Overhead Slides for

Chapter 5

of

Fundamentals of Atmospheric Modeling

by

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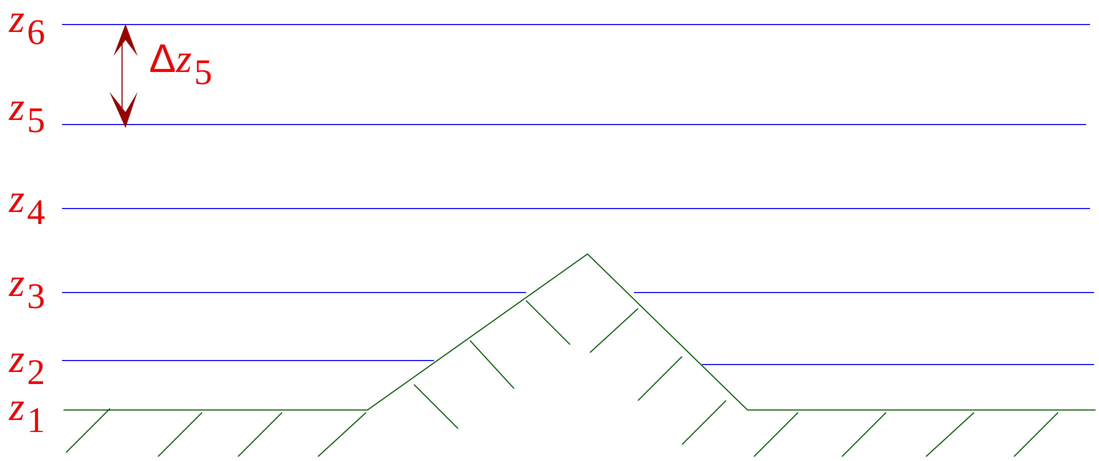
Altitude Coordinate

Fig. 5.1. Heights of altitude-coordinate surfaces.



Equation for Nonhydrostatic Pressure

Decompose pressure into large-scale and perturbation term

$$p_a = \hat{p}_a + p''_a \qquad (5.1)$$

Large-scale atmosphere in hydrostatic balance

$$\frac{1}{\hat{\rho}_a}\frac{\partial \hat{p}_a}{\partial z} = \hat{\alpha}_a \frac{\partial \hat{p}_a}{\partial z} = -g \qquad (5.2)$$

Decompose gravitational and pressure gradient terms

$$g + \frac{1}{\rho_a}\frac{\partial p_a}{\partial z} = g + \left(\hat{\alpha}_a + \alpha''_a\right)\frac{\partial}{\partial z}\left(\hat{p}_a + p''_a\right) \approx \hat{\alpha}_a \frac{\partial p''_a}{\partial z} \frac{\alpha''_a}{\hat{\alpha}_a} g$$

(5.3)

Substitute (5.3) into vertical momentum equation

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\hat{\alpha}_a \frac{\partial p''_a}{\partial z} + \frac{\alpha''_a}{\hat{\alpha}_a} g \qquad (5.4)$$

Take · the sum of (5.4), (4.72), and (4.73)

$$\frac{\partial}{\partial t} \quad \cdot\left(\mathbf{v}\hat{\rho}_a\right) + \quad \cdot\left[\hat{\rho}_a\left(\mathbf{v}\cdot \quad\right)\mathbf{v}\right] = - \quad \cdot\left(\hat{\rho}_a f\mathbf{k}\times\mathbf{v}\right) - \quad {}^2_z \hat{p}_a - \quad {}^2 p''_a + g\frac{\partial}{\partial z}\left(\frac{\alpha''_a}{\hat{\alpha}_a^2}\right)$$

(5.5)

Equation for Nonhydrostatic Pressure

Note that

$$\frac{\alpha''_a}{\hat{\alpha}_a} \approx \frac{\theta''_v}{\hat{\theta}_v} - \frac{c_{v,d}}{c_{p,d}} \frac{p''_a}{\hat{p}_a} \qquad (5.6)$$

Remove local derivative from continuity equation

--> Anelastic continuity equation

$$\nabla \cdot \left(\mathbf{v}\hat{\rho}_a\right) = 0 \qquad (5.7)$$

Substitute (5.6) and (5.7) into (5.5)

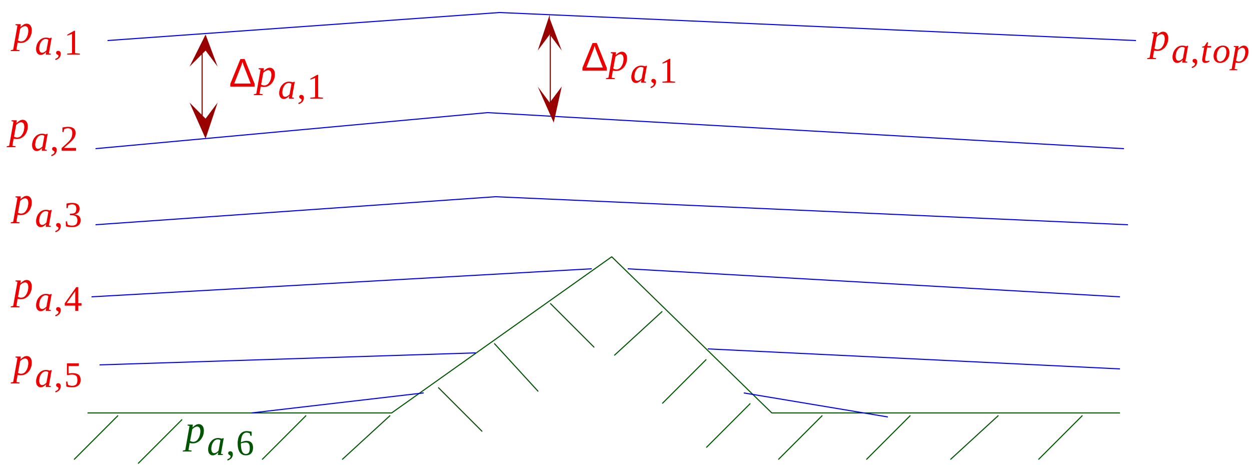
--> Diagnostic equation for nonhydrostatic pressure

$$\nabla^2 p''_a - g \frac{c_{v,d}}{c_{p,d}} \frac{\partial}{\partial z}\left(\hat{\rho}_a \frac{p''_a}{\hat{p}_a}\right) = -\nabla \cdot \left[\hat{\rho}_a (\mathbf{v} \cdot \nabla)\mathbf{v}\right] - \nabla \cdot \left[\hat{\rho}_a f \mathbf{k} \times \mathbf{v}\right]$$

$$- \nabla_z^2 \hat{p}_a + g \frac{\partial}{\partial z}\left(\hat{\rho}_a \frac{\theta''_p}{\hat{\theta}_p}\right) + \nabla \cdot \left(\nabla \cdot \hat{\rho}_a \mathbf{K}_m \nabla\right)\mathbf{v} \qquad (5.8)$$

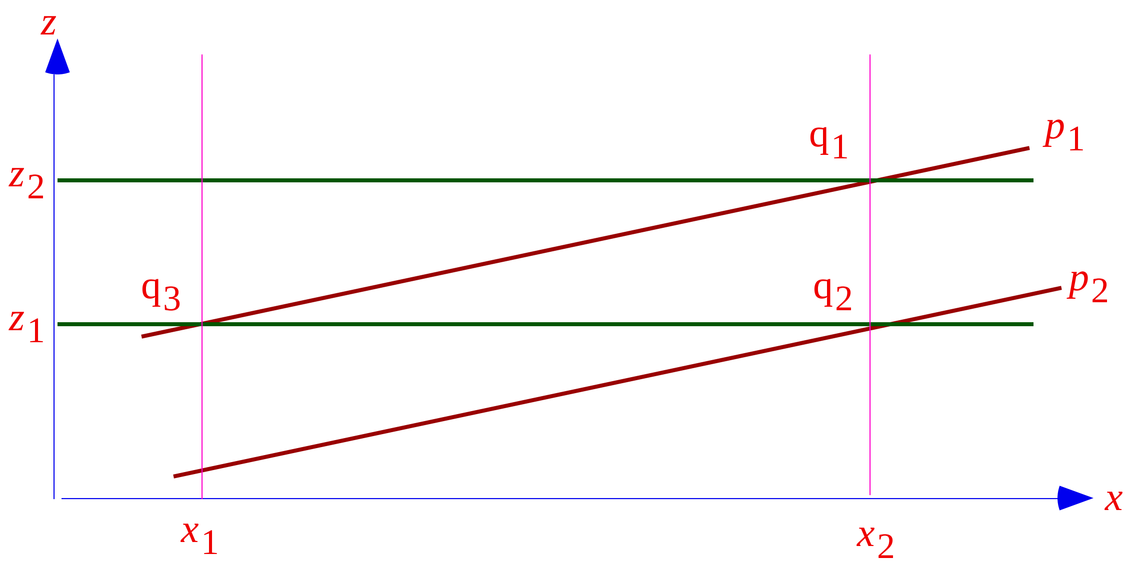
Pressure Coordinate

Fig. 5.2. Heights of pressure-coordinate surfaces.



Intersection of Pressure and Altitude Surfaces

Fig. 5.3.



Gradient Conversion From the Altitude to Pressure Coordinate

Change in mass mixing ratio over distance

$$\frac{q_2 - q_3}{x_2 - x_1} = \frac{q_1 - q_3}{x_2 - x_1} + \left(\frac{p_2 - p_1}{x_2 - x_1}\right)\left(\frac{q_1 - q_2}{p_1 - p_2}\right) \qquad (5.9)$$

Approximate differences as $x_2 - x_1 \rightarrow 0$, $p_1 - p_2 \rightarrow 0$

$$\left(\frac{\partial q}{\partial x}\right)_z = \frac{q_2 - q_3}{x_2 - x_1} \qquad \left(\frac{\partial q}{\partial x}\right)_p = \frac{q_1 - q_3}{x_2 - x_1} \qquad (5.10)$$

$$\left(\frac{\partial p_a}{\partial x}\right)_z = \frac{p_2 - p_1}{x_2 - x_1} \qquad \left(\frac{\partial q}{\partial p_a}\right)_x = \frac{q_1 - q_2}{p_1 - p_2}$$

Gradient conversion from the altitude to pressure coordinate

$$\left(\frac{\partial q}{\partial x}\right)_z = \left(\frac{\partial q}{\partial x}\right)_p + \left(\frac{\partial p_a}{\partial x}\right)_z \left(\frac{\partial q}{\partial p_a}\right)_x \qquad (5.11)$$

General equations

$$\left(\frac{\partial}{\partial x}\right)_z = \left(\frac{\partial}{\partial x}\right)_p + \left(\frac{\partial p_a}{\partial x}\right)_z \left(\frac{\partial}{\partial p_a}\right)_x \qquad (5.12)$$

Gradient Conversion From the Altitude to Pressure Coordinate

Substitute time for distance

$$\left(\frac{\partial}{\partial t}\right)_z = \left(\frac{\partial}{\partial t}\right)_p + \left(\frac{\partial p_a}{\partial t}\right)_z \left(\frac{\partial}{\partial p_a}\right)_t \qquad (5.15)$$

Horizontal gradient operator in the altitude coordinate

$${}_z = \mathbf{i}\left(\frac{\partial}{\partial x}\right)_z + \mathbf{j}\left(\frac{\partial}{\partial y}\right)_z \qquad (4.80)$$

Horizontal gradient operator in the pressure coordinate

$${}_p = \mathbf{i}\left(\frac{\partial}{\partial x}\right)_p + \mathbf{j}\left(\frac{\partial}{\partial y}\right)_p \qquad (5.14)$$

Gradient conversion between the altitude and pressure coordinate

$${}_z = {}_p + {}_z\left(p_a\right)\frac{\partial}{\partial p_a} \qquad (5.13)$$

Geopotential Gradient

Take gradient conversion of geopotential

$$ {}_z\Phi = \quad {}_p\Phi + \quad {}_z\left(p_a\right)\frac{\partial\Phi}{\partial p_a} $$

Note that

$$ {}_z\Phi = 0 $$

Rearrange gradient conversion

$$ {}_z\left(p_a\right) = -\frac{\partial p_a}{\partial\Phi} \quad {}_p\Phi = -\frac{\partial p_a}{g\partial z} \quad {}_p\Phi = \rho_a \quad {}_p\Phi \qquad (5.16) $$

Component directions

$$ \left(\frac{\partial p_a}{\partial x}\right)_z = \rho_a\left(\frac{\partial\Phi}{\partial x}\right)_p \qquad (5.17) $$

$$ \left(\frac{\partial p_a}{\partial y}\right)_z = \rho_a\left(\frac{\partial\Phi}{\partial y}\right)_p \qquad (5.17) $$

Continuity Equation For Air in the Pressure Coordinate

Continuity equation for air in the altitude coordinate

$$\frac{\partial \rho_a}{\partial t} = -\rho_a(\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\rho_a \qquad (3.20)$$

Expand with horizontal operators

$$\left(\frac{\partial \rho_a}{\partial t}\right)_z = -\rho_a\left(\nabla_z \cdot \mathbf{v}_h + \frac{\partial w}{\partial z}\right) - (\mathbf{v}_h \cdot \nabla_z)\rho_a - w\frac{\partial \rho_a}{\partial z} \qquad (5.18)$$

Gradient conversion of velocity

$$\nabla_z \cdot \mathbf{v}_h = \nabla_p \cdot \mathbf{v}_h + \nabla_z(p_a) \cdot \frac{\partial \mathbf{v}_h}{\partial p_a} \qquad (5.19)$$

Substitute gradient conversion and hydrostatic equation

$$\left(\frac{\partial \rho_a}{\partial t}\right)_z = -\rho_a\left(\nabla_p \cdot \mathbf{v}_h + \nabla_z(p_a) \cdot \frac{\partial \mathbf{v}_h}{\partial p_a}\right) - (\mathbf{v}_h \cdot \nabla_z)\rho_a + \rho_a g\frac{\partial (w\rho_a)}{\partial p_a}$$

(5.20)

Continuity Equation For Air in the Pressure Coordinate

Define vertical scalar velocity in the pressure coordinate

$$\omega_p = \frac{dp_a}{dt} = \left(\frac{\partial p_a}{\partial t}\right)_z + (\mathbf{v}\cdot\nabla)p_a = \left(\frac{\partial p_a}{\partial t}\right)_z + (\mathbf{v}_h\cdot\nabla_z)p_a + w\frac{\partial p_a}{\partial z} \tag{5.21}$$

Substitute $dz = -dp_a/\rho_a g$

$$\omega_p = -\left(\rho_a g\frac{\partial z}{\partial t}\right)_z + (\mathbf{v}_h\cdot\nabla_z)p_a - w\rho_a g \tag{5.22}$$

Differentiate vertical velocity with respect to altitude

$$\frac{\partial \omega_p}{\partial z} = -g\left(\frac{\partial \rho_a}{\partial t}\right)_z + \nabla_z(p_a)\cdot\frac{\partial \mathbf{v}_h}{\partial z} + (\mathbf{v}_h\cdot\nabla_z)\frac{\partial p_a}{\partial z} - g\frac{\partial(w\rho_a)}{\partial z} \tag{5.23}$$

Substitute $dz = -dp_a/\rho_a g$

$$\rho_a\frac{\partial \omega_p}{\partial p_a} = \left(\frac{\partial \rho_a}{\partial t}\right)_z + \rho_a\nabla_z(p_a)\cdot\frac{\partial \mathbf{v}_h}{\partial p_a} + (\mathbf{v}_h\cdot\nabla_z)\rho_a - \rho_a g\frac{\partial(w\rho_a)}{\partial p_a} \tag{5.24}$$

Add (5.20) to (5.24) --> continuity equation for air

$$\nabla_p\cdot\mathbf{v}_h + \frac{\partial \omega_p}{\partial p_a} = 0 \tag{5.25}$$

Continuity Equation For Air in the Pressure Coordinate

Expanded continuity equation

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p+\frac{\partial w_p}{\partial p_a}=0 \qquad (5.26)$$

Example 5.1.

Δx	= 5 km	Δy	= 5 km
Δp_a	= -10 mb		
u_1	=-3 (west)	u_2	=-1 m s^{-1} (east)
v_3	= +2 (south)	v_4	=-2 m s^{-1} (north)
$w_{p,5}$	=+0.02 mb s^{-1} (lower)		

---->

$$\frac{(-1+3)\ \mathrm{m\ s^{-1}}}{5000\ \mathrm{m}}+\frac{(-2-2)\ \mathrm{m\ s^{-1}}}{5000\ \mathrm{m}}+\frac{(w_{p,6}-0.02)\ \mathrm{mb\ s^{-1}}}{-10\ \mathrm{mb}}=0$$

----> $w_{p,6}$ = +0.016 mb s^{-1} (downward)

Total Derivative in the Pressure Coordinate

Total derivative in Cartesian-altitude coordinate

$$\frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\partial}{\partial t}\right)_z + \left(\mathbf{v}_h \cdot \nabla_z\right) + w\frac{\partial}{\partial z} \qquad (5.27)$$

Substitute time and horizontal gradient conversions

$$\frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\partial}{\partial t}\right)_p + \left(\frac{\partial p_a}{\partial t}\right)_z \frac{\partial}{\partial p_a} + \left(\mathbf{v}_h \cdot \nabla_p\right) + \left[\left(\mathbf{v}_h \cdot \nabla_z\right)p_a\right]\frac{\partial}{\partial p_a} + w\frac{\partial}{\partial z} \qquad (5.28)$$

Vertical velocity in altitude coordinate from (5.21)

$$w = \frac{\left(\frac{\partial p_a}{\partial t}\right)_z + \left(\mathbf{v}_h \cdot \nabla_z\right)p_a - w_p}{\rho_a g} \qquad (5.29)$$

Substitute (5.29) and hydrostatic equation into (5.28)

--> total derivative in Cartesian-pressure coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\partial}{\partial t}\right)_p + \left(\mathbf{v}_h \cdot \nabla_p\right) + w_p \frac{\partial}{\partial p_a} \qquad (5.30)$$

Species Continuity Equation in the Pressure Coordinate

Species continuity equation in the altitude coordinate

$$\frac{dq}{dt} = \frac{\left(\nabla \cdot \rho_a \mathbf{K}_h \nabla\right) q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n$$

Apply total derivative in Cartesian-pressure coordinates

$$\frac{dq}{dt} = \left(\frac{\partial q}{\partial t}\right)_p + \left(\mathbf{v}_h \cdot \nabla_p\right) q + w_p \frac{\partial q}{\partial p_a} = \frac{\left(\nabla \cdot \rho_a \mathbf{K}_h \nabla\right) q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n \tag{5.31}$$

Convert mass mixing ratio to number concentration

$$q = \frac{Nm}{\rho_a A} \tag{5.32}$$

Thermodynamic Energy Equation in the Pressure Coordinate

Thermodynamic energy equation in the altitude coordinate

$$\frac{d\theta_v}{dt} = \frac{(\nabla \cdot \rho_a \mathbf{K}_h \nabla)\theta_v}{\rho_a} + \frac{\theta_v}{c_{p,d}T_v}\sum_{n=1}^{N_{e,h}}\frac{dQ_n}{dt}$$

Apply total derivative in Cartesian-pressure coordinates

$$\left(\frac{\partial\theta_v}{\partial t}\right)_p + (\mathbf{v}_h \cdot \nabla_p)\theta_v + w_p\frac{\partial\theta_v}{\partial p_a} = \frac{(\nabla \cdot \rho_a \mathbf{K}_h \nabla)\theta_v}{\rho_a} + \frac{\theta_v}{c_{p,d}T_v}\sum_{n=1}^{N_{e,h}}\frac{dQ_n}{dt}$$

(5.34)

Horizontal Momentum Equations in the Pressure Coordinate

Horizontal momentum equation in the altitude coordinate

$$\frac{d\mathbf{v}_h}{dt} = -f\mathbf{k}\times\mathbf{v}_h - \frac{1}{\rho_a}\nabla_z\left(p_a\right) + \frac{\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)\mathbf{v}_h}{\rho_a}$$

Substitute

$$\nabla_z\left(p_a\right) = \rho_a\nabla_p\Phi$$

and apply total derivative in Cartesian-pressure coordinates -->

$$\left(\frac{\partial\mathbf{v}_h}{\partial t}\right)_p + \left(\mathbf{v}_h\cdot\nabla_p\right)\mathbf{v}_h + w_p\frac{\partial\mathbf{v}_h}{\partial p_a} = -f\mathbf{k}\times\mathbf{v}_h - \nabla_p\Phi + \frac{\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)\mathbf{v}_h}{\rho_a}$$

(5.35)

Vertical Momentum Equation in the Pressure Coordinate

Assume hydrostatic equilibrium

$$\frac{\partial p_a}{\partial z} = -\rho_a g$$

Substitute

$$g = \partial\Phi / \partial z \qquad p_a = \rho_a R' T_v \qquad T_v = \theta_v P$$

Hydrostatic equation in the pressure coordinate

$$\frac{\partial\Phi}{\partial p_a} = -\frac{R' T_v}{p_a} = -\frac{R' \theta_v P}{p_a} = -\frac{R' \theta_v}{p_a}\left(\frac{p_a}{1000\ \text{mb}}\right)^{\kappa} \tag{5.37}$$

Substitute $\kappa = R'/c_{p,d}$

$$d\Phi = -c_{p,d}\theta_v d\left[\left(\frac{p_a}{1000}\right)^{\kappa}\right] = -c_{p,d}\theta_v dP \tag{5.38}$$

Geostrophic Wind in the Pressure Coordinate

Substitute

$$\left(\frac{\partial p_a}{\partial x}\right)_z = \rho_a \left(\frac{\partial \Phi}{\partial x}\right)_p \qquad \left(\frac{\partial p_a}{\partial y}\right)_z = \rho_a \left(\frac{\partial \Phi}{\partial y}\right)_p \qquad (5.17)$$

into

$$v_g = \frac{1}{f\rho_a}\frac{\partial p_a}{\partial x} \qquad u_g = -\frac{1}{f\rho_a}\frac{\partial p_a}{\partial y} \qquad (4.78)$$

--> Geostrophic wind in the pressure coordinate

$$v_g = \frac{1}{f}\left(\frac{\partial \Phi}{\partial x}\right)_p \qquad u_g = -\frac{1}{f}\left(\frac{\partial \Phi}{\partial y}\right)_p \qquad (5.39)$$

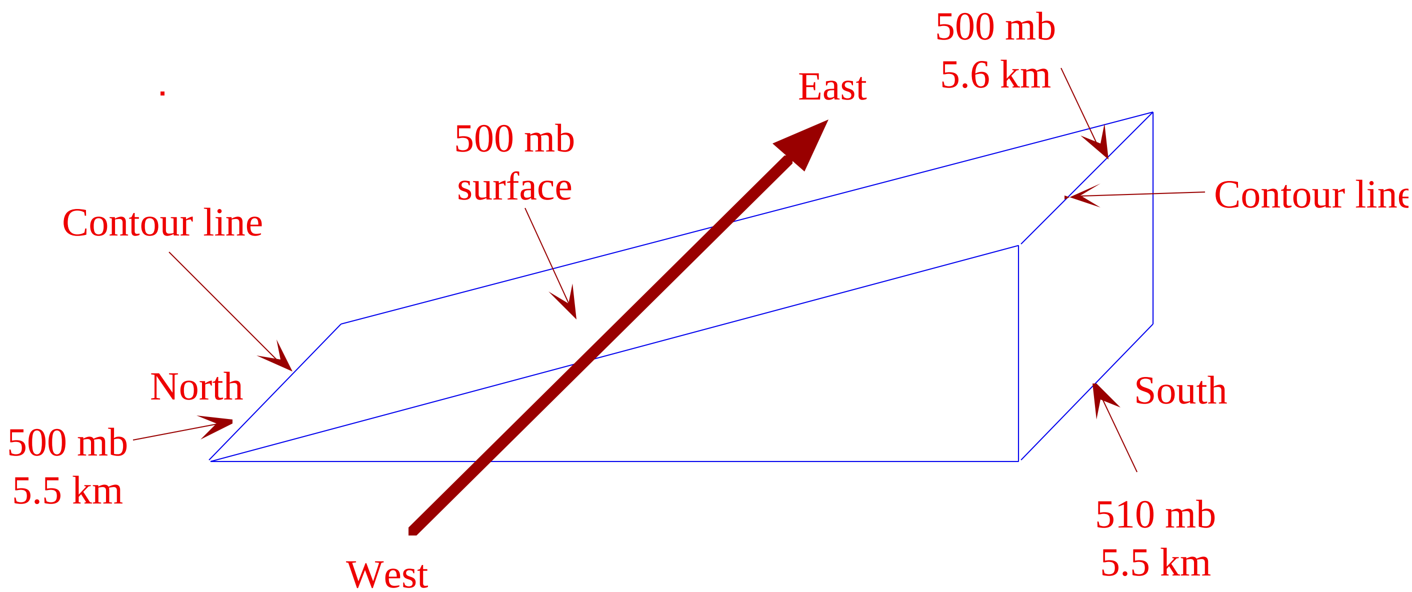
Vector form

$$\mathbf{v}_g = \mathbf{i}u_g + \mathbf{j}v_g = -\mathbf{i}\frac{1}{f}\left(\frac{\partial \Phi}{\partial y}\right)_p + \mathbf{j}\frac{1}{f}\left(\frac{\partial \Phi}{\partial x}\right)_p = \frac{1}{f}\left(\mathbf{k} \times \nabla_p \Phi\right)$$

(5.40)

Geostrophic Wind on a Constant Pressure Surface

Fig. 5.4.



The Sigma-Pressure Coordinate

Definition of a sigma level

$$\sigma = \frac{p_a - p_{a,top}}{p_{a,surf} - p_{a,top}} = \frac{p_a - p_{a,top}}{\pi_a} \qquad (5.41)$$

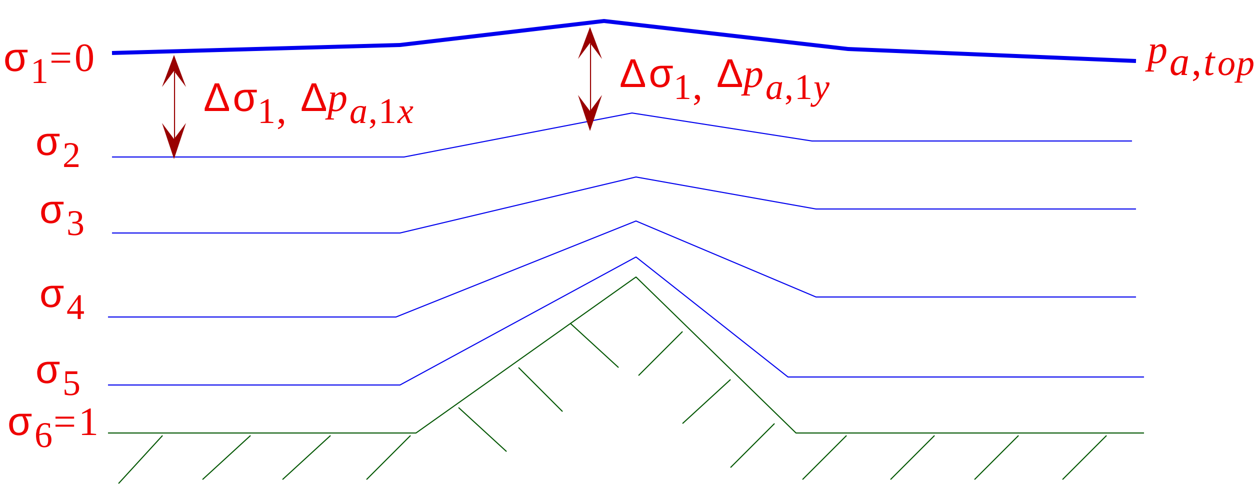
Pressure difference between column surface and top

$$\pi_a = p_{a,surf} - p_{a,top}$$

Pressure at a given sigma level

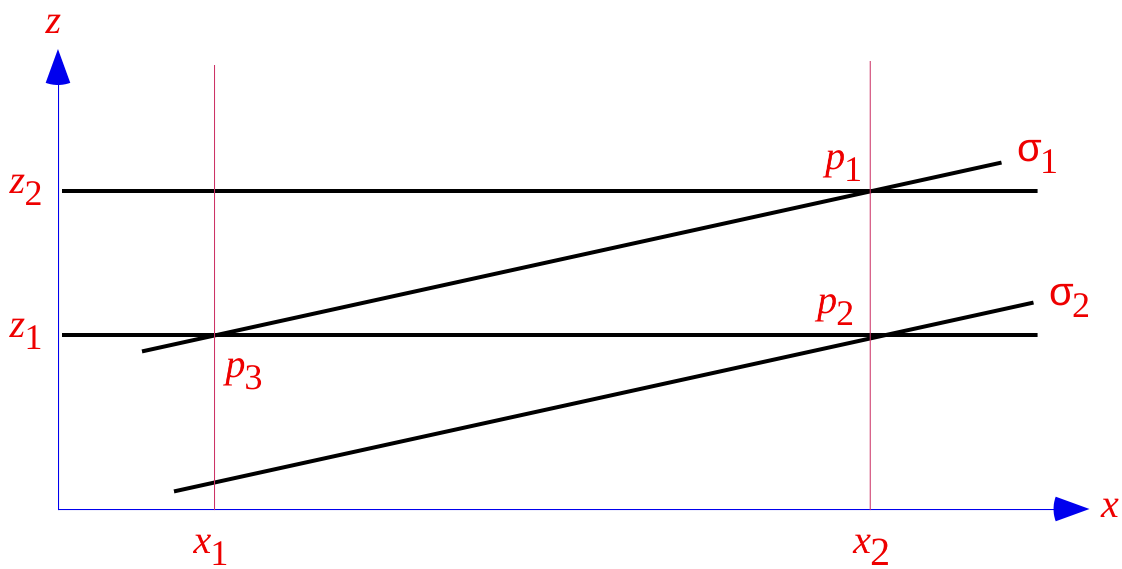
$$p_a = p_{a,top} + \sigma\pi_a \qquad (5.42)$$

Fig. 5.5. Heights of sigma-pressure coordinate surfaces.



Intersection of Sigma-Pressure and Altitude Surfaces

Fig. 5.6.



Gradient Conversion From the Altitude to Sigma-Pressure Coordinate

Change in pressure per unit distance

$$\frac{p_2 - p_3}{x_2 - x_1} = \frac{p_1 - p_3}{x_2 - x_1} + \left(\frac{\sigma_2 - \sigma_1}{x_2 - x_1}\right)\left(\frac{p_1 - p_2}{\sigma_1 - \sigma_2}\right) \qquad (5.43)$$

Approximate differences

$$\left(\frac{\partial p_a}{\partial x}\right)_z = \frac{p_2 - p_3}{x_2 - x_1} \qquad \left(\frac{\partial p_a}{\partial x}\right)_\sigma = \frac{p_1 - p_3}{x_2 - x_1} \qquad (5.44)$$

$$\left(\frac{\partial \sigma}{\partial x}\right)_z = \frac{\sigma_2 - \sigma_1}{x_2 - x_1} \qquad \left(\frac{\partial p_a}{\partial \sigma}\right)_x = \frac{p_1 - p_2}{\sigma_1 - \sigma_2}$$

Gradient conversion from z to σ-p coordinate

$$\left(\frac{\partial p_a}{\partial x}\right)_z = \left(\frac{\partial p_a}{\partial x}\right)_\sigma + \left(\frac{\partial \sigma}{\partial x}\right)_z \left(\frac{\partial p_a}{\partial \sigma}\right)_x \qquad (5.45)$$

Conversion in gradient operator notation

$$\nabla_z (p_a) = \nabla_\sigma (p_a) + (\nabla_z \sigma)\frac{\partial p_a}{\partial \sigma} \qquad (5.46)$$

where

$$\nabla_\sigma = \left(\mathbf{i}\frac{\partial}{\partial x}\right)_\sigma + \left(\mathbf{j}\frac{\partial}{\partial y}\right)_\sigma \qquad (5.47)$$

Gradient Conversion From theAltitude to Sigma-Pressure Coordinate

Generalize gradient operator

$$ {}_{z} = {}_{\sigma} + {}_{z}(\sigma)\frac{\partial}{\partial\sigma} \qquad (5.48) $$

Definition of sigma

$$ \sigma = \left(p_a - p_{a,top}\right)/\pi_a $$

Gradient of sigma

$$ {}_{z}\sigma = \left(p_a - p_{a,top}\right) {}_{z}\left(\frac{1}{\pi_a}\right) + \frac{{}_{z}\left(p_a\right)}{\pi_a} = -\frac{\sigma}{\pi_a} {}_{z}\left(\pi_a\right) + \frac{{}_{z}\left(p_a\right)}{\pi_a} $$

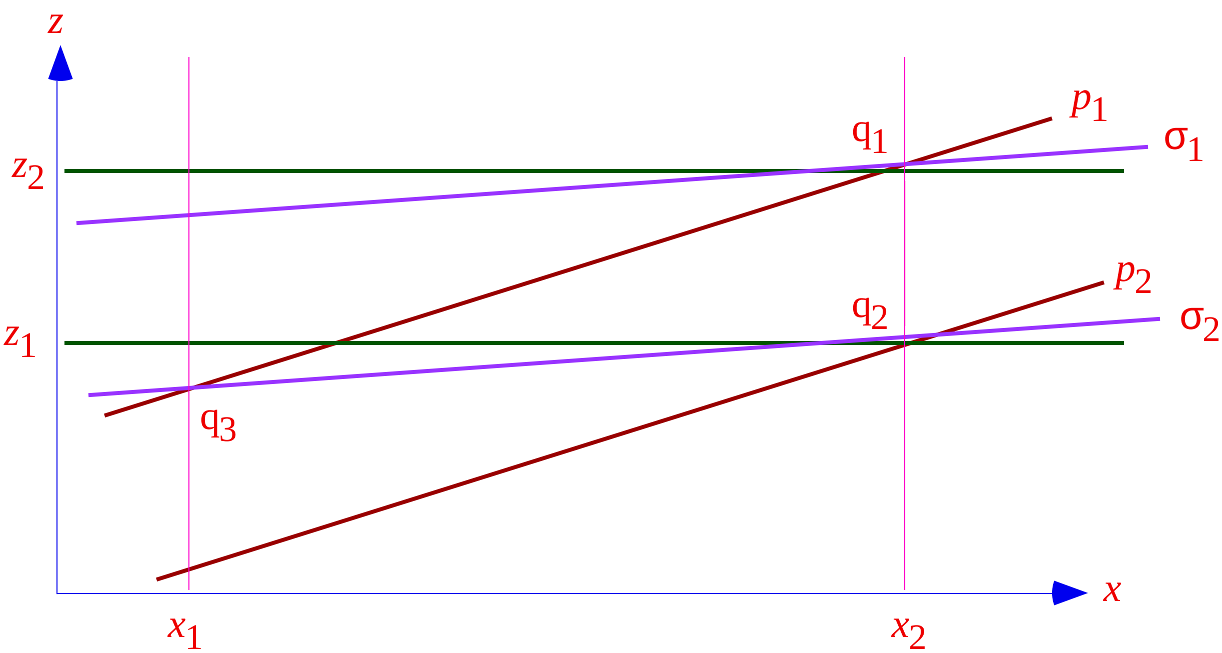
(5.49)

Substitute into (5.48)

$$ \boxed{{}_{z} = {}_{\sigma} - \left[\frac{\sigma}{\pi_a} {}_{z}\left(\pi_a\right) - \frac{{}_{z}\left(p_a\right)}{\pi_a}\right]\frac{\partial}{\partial\sigma}} \qquad (5.50) $$

Intersection of Pressure, Altitude, and Sigma-Pressure Surfaces

Fig. 5.7.



Gradient Conversion From the Pressure to Sigma-Pressure Coordinate

Change in mixing ratio per unit distance

$$\frac{q_1 - q_3}{x_2 - x_1} = \frac{q_2 - q_3}{x_2 - x_1} + \left(\frac{\sigma_1 - \sigma_2}{x_2 - x_1}\right)\left(\frac{q_1 - q_2}{\sigma_1 - \sigma_2}\right) \qquad (5.51)$$

Gradient conversion from p to σ-p coordinate

$$\left(\frac{\partial q}{\partial x}\right)_p = \left(\frac{\partial q}{\partial x}\right)_\sigma + \left(\frac{\partial \sigma}{\partial x}\right)_p \left(\frac{\partial q}{\partial \sigma}\right)_x \qquad (5.52)$$

Generalize

$${}_p = \ {}_\sigma + \ {}_p(\sigma)\frac{\partial}{\partial \sigma} \qquad (5.53)$$

Take gradient of sigma along surface of constant pressure

$${}_p(\sigma) = \left(p_a - p_{top}\right) \ {}_p\left(\frac{1}{\pi_a}\right) + \frac{{}_p\left(p_a - p_{top}\right)}{\pi_a} = -\frac{\sigma}{\pi_a} \ {}_p\left(\pi_a\right) \qquad (5.54)$$

where $\quad {}_p\left(p_a\right) = 0 \qquad {}_p\left(p_{top}\right) = 0$

$${}_p\left(\pi_a\right) = \ {}_\sigma\left(\pi_a\right) = \ {}_z\left(\pi_a\right)$$

Substitute (5.54) into (5.53)

$${}_p = \ {}_\sigma - \frac{\sigma}{\pi_a} \ {}_\sigma\left(\pi_a\right)\frac{\partial}{\partial \sigma} \qquad (5.55)$$

Continuity Equation For Air in the Sigma-Pressure Coordinate

Continuity equation for air in the pressure coordinate

$$p \cdot \mathbf{v}_h + \frac{\partial w_p}{\partial p_a} = 0$$

Substitute gradient conversion and $\partial p_a / \partial \sigma = \pi_a$

$$\sigma \cdot \mathbf{v}_h - \frac{\sigma}{\pi_a} \quad \sigma(\pi_a) \cdot \frac{\partial \mathbf{v}_h}{\partial \sigma} + \frac{1}{\pi_a}\frac{\partial w_p}{\partial \sigma} = 0 \qquad (5.56)$$

Vertical velocity in the pressure coordinate

$$w_p = \frac{dp_a}{dt} = \sigma\frac{d\pi_a}{dt} + \frac{d\sigma}{dt}\pi_a = \sigma\frac{d\pi_a}{dt} + \dot{\sigma}\pi_a \qquad (5.58)$$

where

$$p_a = p_{a,top} + \pi_a \sigma$$

Vertical velocity in the sigma-pressure coordinate

$$\dot{\sigma} = \frac{d\sigma}{dt} \qquad (5.57)$$

Continuity Equation For Air in the Sigma-Pressure Coordinate

Material time derivative in the sigma-pressure coordinate

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_\sigma + \left(\mathbf{v}_h \cdot \nabla_\sigma\right) + \dot{\sigma}\frac{\partial}{\partial \sigma} \qquad (5.59)$$

Substituting the total derivative of π_a into (5.58)

$$w_p = \sigma\left[\left(\frac{\partial \pi_a}{\partial t}\right)_\sigma + \left(\mathbf{v}_h \cdot \nabla_\sigma\right)\pi_a\right] + \dot{\sigma}\pi_a \qquad (5.60)$$

Take partial derivative

$$\frac{\partial w_p}{\partial \sigma} = \left(\frac{\partial \pi_a}{\partial t}\right)_\sigma + \left(\mathbf{v}_h \cdot \nabla_\sigma\right)\pi_a + \sigma\nabla_\sigma\left(\pi_a\right) \cdot \frac{\partial \mathbf{v}_h}{\partial \sigma} + \pi_a\frac{\partial \dot{\sigma}}{\partial \sigma} \qquad (5.61)$$

Substitute into (5.56) --> continuity equation for air

$$\left(\frac{\partial \pi_a}{\partial t}\right)_\sigma + \nabla_\sigma \cdot \left(\mathbf{v}_h \pi_a\right) + \pi_a\frac{\partial \dot{\sigma}}{\partial \sigma} = 0 \qquad (5.62)$$

Convert to spherical-sigma-pressure coordinates

$$R_e^2\cos\phi\left(\frac{\partial \pi_a}{\partial t}\right)_\sigma + \left[\frac{\partial}{\partial \lambda_e}\left(u\pi_a R_e\right) + \frac{\partial}{\partial \phi}\left(v\pi_a R_e\cos\phi\right)\right]_\sigma + \pi_a R_e^2\cos\phi\frac{\partial \dot{\sigma}}{\partial \sigma} = 0 \qquad (5.63)$$

Column Pressure From the Continuity Equation

Continuity equation for air

$$\left(\frac{\partial \pi_a}{\partial t}\right)_\sigma + {}_\sigma \cdot \left(\mathbf{v}_h \pi_a\right) + \pi_a \frac{\partial \dot{\sigma}}{\partial \sigma} = 0 \qquad (5.62)$$

Rearrange and integrate

$$\int_0^1 \left(\frac{\partial \pi_a}{\partial t}\right)_\sigma d\sigma = - {}_\sigma \cdot \int_0^1 \left(\mathbf{v}_h \pi_a\right) d\sigma - \pi_a \int_0^0 d\dot{\sigma} \qquad (5.64)$$

Prognostic equation for column pressure

$$\left(\frac{\partial \pi_a}{\partial t}\right)_\sigma = - {}_\sigma \cdot \int_0^1 \left(\mathbf{v}_h \pi_a\right) d\sigma \qquad (5.65)$$

Analogous equation in spherical-sigma-pressure coordinates

$$R_e^2 \cos\phi \left(\frac{\partial \pi_a}{\partial t}\right)_\sigma = -\int_0^1 \left[\frac{\partial}{\partial \lambda_e}\left(u \pi_a R_e\right) + \frac{\partial}{\partial \phi}\left(v \pi_a R_e \cos\phi\right)\right]_\sigma d\sigma \qquad (5.66)$$

Vertical Velocity From Continuity Equation

Continuity equation for air

$$\left(\frac{\partial \pi_a}{\partial t}\right)_\sigma + \nabla_\sigma \cdot \left(\mathbf{v}_h \pi_a\right) + \pi_a \frac{\partial \dot{\sigma}}{\partial \sigma} = 0 \qquad (5.62)$$

Rearrange and integrate

$$\pi_a \int_0^{\dot{\sigma}} d\dot{\sigma} = -\nabla_\sigma \cdot \int_0^\sigma \left(\mathbf{v}_h \pi_a\right) d\sigma - \int_0^\sigma \left(\frac{\partial \pi_a}{\partial t}\right)_\sigma d\sigma \qquad (5.67)$$

Diagnostic equation for vertical velocity

$$\dot{\sigma}\pi_a = -\nabla_\sigma \cdot \int_0^\sigma \left(\mathbf{v}_h \pi_a\right) d\sigma - \sigma\left(\frac{\partial \pi_a}{\partial t}\right)_\sigma \qquad (5.68)$$

Analogous equation in spherical-sigma-pressure coordinates

$$\dot{\sigma}\pi_a R_e^2 \cos\phi = -\int_0^\sigma \left[\frac{\partial}{\partial \lambda_e}\left(u\pi_a R_e\right) + \frac{\partial}{\partial \phi}\left(v\pi_a R_e \cos\phi\right)\right]_\sigma d\sigma - \sigma R_e^2 \cos\phi \left(\frac{\partial \pi_a}{\partial t}\right)_\sigma$$

(5.69)

Species Continuity Equation in Spherical-Sigma-Pressure Coordinates

Species continuity equation in Cartesian-altitude coordinates

$$\frac{\partial q}{\partial t} + (\mathbf{v} \cdot \nabla) q = \frac{1}{\rho_a} (\nabla \cdot \rho_a \mathbf{K}_h \nabla) q + \sum_{n=1}^{N_{e,t}} R_n \quad (3.54)$$

Apply material time derivative is sigma-pressure coordinate

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_\sigma + \mathbf{v}_h \cdot \nabla_\sigma + \dot{\sigma}\frac{\partial}{\partial \sigma}$$

--> Equation in Cartesian-sigma-pressure coordinates

$$\frac{dq}{dt} = \left(\frac{\partial q}{\partial t}\right)_\sigma + (\mathbf{v}_h \cdot \nabla_\sigma) q + \dot{\sigma}\frac{\partial q}{\partial \sigma} = \frac{(\nabla \cdot \rho_a \mathbf{K}_h \nabla) q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n$$

(5.70)

Combine with continuity equation for air

$$\left[\frac{\partial (\pi_a q)}{\partial t}\right]_\sigma + \nabla_\sigma \cdot (\mathbf{v}_h \pi_a q) + \pi_a \frac{\partial (\dot{\sigma} q)}{\partial \sigma} = \pi_a \left[\frac{(\nabla \cdot \rho_a \mathbf{K}_h \nabla) q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n\right]$$

(5.72)

Apply spherical-coordinate transformations

$$R_e^2 \cos\phi \left[\frac{\partial}{\partial t}(\pi_a q)\right]_\sigma + \left[\frac{\partial}{\partial \lambda_e}(u \pi_a q R_e) + \frac{\partial}{\partial \phi}(v \pi_a q R_e \cos\phi)\right]_\sigma$$

$$+ \pi_a R_e^2 \cos\phi \frac{\partial}{\partial \sigma}(\dot{\sigma} q) = \pi_a R_e^2 \cos\phi \left[\frac{(\nabla \cdot \rho_a \mathbf{K}_h \nabla) q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n\right]$$

(5.73)

Thermodynamic Energy Equation in Spherical-Sigma-Pressure Coordinates

Therm. energy equation in Cartesian-altitude coordinates

$$\frac{\partial \theta_v}{\partial t} + (\mathbf{v} \cdot \nabla)\theta_v = \frac{1}{\rho_a}(\nabla \cdot \rho_a \mathbf{K}_h \nabla)\theta_v + \frac{\theta_v}{c_{p,d} T} \sum_{n=1}^{N_{e,h}} \frac{dQ_n}{dt} \qquad (3.76)$$

Apply the sigma-pressure coordinate material time derivative

$$\left(\frac{\partial \theta_v}{\partial t}\right)_\sigma + (\mathbf{v}_h \cdot \nabla_\sigma)\theta_v + \dot{\sigma}\frac{\partial \theta_v}{\partial \sigma} = \frac{(\nabla \cdot \rho_a \mathbf{K}_h \nabla)\theta_v}{\rho_a} + \frac{\theta_v}{c_{p,d} T_v} \sum_{n=1}^{N_{e,h}} \frac{dQ_n}{dt}$$

(5.74)

Combine with continuity equation for air

$$\left[\frac{\partial(\pi_a \theta_v)}{\partial t}\right]_\sigma + \nabla_\sigma \cdot (\mathbf{v}_h \pi_a \theta_v) + \pi_a \frac{\partial(\dot{\sigma}\theta_v)}{\partial \sigma} \qquad (5.75)$$

$$= \pi_a \left[\frac{(\nabla \cdot \rho_a \mathbf{K}_h \nabla)\theta_v}{\rho_a} + \frac{\theta_v}{c_{p,d} T_v} \sum_{n=1}^{N_{e,h}} \frac{dQ_n}{dt}\right]$$

Apply spherical-coordinate transformations

$$R_e^2 \cos\phi \left[\frac{\partial}{\partial t}(\pi_a \theta_v)\right]_\sigma + \left[\frac{\partial}{\partial \lambda_e}(u \pi_a \theta_v R_e) + \frac{\partial}{\partial \phi}(v \pi_a \theta_v R_e \cos\phi)\right]_\sigma$$

$$+ \pi_a R_e^2 \cos\phi \frac{\partial}{\partial \sigma}(\dot{\sigma}\theta_v) = \pi_a R_e^2 \cos\phi \left[\frac{(\nabla \cdot \rho_a \mathbf{K}_h \nabla)\theta_v}{\rho_a} + \frac{\theta_v}{c_{p,d} T_v} \sum_{n=1}^{N_{e,h}} \frac{dQ_n}{dt}\right]$$

(5.76)

Momentum Equation in the Sigma-Pressure Coordinate

Horizontal momentum equation in Cartesian-altitude coordinates

$$\frac{d\mathbf{v}}{dt} = -f\mathbf{k}\times\mathbf{v} - \nabla\Phi - \frac{1}{\rho_a}\nabla p_a + \frac{\eta_a}{\rho_a}\nabla^2\mathbf{v} + \frac{1}{\rho_a}\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)\mathbf{v} \tag{4.70}$$

Material time derivative of velocity

$$\frac{d\mathbf{v}_h}{dt} = \left(\frac{\partial\mathbf{v}_h}{\partial t}\right)_\sigma + \left(\mathbf{v}_h\cdot\nabla_\sigma\right)\mathbf{v}_h + \dot{\sigma}\frac{\partial\mathbf{v}_h}{\partial\sigma}$$

Apply to horizontal momentum equation

$$\left(\frac{\partial\mathbf{v}_h}{\partial t}\right)_\sigma + \left(\mathbf{v}_h\cdot\nabla_\sigma\right)\mathbf{v}_h + \dot{\sigma}\frac{\partial\mathbf{v}_h}{\partial\sigma} + f\mathbf{k}\times\mathbf{v}_h = -\frac{1}{\rho_a}\nabla_z\left(p_a\right) + \frac{\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)\mathbf{v}_h}{\rho_a} \tag{5.77}$$

Pressure gradient term

$$\frac{1}{\rho_a}\nabla_z\left(p_a\right) = \nabla_p\Phi = \nabla_\sigma\Phi - \frac{\sigma}{\pi_a}\nabla_\sigma\left(\pi_a\right)\frac{\partial\Phi}{\partial\sigma} \tag{5.78}$$

Substitute into momentum equation

$$\left(\frac{\partial\mathbf{v}_h}{\partial t}\right)_\sigma + \left(\mathbf{v}_h\cdot\nabla_\sigma\right)\mathbf{v}_h + \dot{\sigma}\frac{\partial\mathbf{v}_h}{\partial\sigma} + f\mathbf{k}\times\mathbf{v}_h = -\nabla_\sigma\Phi + \frac{\sigma}{\pi_a}\nabla_\sigma\left(\pi_a\right)\frac{\partial\Phi}{\partial\sigma} + \frac{\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)\mathbf{v}_h}{\rho_a} \tag{5.79}$$

Coupling Horizontal and Vertical Momentum Equations

Hydrostatic equation in the pressure coordinate

$$\frac{\partial \Phi}{\partial \sigma} = -\frac{\pi_a R' T_v}{p_a} = -\frac{\pi_a}{\rho_a} = -\alpha_a \pi_a \qquad (5.80)$$

Re-derive specific density

$$\alpha_a = \frac{R' T_v}{p_a} = \frac{\kappa c_{p,d} \theta_v P}{p_a} = c_{p,d} \theta_v \frac{\partial P}{\partial p_a} = \frac{c_{p,d} \theta_v}{\pi_a} \frac{\partial P}{\partial \sigma} \qquad (5.82)$$

Combine terms above with momentum and continuity equations

$$\left[\frac{\partial (\mathbf{v}_h \pi_a)}{\partial t}\right]_\sigma + \mathbf{v}_h \nabla_\sigma \cdot (\mathbf{v}_h \pi_a) + \pi_a (\mathbf{v}_h \cdot \nabla_\sigma) \mathbf{v}_h + \pi_a \frac{\partial}{\partial \sigma} (\dot{\sigma} \mathbf{v}_h)$$

$$= -\pi_a f \mathbf{k} \times \mathbf{v}_h - \pi_a \nabla_\sigma \Phi - \sigma c_{p,d} \theta_v \frac{\partial P}{\partial \sigma} \nabla_\sigma (\pi_a)$$

$$+ \pi_a \frac{(\nabla \cdot \rho_a \mathbf{K}_m \nabla) \mathbf{v}_h}{\rho_a} \qquad (5.83)$$

Expand advection terms

$$\mathbf{v}_h \nabla_\sigma \cdot (\mathbf{v}_h \pi_a) = \mathbf{i} u \left[\frac{\partial (u \pi_a)}{\partial x} + \frac{\partial (v \pi_a)}{\partial y}\right] + \mathbf{j} v \left[\frac{\partial (u \pi_a)}{\partial x} + \frac{\partial (v \pi_a)}{\partial y}\right] \qquad (5.84)$$

$$\pi_a (\mathbf{v}_h \cdot \nabla_\sigma) \mathbf{v}_h = \mathbf{i} \pi_a \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) + \mathbf{j} \pi_a \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}\right) \qquad (5.85)$$

Momentum Equation in Spherical-Sigma-Pressure Coordinates

$$R_e^2\cos\phi\left[\frac{\partial}{\partial t}(\pi_a u)\right]_\sigma + \left[\frac{\partial}{\partial\lambda_e}\left(\pi_a u^2 R_e\right) + \frac{\partial}{\partial\phi}\left(\pi_a uvR_e\cos\phi\right)\right]_\sigma + \pi_a R_e^2\cos\phi\frac{\partial}{\partial\sigma}(\dot{\sigma}u)$$

$$= \pi_a uvR_e\sin\phi + \pi_a fvR_e^2\cos\phi - R_e\left(\pi_a\frac{\partial\Phi}{\partial\lambda_e} + \sigma c_{p,d}\theta_v\frac{\partial P}{\partial\sigma}\frac{\partial\pi_a}{\partial\lambda_e}\right)_\sigma$$

$$+ R_e^2\cos\phi\frac{\pi_a}{\rho_a}\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)u \qquad (5.86)$$

$$R_e^2\cos\phi\left[\frac{\partial}{\partial t}(\pi_a v)\right]_\sigma + \left[\frac{\partial}{\partial\lambda_e}\left(\pi_a uvR_e\right) + \frac{\partial}{\partial\phi}\left(v^2\pi_a R_e\cos\phi\right)\right]_\sigma + \pi_a R_e^2\cos\phi\frac{\partial}{\partial\sigma}(\dot{\sigma}v)$$

$$= -\pi_a u^2 R_e\sin\phi - \pi_a fuR_e^2\cos\phi - R_e\cos\phi\left(\pi_a\frac{\partial\Phi}{\partial\phi} + \sigma c_{p,d}\theta_v\frac{\partial P}{\partial\sigma}\frac{\partial\pi_a}{\partial\phi}\right)_\sigma$$

$$+ R_e^2\cos\phi\frac{\pi_a}{\rho_a}\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)v \qquad (5.87)$$

The Sigma-Altitude Coordinate

Sigma-altitude value

$$s = \frac{z_{top} - z}{z_{top} - z_{surf}} = \frac{z_{top} - z}{Z_t} \qquad (5.89)$$

Altitude difference between column top and surface

$$Z_t = z_{top} - z_{surf}$$

Altitude of a sigma surface

$$z = z_{top} - Z_t s \qquad (5.90)$$

Gradient Conversion From the Altitude to Sigma-Altitude Coordinate

Gradient conversion between *z* and *s*-*z* coordinate

$$ {}_{z} = {}_{s} + {}_{z}(s)\frac{\partial}{\partial s} \qquad (5.91) $$

Horiz. gradient of sigma along const. altitude surface

$$ {}_{z}(s) = -\frac{z_{top} - z}{Z_t^2} \quad {}_{z}(Z_t) = -\frac{s}{Z_t} \quad {}_{z}(Z_t) \qquad (5.92) $$

Substitute into gradient conversion

$$ {}_{z} = {}_{s} - \frac{s}{Z_t} \quad {}_{z}(Z_t)\frac{\partial}{\partial s} \qquad (5.93) $$

Conversions in the Sigma-Altitude Coordinate

Time-derivative conversion between *z* and *σ*-*z* coordinate

$$\left(\frac{\partial}{\partial t}\right)_z = \left(\frac{\partial}{\partial t}\right)_\sigma \qquad (5.94)$$

Scalar velocity in the sigma-altitude coordinate

$$\dot{\sigma} = \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \left(\mathbf{v}_h \cdot \nabla_z\right)\sigma + w\frac{\partial \sigma}{\partial z} = \left(\mathbf{v}_h \cdot \nabla_z\right)\sigma - \frac{w}{Z_t} \qquad (5.95)$$

where

$$\frac{\partial \sigma}{\partial z} = -\frac{1}{Z_t}$$

Material time derivative in the sigma-altitude coordinate

$$\frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\partial}{\partial t}\right)_\sigma + \left(\mathbf{v}_h \cdot \nabla_\sigma\right) + \dot{\sigma}\frac{\partial}{\partial \sigma} \qquad (5.96)$$

Continuity Equation For Air in the Sigma-Altitude Coordinate

Continuity equation for air in the *z* coordinate

$$\left(\frac{\partial \rho_a}{\partial t}\right)_z = -\rho_a\left(\nabla_z \cdot \mathbf{v}_h + \frac{\partial w}{\partial z}\right) - \left(\mathbf{v}_h \cdot \nabla_z\right)\rho_a - w\frac{\partial \rho_a}{\partial z}$$

Apply gradient conversion to horizontal velocity

$$\nabla_z \cdot \mathbf{v}_h = \nabla_s \cdot \mathbf{v}_h + \nabla_z(s)\frac{\partial \mathbf{v}_h}{\partial s}$$

Apply gradient conversion to dry air density

$$\nabla_z(\rho_a) = \nabla_s(\rho_a) + \nabla_z(s)\frac{\partial \rho_a}{\partial s}$$

Substitute these two terms into continuity equation above

$$\left(\frac{\partial \rho_a}{\partial t}\right)_s = -\rho_a\left[\nabla_s \cdot \mathbf{v}_h + \nabla_z(s)\frac{\partial \mathbf{v}_h}{\partial s} + \frac{\partial w}{\partial z}\right] - \mathbf{v}_h \cdot \left[\nabla_s(\rho_a) + \nabla_z(s)\frac{\partial \rho_a}{\partial s}\right] - w\frac{\partial \rho_a}{\partial z} \qquad (5.97)$$

Continuity Equation For Air in the Sigma-Altitude Coordinate

Rewrite vertical velocity equation

$$w = Z_t\left[\mathbf{v}_h \cdot \nabla_z(s) - \dot{s}\right]$$

Differentiate with respect to altitude

$$\frac{\partial w}{\partial z} = Z_t\left[\nabla_z(s)\frac{\partial \mathbf{v}_h}{\partial z} + \left(\mathbf{v}_h \cdot \nabla_z\right)\frac{\partial s}{\partial z} - \frac{\partial \dot{s}}{\partial z}\right] \qquad (5.98)$$

Substitute $\partial s/\partial z = -1/Z_t$

$$\frac{\partial w}{\partial z} = \frac{\partial \dot{s}}{\partial s} - \nabla_z(s)\frac{\partial \mathbf{v}_h}{\partial s} + \frac{1}{Z_t}\left(\mathbf{v}_h \cdot \nabla_z\right)Z_t \qquad (5.99)$$

Substitute $w = Z_t\left[\mathbf{v}_h \cdot \nabla_z(s) - \dot{s}\right]$, (5.99) and $\partial s/\partial z = -1/Z_t$ into (5.97)

$$\left(\frac{\partial \rho_a}{\partial t}\right)_s = -\rho_a\left[\nabla_s \cdot \mathbf{v}_h + \frac{\partial \dot{s}}{\partial s} + \frac{1}{Z_t}\left(\mathbf{v}_h \cdot \nabla_z\right)Z_t\right] - \left(\mathbf{v}_h \cdot \nabla_s\right)\rho_a - \dot{s}\frac{\partial \rho_a}{\partial s}$$

(5.100)

Continuity Equation For Air in the Sigma-Altitude Coordinate

Substitute $\nabla_z(Z_t) = \nabla_s(Z_t)$ and compress -->

Nonhydrostatic continuity equation for air in *s*-*z* coordinate

$$\left(\frac{\partial \rho_a}{\partial t}\right)_s = -\frac{1}{Z_t}\nabla_s \cdot \left(\mathbf{v}_h \rho_a Z_t\right) - \frac{\partial}{\partial s}\left(\dot{s}\rho_a\right)$$

$$= -\frac{1}{Z_t}\left[\frac{\partial\left(u\rho_a Z_t\right)}{\partial x} + \frac{\partial\left(v\rho_a Z_t\right)}{\partial y}\right]_s - \frac{\partial\left(\dot{s}\rho_a\right)}{\partial s} \qquad (5.101)$$

Hydrostatic equation in the sigma-altitude coordinate

$$\rho'_a = -\frac{1}{g}\frac{\partial p'_a}{\partial z} = \frac{1}{Z_t g}\frac{\partial p'_a}{\partial s} \qquad (5.102)$$

Substitute into (5.101) --> Hydrostatic continuity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial p'_a}{\partial s}\right) = -\nabla_s \cdot \left(\mathbf{v}_h \frac{\partial p'_a}{\partial s}\right) - \frac{\partial}{\partial s}\left(\dot{s}\frac{\partial p'_a}{\partial s}\right) \qquad (5.103)$$

Species Continuity Equation in the Sigma-Altitude Coordinate

Apply material derivative in the *s*-*z* coordinate to the continuity equation for a trace species in the *z* coordinate

$$\left(\frac{dq}{dt}\right)_s = \left(\frac{\partial q}{\partial t}\right)_s + \left(\mathbf{v}_h \cdot \nabla_s\right) q + \dot{s}\frac{\partial q}{\partial s} = \frac{\left(\nabla \cdot \rho_a \mathbf{K}_h \nabla\right) q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n$$

(5.104)

Thermodynamic Energy Equation in the Sigma-Altitude Coordinate

Apply material derivative in the *s*-*z* coordinate to the thermodynamic energy equation in the *z* coordinate

$$\left(\frac{\partial \theta_v}{\partial t}\right)_s + \left(\mathbf{v}_h \cdot \nabla_s\right)\theta_v + \dot{s}\frac{\partial \theta_v}{\partial s} = \frac{\left(\nabla \cdot \rho_a \mathbf{K}_h \nabla\right)\theta_v}{\rho_a} + \frac{\theta_v}{c_{p,d} T_v}\sum_{n=1}^{N_{e,h}} \frac{\mathrm{d}Q_n}{\mathrm{d}t}$$

(5.106)

Horizontal Momentum Equation in the Sigma-Altitude Coordinate

Horizontal equation in the *z* coordinate

$$\frac{d\mathbf{v}_h}{dt} = -f\mathbf{k}\times\mathbf{v}_h - \frac{1}{\rho_a}\nabla_z(p_a) + \frac{(\nabla\cdot\rho_a\mathbf{K}_m\nabla)\mathbf{v}_h}{\rho_a}$$

Apply material time derivative of velocity

$$\left(\frac{\partial\mathbf{v}_h}{\partial t}\right)_\sigma + (\mathbf{v}_h\cdot\nabla_\sigma)\mathbf{v}_h + \dot{\sigma}\frac{\partial\mathbf{v}_h}{\partial\sigma} + f\mathbf{k}\times\mathbf{v}_h = -\frac{1}{\rho_a}\nabla_z(p_a) + \frac{(\nabla_z\cdot\rho_a\mathbf{K}_m\nabla_z)\mathbf{v}_h}{\rho_a} \quad (5.107)$$

Gradient conversion of pressure

$$\nabla_z(p_a) = \nabla_\sigma(p_a) - \frac{\sigma}{Z_t}\nabla_z(Z_t)\frac{\partial p_a}{\partial\sigma} \quad (5.108)$$

Substitute gradient conversion

$$\left(\frac{\partial\mathbf{v}_h}{\partial t}\right)_\sigma + (\mathbf{v}_h\cdot\nabla_\sigma)\mathbf{v}_h + \dot{\sigma}\frac{\partial\mathbf{v}_h}{\partial\sigma} \quad (5.109)$$

$$= -f\mathbf{k}\times\mathbf{v}_h - \frac{1}{\rho_a}\left[\nabla_\sigma(p_a) - \frac{\sigma}{Z_t}\nabla_z(Z_t)\frac{\partial p_a}{\partial\sigma} - (\nabla\cdot\rho_a\mathbf{K}_m\nabla)\mathbf{v}_h\right]$$

Vertical Momentum Equation in the Sigma-Altitude Coordinate

Substitute $\partial\sigma/\partial z = -1/Z_t$ into vertical momentum eq. in *z* coordinate

$$\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right)_\sigma + \dot{\sigma}\frac{\partial w}{\partial \sigma} = -g + \frac{1}{Z_t\rho_a}\frac{\partial p_a}{\partial \sigma} + \frac{(\nabla\cdot\rho_a\mathbf{K}_m\nabla)w}{\rho_a}$$

(5.113)

Substitute

$$w = Z_t\left[\mathbf{v}_h\cdot\nabla_z(\sigma) - \dot{\sigma}\right]$$

Another form of vertical momentum equation

$$\left[\left(\frac{\partial}{\partial t}\right)_\sigma + u\left(\frac{\partial}{\partial x}\right)_\sigma + v\left(\frac{\partial}{\partial y}\right)_\sigma + \dot{\sigma}\frac{\partial}{\partial \sigma}\right]\left[Z_t u\left(\frac{\partial \sigma}{\partial x}\right)_z + Z_t v\left(\frac{\partial \sigma}{\partial y}\right)_z - Z_t\dot{\sigma}\right] = -g + \frac{1}{Z_t\rho_a}\frac{\partial p_a}{\partial \sigma}$$

$$+ \frac{1}{\rho_a}(\nabla\cdot\rho_a\mathbf{K}_m\nabla)\left[Z_t u\left(\frac{\partial \sigma}{\partial x}\right)_z + Z_t v\left(\frac{\partial \sigma}{\partial y}\right)_z - Z_t\dot{\sigma}\right]$$

(5.114)