end{cases} \qquad (4.106)$$

# Inertial Lamb and Gravity Waves

Inertial Lamb waves

$$\nu_\alpha^2 = f^2 + c_s^2 \tilde{k}^2 \qquad (4.107)$$

Inertial gravity waves

$$\nu_\alpha^2 = f^2 + \frac{N_{bv}^2 \tilde{k}^2}{\tilde{k}^2 + \tilde{m}^2 + \nu_c^2 / c_s^2} \qquad (4.108)$$

Rossby radius of deformation

$$\lambda_R = \frac{\sqrt{g h_e}}{f}$$

Equivalent depth

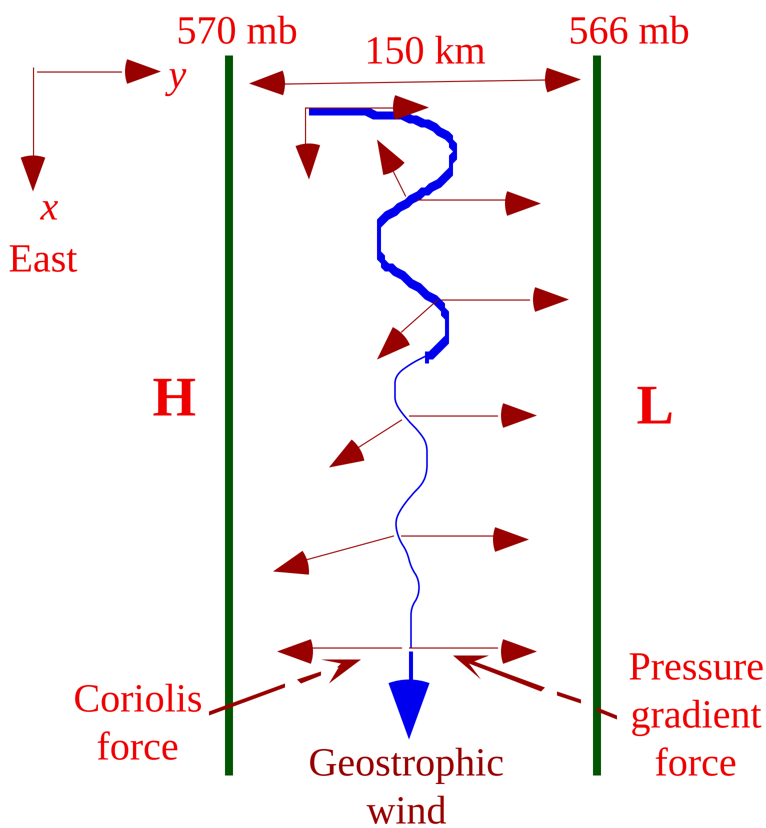
$$h_e = \begin{cases} \dfrac{c_s^2}{g} & \text{inertia Lamb waves} \\ \dfrac{N_{bv}^2 / g}{\tilde{m}^2 + \nu_c^2 / c_s^2} & \text{inertia gravity waves} \end{cases} \qquad (4.109)$$

$L > \lambda_R$ --> velocity field adjusts to pressure field

$L < \lambda_R$ --> pressure field adjusts to velocity field

# Geostrophic Adjustment

Fig. 4.19. Geostrophic adjustment of the velocity field to the pressure field at one altitude when $L > \lambda_R$ and when the domain is assumed large enough to allow the energy of the oscillations to disperse away and decay.



# Vorticity

Relative vorticity

$$\zeta_r = \quad \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

(4.110)

Vertical component of relative vorticity

$$\zeta_{r,z} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Absolute vorticity

$$\zeta_{a,z} = f + \zeta_{r,z}$$

Potential vorticity

$$P_v = \frac{f + \zeta_{r,z}}{\Delta z_t} = \frac{f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}{\Delta z_t} = \text{constant} \qquad (4.111)$$

# Rossby Waves

Horizontal momentum equations

$$\frac{du}{dt} = fv - \frac{1}{\rho_a}\left(\frac{\partial p_a}{\partial x}\right)_z \qquad \frac{dv}{dt} = -fu - \frac{1}{\rho_a}\left(\frac{\partial p_a}{\partial y}\right)_z \qquad (4.112)$$

Midlatitude beta plane approximation

$$f = f_0 + \beta(y - y_0)$$

$$\beta = \frac{\partial f}{\partial y} = 2\Omega\frac{\partial \phi}{\partial y}\cos\phi \approx \frac{2\Omega}{R_e}\cos\phi \qquad (4.113)$$

Geopotential gradients on surfaces of constant pressure

$$\left(\frac{\partial p_a}{\partial x}\right)_z = \rho_a\left(\frac{\partial \Phi}{\partial x}\right)_p \qquad \left(\frac{\partial p_a}{\partial y}\right)_z = \rho_a\left(\frac{\partial \Phi}{\partial y}\right)_p \qquad (4.114)$$

Divide *u*, *v*, etc. into geostrophic / ageostrophic components

$$u = u_g + u_a \qquad \Phi = \Phi_g + \Phi_a$$

# Rossby Waves

Rewrite momentum equations

$$\frac{d(u_g + u_a)}{dt} = [f_0 + \beta(y - y_0)](v_g + v_a) - \left[\frac{\partial(\Phi_g + \Phi_a)}{\partial x}\right]_p$$

(4.115)

$$\frac{d(v_g + v_a)}{dt} = -[f_0 + \beta(y - y_0)](u_g + u_a) - \left[\frac{\partial(\Phi_g + \Phi_a)}{\partial y}\right]_p$$

(4.116)

Combine geostrophic wind with geopotential gradients

$$v_g = \frac{1}{f_0}\left(\frac{\partial \Phi_g}{\partial x}\right)_p \qquad u_g = -\frac{1}{f_0}\left(\frac{\partial \Phi_g}{\partial y}\right)_p \qquad (4.117)$$

Substitute --> quasi-geostrophic momentum equations

$$\frac{du_g}{dt} = f_0 v_a + \beta(y - y_0)v_g - \left(\frac{\partial \Phi_a}{\partial x}\right)_p \qquad (4.118)$$

$$\frac{dv_g}{dt} = -f_0 u_a - \beta(y - y_0)u_g - \left(\frac{\partial \Phi_a}{\partial y}\right)_p \qquad (4.119)$$

Subtract $\partial/\partial y$ of (4.118) from $\partial/\partial x$ of (4.119)

$$\frac{d}{dt}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g \qquad (4.120)$$

# Rossby Waves

Vertical velocity

$$w = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{g}\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{4.121}$$

Substitute (4.121), $u = u_g + u_a$, $v = v_g + v_a$, and

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$$ (Geostrophic wind is nondivergent)

into continuity equation for incompressible atmosphere

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

to obtain

$$\frac{1}{g}\frac{\partial}{\partial z}\left(\frac{\mathrm{d}\Phi_g}{\mathrm{d}t}\right) = -\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) \tag{4.122}$$

Integrate from surface to mean tropopause height, $\Delta z_t$

$$\frac{\mathrm{d}\Phi_g}{\mathrm{d}t} = -g\Delta z_t\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) \tag{4.123}$$

# Rossby Waves

Substitute (4.123) into (4.120)

$$\frac{d}{dt}\left(\zeta_g - \frac{f_0}{g\Delta z_t}\Phi_g\right) = -\beta v_g \qquad (4.124)$$

Geostrophic potential vorticity

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi_g}{\partial x^2} + \frac{\partial^2 \Phi_g}{\partial y^2}\right)_p = \frac{\nabla_p^2 \Phi_g}{f_0} \qquad (4.125)$$

Expand (4.124) -- quasi-geostrophic potential vorticity equation

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\nabla_p^2 \Phi_g - \frac{f_0^2}{g\Delta z_t}\Phi_g\right) + \beta\left(\frac{\partial \Phi_g}{\partial x}\right)_p = 0 \qquad (4.126)$$

Wave solution

$$\Phi_g = \Phi_{g,0}\sin\left(\tilde{k}x + \tilde{l}y - \nu_\alpha t\right)$$

Dispersion relationship by substituting wave solution

$$\nu_\alpha = \left(\bar{u} - \frac{\beta}{\tilde{k}^2 + \tilde{l}^2 + \lambda_R^{-2}}\right)\tilde{k} \qquad (4.127)$$

$$\lambda_R = \frac{\sqrt{g\Delta z_t}}{f_0}$$