Overhead Slides for

Chapter 7

of

Fundamentals of Atmospheric Modeling

by

Mark Z. Jacobson

Department of
Civil & Environmental Engineering
Stanford University
Stanford, CA 94305-4020

January 30, 2002

Vertical Model Grid

Fig. 7.1. Location of variable values in a vertical grid.

___Model top boundary__ $\dot{\sigma}_{1/2} = 0, \quad \sigma_{1/2} = 0, \quad p_{a,top}$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ $q_1, \quad \theta_{v,1}, \quad u_1, \quad v_1, \quad p_{a,1}$

______________________ $\dot{\sigma}_{1+1/2}, \quad \sigma_{1+1/2}, \quad p_{a,1+1/2}$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ $q_2, \quad \theta_{v,2}, \quad u_2, \quad v_2, \quad p_{a,2}$

______________________ $\dot{\sigma}_{k-1/2}, \quad \sigma_{k-1/2}, \quad p_{a,k-1/2}$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ $q_k, \quad \theta_{v,k}, \quad u_k, \quad v_k, \quad p_{a,k}$

______________________ $\dot{\sigma}_{k+1/2}, \quad \sigma_{k+1/2}, \quad p_{a,k+1/2}$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ $q_{k+1}, \quad \theta_{v,k+1}, \quad u_{k+1}, \quad v_{k+1}, \quad p_{a,k+1}$

______________________ $\dot{\sigma}_{N_L-1/2}, \quad \sigma_{N_L-1/2}, \quad p_{a,N_L-1/2}$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ $q_{N_L}, \quad \theta_{v,N_L}, \quad u_{N_L}, \quad v_{N_L}, \quad p_{a,N_L}$

Model bottom boundary $\dot{\sigma}_{N_L+1/2} = 0, \quad \sigma_{N_L+1/2} = 1, \quad p_{a,surf}$

Estimating Sigma Levels

Estimate altitude at bottom boundary of each test layer

$$z_{k+1/2,test} = z_{top,test}\left(1 - \frac{k}{N_L}\right) \qquad \text{for } k = 0,\ldots,N_L \qquad (7.2)$$

Find pressures from table --> sigma values

$$\sigma_{k+1/2} = \frac{p_{a,k+1/2,test} - p_{a,top,test}}{p_{a,N_L+1/2,test} - p_{a,top,test}} \qquad \text{for } k = 1,\ldots,N_L \qquad (7.3)$$

Pressure at the bottom boundary of a model layer

$$p_{a,k+1/2} = p_{a,top} + \sigma_{k+1/2}\pi_a \qquad (7.4)$$

Layer-Midpoint Pressure

Example layers

————————	$p_{a,k-1/2}$	= 700 mb
– – – – – – – – –	$\theta_{v,k}$	= 308 K
————————	$p_{a,k+1/2}$	= 750 mb
– – – – – – – – –	$\theta_{v,k+1}$	= 303 K
————————	$p_{a,k+3/2}$	= 800 mb

Pressure at the mass-center of a layer

$$p_{a,k} = p_{a,k-1/2} + 0.5\left(p_{a,k+1/2} - p_{a,k-1/2}\right) \qquad (7.5)$$

Layer-Midpoint Pressure

Pressure where mass-weighted mean of P is located

When pot. virtual temp. increases monotonically with height

$$p_{a,k} = (1000 \text{ mb}) P_k^{1/\kappa} \qquad (7.8)$$

Mass-weighed mean of P (7.6)

$$P_k = \frac{1}{p_{a,k+1/2} - p_{a,k-1/2}} \int_{p_{a,k-1/2}}^{p_{a,k+1/2}} P \, dp_a = \frac{1}{1+\kappa} \left(\frac{P_{k+1/2} p_{a,k+1/2} - P_{k-1/2} p_{a,k-1/2}}{p_{a,k+1/2} - p_{a,k-1/2}} \right)$$

Values of P at layer boundaries

$$P_{k+1/2} = \left(p_{a,k+1/2} / 1000 \text{ mb} \right)^{\kappa} \qquad (7.7)$$

Consistent formula for potential virtual temperature at a layer boundary

$$\theta_{v,k+1/2} = \frac{\left(P_{k+1/2} - P_k \right) \theta_{v,k} + \left(P_{k+1} - P_{k+1/2} \right) \theta_{v,k+1}}{P_{k+1} - P_k}$$

(7.9)

Finite Differencing the Continuity Equation for Air

Prognostic equation for column pressure

$$R_e^2 \cos\phi \left(\frac{\partial \pi_a}{\partial t}\right)_\sigma = -\int_0^1 \left[\frac{\partial}{\partial \lambda_e}\left(u\pi_a R_e\right) + \frac{\partial}{\partial \phi}\left(v\pi_a R_e \cos\phi\right)\right]_\sigma d\sigma$$

(7.10)

First-order in time, second-order in space approximation

$$\left(R_e^2 \cos\phi \Delta\lambda_e \Delta\phi\right)_{i,j}\left(\frac{\pi_{a,t} - \pi_{a,t-h}}{h}\right)_{i,j}$$

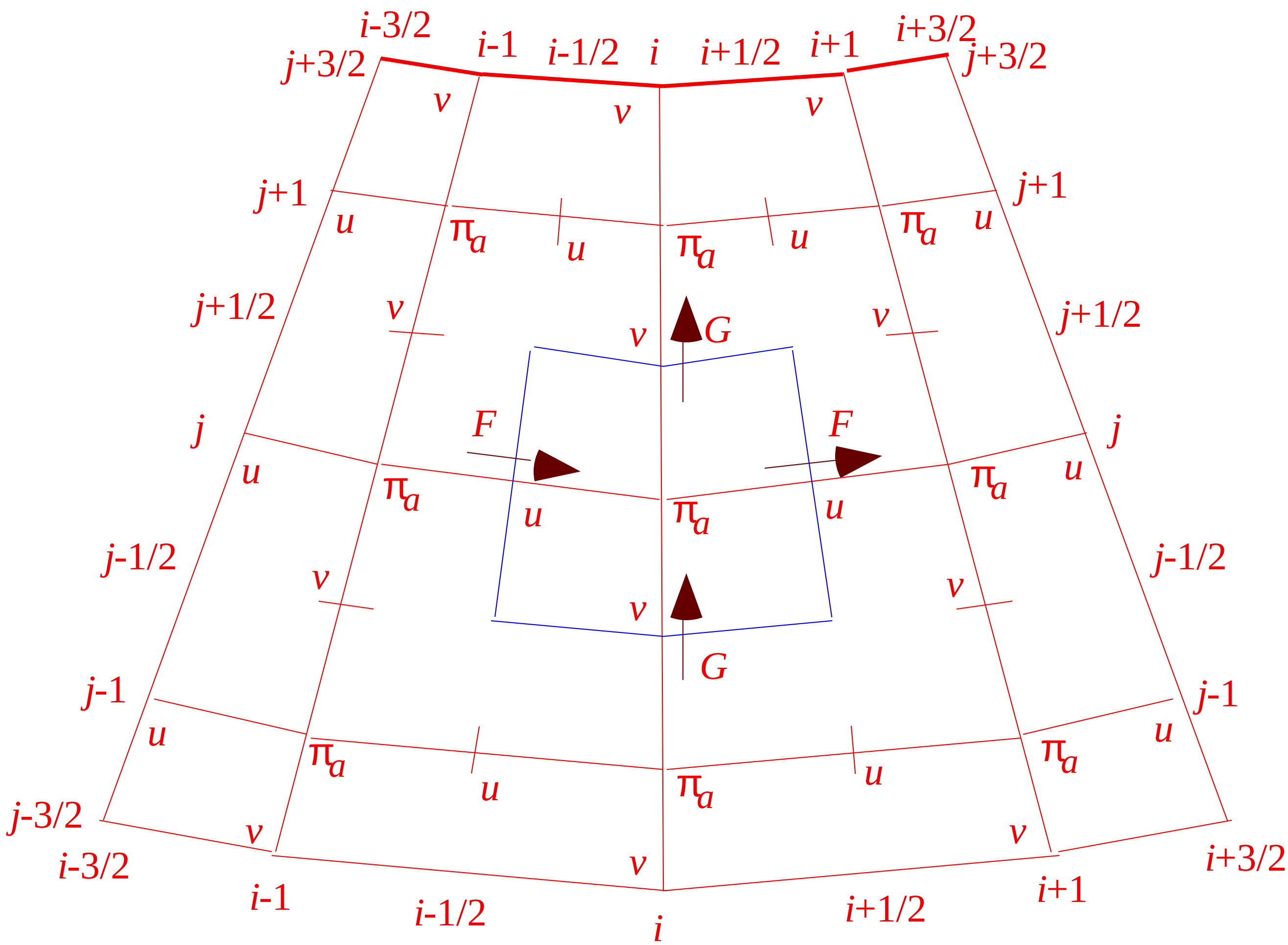
(7.11)

$$= -\sum_{k=1}^{N_L}\left[\frac{\left(u\pi_a R_e \Delta\phi\Delta\lambda_e\Delta\sigma\right)_{i+1/2,j} - \left(u\pi_a R_e \Delta\phi\Delta\lambda_e\Delta\sigma\right)_{i-1/2,j}}{\Delta\lambda_e}\right]_{k,t-h}$$

$$-\sum_{k=1}^{N_L}\left[\frac{\left(v\pi_a R_e \cos\phi\Delta\phi\Delta\lambda_e\Delta\sigma\right)_{i,j+1/2} - \left(v\pi_a R_e \cos\phi\Delta\phi\Delta\lambda_e\Delta\sigma\right)_{i,j-1/2}}{\Delta\phi}\right]_{k,t-h}$$

Arakawa C Grid

Fig. 7.2.



Prognostic Equation For Column Pressure

Define horizontal fluxes in domain interior

$$F_{i+1/2,j,k,t-h} = \left[\frac{\pi_{a,i,j} + \pi_{a,i+1,j}}{2}\left(uR_e\Delta\phi\right)_{i+1/2,j,k}\right]_{t-h} \quad (7.13)$$

$$G_{i,j+1/2,k,t-h} = \left[\frac{\pi_{a,i,j} + \pi_{a,i,j+1}}{2}\left(vR_e\cos\phi\Delta\lambda_e\right)_{i,j+1/2,k}\right]_{t-h} \quad (7.14)$$

Define horizontal fluxes at eastern and northern boundaries

$$F_{I+1/2,j,k,t-h} = \left[\pi_{a,I,j}\left(uR_e\Delta\phi\right)_{I+1/2,j,k}\right]_{t-h} \quad (7.15)$$

$$G_{i,J+1/2,k,t-h} = \left[\pi_{a,i,J}\left(vR_e\cos\phi\Delta\lambda_e\right)_{i,J+1/2,k}\right]_{t-h} \quad (7.16)$$

Prognostic equation for column pressure

$$\pi_{a,i,j,t} = \pi_{a,i,j,t-h} - \frac{h}{\left(R_e^2\cos\phi\Delta\lambda_e\Delta\phi\right)_{i,j}} \times \sum_{k=1}^{N_L}\left[\left(F_{i+1/2,j} - F_{i-1/2,j} + G_{i,j+1/2} - G_{i,j-1/2}\right)_{k,t-h}\Delta\sigma_k\right] \quad (7.12)$$

Vertical Velocity

Diagnostic equation for vertical velocity

$$\dot{\sigma}\pi_a R_e^2 \cos\phi = -\int_0^{\sigma}\left[\frac{\partial}{\partial\lambda_e}\left(u\pi_a R_e\right) + \frac{\partial}{\partial\phi}\left(v\pi_a R_e\cos\phi\right)\right] d\sigma - \sigma R_e^2\cos\phi\left(\frac{\partial\pi_a}{\partial t}\right)_{\sigma} \qquad (7.17)$$

Finite difference equation

$$\begin{aligned}
\left(\dot{\sigma}\pi_a R_e^2\cos\phi\Delta\lambda_e\Delta\phi\right)_{i,j,k+1/2,t} = \\
= -\sum_{l=1}^{k}\left[\frac{\left(u\pi_a R_e\Delta\lambda_e\Delta\phi\Delta\sigma\right)_{i-1/2,j} - \left(u\pi_a R_e\Delta\lambda_e\Delta\phi\Delta\sigma\right)_{i+1/2,j}}{\Delta\lambda_e}\right]_{l,t-h} \\
-\sum_{l=1}^{k}\left[\frac{\left(v\pi_a R_e\cos\phi\Delta\lambda_e\Delta\phi\Delta\sigma\right)_{i,j-1/2} - \left(v\pi_a R_e\cos\phi\Delta\lambda_e\Delta\phi\Delta\sigma\right)_{i,j+1/2}}{\Delta\phi}\right]_{l,t-h} \\
-\sigma_{k+1/2}\left(R_e^2\cos\phi\Delta\lambda_e\Delta\phi\right)_{i,j}\left(\frac{\pi_{a,t}-\pi_{a,t-h}}{h}\right)_{i,j} \qquad (7.18)
\end{aligned}$$

Substitute fluxes and rearrange --> vertical velocity

$$\begin{aligned}
\dot{\sigma}_{i,j,k+1/2,t} = -\frac{1}{\left(\pi_a R_e^2\cos\phi\Delta\lambda_e\Delta\phi\right)_{i,j,t}} \\
\times\sum_{l=1}^{k}\left[\left(F_{i+1/2,j} - F_{i-1/2,j} + G_{i,j+1/2} - G_{i,j-1/2}\right)_{l,t-h}\Delta\sigma_l\right] \\
-\sigma_{k+1/2}\left(\frac{\pi_{a,t}-\pi_{a,t-h}}{h\pi_{a,t}}\right)_{i,j} \qquad (7.19)
\end{aligned}$$

Species Continuity Equation

$$R_e^2\cos\phi\left[\frac{\partial}{\partial t}\left(\pi_a q\right)\right]_\sigma + \left[\frac{\partial}{\partial\lambda_e}\left(u\pi_a q R_e\right) + \frac{\partial}{\partial\phi}\left(v\pi_a q R_e\cos\phi\right)\right]_\sigma$$

$$+\pi_a R_e^2\cos\phi\frac{\partial}{\partial\sigma}\left(\dot{\sigma}q\right) = \pi_a R_e^2\cos\phi\left[\frac{\left(\quad\cdot\ \rho_a\mathbf{K}_h\quad\right)q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n\right]$$

(7.20)

Finite difference form

$$\left(R_e^2\cos\phi\Delta\lambda_e\Delta\phi\right)_{i,j}\left(\frac{\pi_{a,t}q_t - \pi_{a,t-h}q_{t-h}}{h}\right)_{i,j,k}$$

$$+\frac{\left(u\pi_a q R_e\Delta\lambda_e\Delta\phi\right)_{i+1/2,j,k,t-h} - \left(u\pi_a q R_e\Delta\lambda_e\Delta\phi\right)_{i-1/2,j,k,t-h}}{\Delta\lambda_e}$$

$$+\frac{\left(v\pi_a q R_e\cos\phi\Delta\lambda_e\Delta\phi\right)_{i,j+1/2,k,t-h} - \left(v\pi_a q R_e\cos\phi\Delta\lambda_e\Delta\phi\right)_{i,j-1/2,k,t-h}}{\Delta\phi}$$

$$+\left[\pi_{a,t}R_e^2\cos\phi\Delta\lambda_e\Delta\phi\frac{\left(\dot{\sigma}_t q_{t-h}\right)_{k+1/2} - \left(\dot{\sigma}_t q_{t-h}\right)_{k-1/2}}{\Delta\sigma_k}\right]_{i,j}$$

$$=\left\{\pi_a R_e^2\cos\phi\Delta\lambda_e\Delta\phi\left[\frac{\left(\ _z\cdot\ \rho_a\mathbf{K}_h\ _z\right)q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n\right]\right\}_{i,j,k,t-h}$$

(7.21)

Species Continuity Equation

Substitute fluxes --> equation for moist-air mass mixing ratio

$$
\begin{aligned}
q_{i,j,k,t} = \frac{(\pi_a q)_{i,j,k,t-h}}{\pi_{a,i,j,t}} + \frac{h}{\left(\pi_{a,t} R_e^2 \cos\phi \Delta\lambda_e \Delta\phi\right)_{i,j}} \\
\times \left\{ \left[F_{i-1/2,j} \frac{q_{i-1,j} + q_{i,j}}{2} - F_{i+1/2,j} \frac{q_{i,j} + q_{i+1,j}}{2} + G_{i,j-1/2} \frac{q_{i,j-1} + q_{i,j}}{2} - G_{i,j+1/2} \frac{q_{i,j} + q_{i,j+1}}{2} \right]_{k,t-h} \right. \\
+ \left[\pi_{a,t} R_e^2 \cos\phi \Delta\lambda_e \Delta\phi \frac{\left(\dot{\sigma}_t q_{t-h}\right)_{k-1/2} - \left(\dot{\sigma}_t q_{t-h}\right)_{k+1/2}}{\Delta\sigma_k} \right]_{i,j} \\
\left. + \left[\pi_a R_e^2 \cos\phi \Delta\lambda_e \Delta\phi \left(\frac{\left(\nabla_z \cdot \rho_a \mathbf{K}_h \nabla_z\right) q}{\rho_a} + \sum_{n=1}^{N_{e,t}} R_n \right) \right]_{i,j,k,t-h} \right\}
\end{aligned}
$$

(7.22)

Interpolate mixing ratios to vertical layer top and bottom

$$
q_{i,j,k-1/2} = \frac{\ln q_{i,j,k-1} - \ln q_{i,j,k}}{\left(1/q_{i,j,k}\right) - \left(1/q_{i,j,k-1}\right)}
$$

$$
q_{i,j,k+1/2} = \frac{\ln q_{i,j,k} - \ln q_{i,j,k+1}}{\left(1/q_{i,j,k+1}\right) - \left(1/q_{i,j,k}\right)} \qquad (7.23)
$$

Thermodynamic Energy Equation

$$R_e^2\cos\phi\left[\frac{\partial}{\partial t}\left(\pi_a\theta_v\right)\right]_\sigma+\left[\frac{\partial}{\partial\lambda_e}\left(u\pi_a\theta_v R_e\right)+\frac{\partial}{\partial\phi}\left(v\pi_a\theta_v R_e\cos\phi\right)\right]$$

$$+\pi_a R_e^2\cos\phi\,\frac{\partial}{\partial\sigma}\left(\dot{\sigma}\theta_v\right)=\pi_a R_e^2\cos\phi\left[\frac{\left(\nabla\cdot\rho_a\mathbf{K}_h\nabla\right)\theta_v}{\rho_a}+\frac{\theta_v}{c_{p,d}T_v}\sum_{n=1}^{N_{e,h}}\frac{dQ_n}{dt}\right]$$

(7.24)

Write in finite difference form and substitute fluxes

$$\theta_{v,i,j,k,t}=\frac{\left(\pi_a\theta_v\right)_{i,j,k,t-h}}{\pi_{a,i,j,t}}+\frac{h}{\left(\pi_{a,t}R_e^2\cos\phi\,\Delta\lambda_e\Delta\phi\right)_{i,j}}$$

$$\times\left\{\begin{array}{l}\left(\begin{array}{l}F_{i-1/2,j}\dfrac{\theta_{v,i-1,j}+\theta_{v,i,j}}{2}-F_{i+1/2,j}\dfrac{\theta_{v,i,j}+\theta_{v,i+1,j}}{2}\\+G_{i,j-1/2}\dfrac{\theta_{v,i,j-1}+\theta_{v,i,j}}{2}-G_{i,j+1/2}\dfrac{\theta_{v,i,j}+\theta_{v,i,j+1}}{2}\end{array}\right)_{k,t-h}\\+\left[\pi_{a,t}R_e^2\cos\phi\,\Delta\lambda_e\Delta\phi\,\dfrac{\left(\dot{\sigma}_t\theta_{v,t-h}\right)_{k-1/2}-\left(\dot{\sigma}_t\theta_{v,t-h}\right)_{k+1/2}}{\Delta\sigma_k}\right]_{i,j}\\+\left(\pi_a R_e^2\cos\phi\,\Delta\lambda_e\Delta\phi\left[\dfrac{\left(\nabla_z\cdot\rho_a\mathbf{K}_h\nabla_z\right)\theta_v}{\rho_a}+\dfrac{\theta_v}{c_{p,d}T_v}\displaystyle\sum_{n=1}^{N_{e,h}}\dfrac{dQ_n}{dt}\right]\right)_{i,j,k,t-h}\end{array}\right\}$$

(7.25)

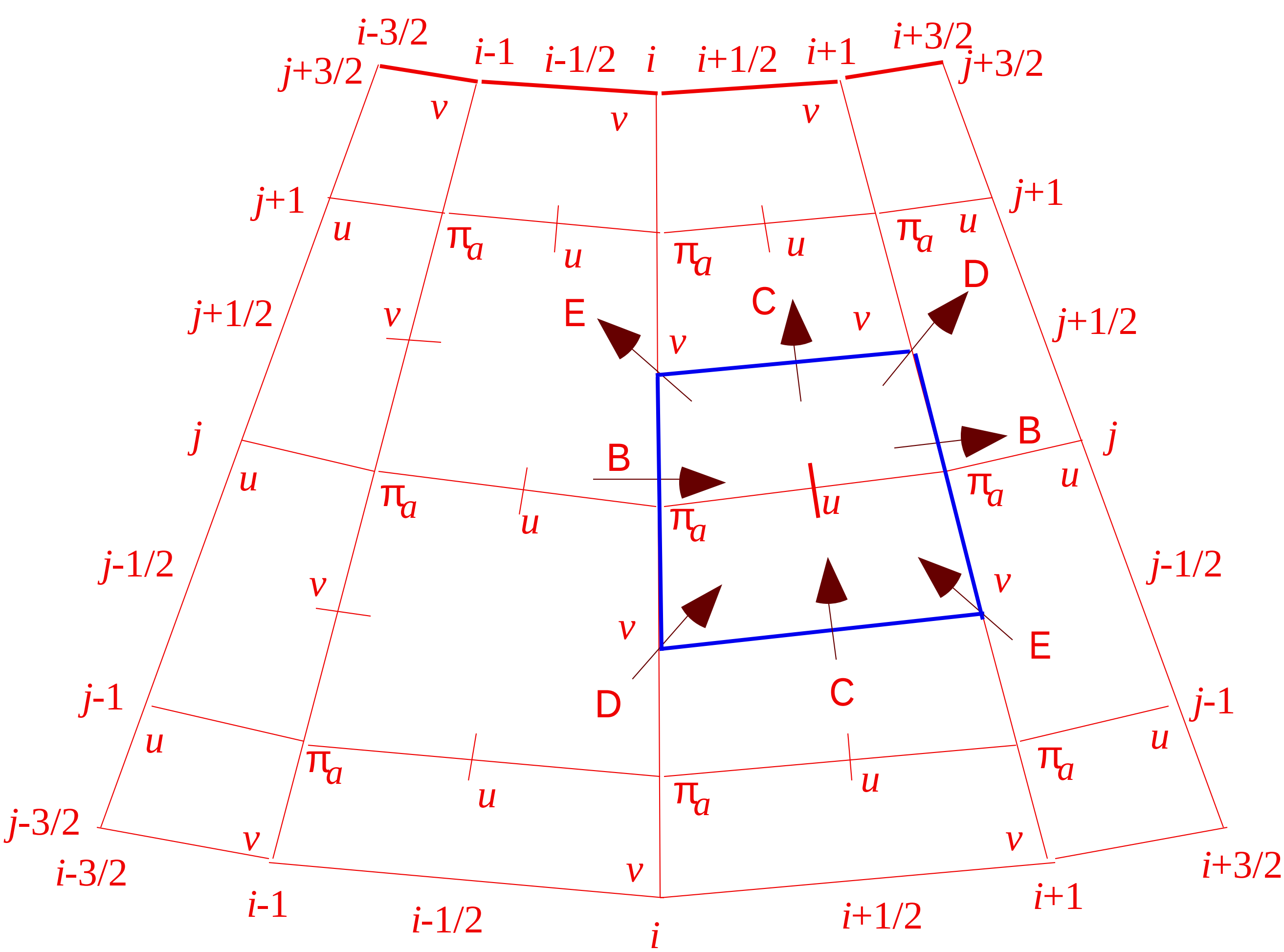
The *u*-Momentum Equation

$$R_e^2\cos\phi\left[\frac{\partial}{\partial t}(\pi_a u)\right]_\sigma + \left[\frac{\partial}{\partial\lambda_e}\left(\pi_a u^2 R_e\right) + \frac{\partial}{\partial\phi}(\pi_a u v R_e\cos\phi)\right]_\sigma + \pi_a R_e^2\cos\phi\frac{\partial}{\partial\sigma}(\dot{\sigma}u)$$

$$= \pi_a u v R_e\sin\phi + \pi_a f v R_e^2\cos\phi - R_e\left(\pi_a\frac{\partial\Phi}{\partial\lambda_e} + \sigma c_{p,d}\theta_v\frac{\partial P}{\partial\sigma}\frac{\partial\pi_a}{\partial\lambda_e}\right)_\sigma$$

$$+ R_e^2\cos\phi\frac{\pi_a}{\rho_a}\left(\ \cdot\rho_a\mathbf{K}_m\ \right)u \qquad (7.26)$$

Fig. 7.3. Grid for differencing the *u*-momentum equation.



The *u*-Momentum Equation

Time difference term (7.27)

$$u_{i+1/2,j,k,t} = \frac{\left(\pi_{a,t-h}\Delta A\right)_{i+1/2,j}}{\left(\pi_{a,t}\Delta A\right)_{i+1/2,j}} u_{i+1/2,j,k,t-h} + \frac{h}{\left(\pi_{a,t}\Delta A\right)_{i+1/2,j}} \times \Bigg\{$$

Horizontal advection terms (7.28)

$$\left(\begin{array}{l} B_{i,j}\dfrac{u_{i-1/2,j}+u_{i+1/2,j}}{2} - B_{i+1,j}\dfrac{u_{i+1/2,j}+u_{i+3/2,j}}{2} \\ + C_{i+1/2,j-1/2}\dfrac{u_{i+1/2,j-1}+u_{i+1/2,j}}{2} - C_{i+1/2,j+1/2}\dfrac{u_{i+1/2,j}+u_{i+1/2,j+1}}{2} \\ + D_{i,j-1/2}\dfrac{u_{i-1/2,j-1}+u_{i+1/2,j}}{2} - D_{i+1,j+1/2}\dfrac{u_{i+1/2,j}+u_{i+3/2,j+1}}{2} \\ + E_{i+1,j-1/2}\dfrac{u_{i+3/2,j-1}+u_{i+1/2,j}}{2} - E_{i,j+1/2}\dfrac{u_{i+1/2,j}+u_{i-1/2,j+1}}{2} \end{array} \right)_{k,t-h}$$

Vertical velocity term (7.29)

$$+\frac{1}{\Delta\sigma_k}\left(\pi_{a,t}\Delta A\dot{\sigma}_{k-1/2,t}u_{k-1/2,t-h} - \pi_{a,t}\Delta A\dot{\sigma}_{k+1/2,t}u_{k+1/2,t-h}\right)_{i+1/2,j}$$

The u-Momentum Equation

Coriolis and spherical grid conversion terms (7.30)

$$+\frac{R_e\left(\Delta\lambda_e\Delta\phi\right)_{i+1/2,j}}{2}\times\left[\begin{array}{l}\pi_{a,i,j}\dfrac{v_{i,j-1/2}+v_{i,j+1/2}}{2}\left(f_j R_e\cos\phi_j+\dfrac{u_{i-1/2,j}+u_{i+1/2,j}}{2}\sin\phi_j\right)\\ +\pi_{a,i+1,j}\dfrac{v_{i+1,j-1/2}+v_{i+1,j+1/2}}{2}\left(f_j R_e\cos\phi_j+\dfrac{u_{i+1/2,j}+u_{i+3/2,j}}{2}\sin\phi_j\right)\end{array}\right]_{k,t-h}$$

Pressure gradient terms (7.31)

$$-R_e\Delta\phi_{i+1/2,j}\left[\begin{array}{l}\left(\Phi_{i+1,j,k}-\Phi_{i,j,k}\right)\dfrac{\pi_{a,i,j}+\pi_{a,i+1,j}}{2}+\left(\pi_{a,i+1,j}-\pi_{a,i,j}\right)\\ \times\dfrac{c_{p,d}}{2}\left(\begin{array}{l}\left[\theta_{v,k}\dfrac{\sigma_{k+1/2}\left(P_{k+1/2}-P_k\right)+\sigma_{k-1/2}\left(P_k-P_{k-1/2}\right)}{\Delta\sigma_k}\right]_{i,j}\\ +\left[\theta_{v,k}\dfrac{\sigma_{k+1/2}\left(P_{k+1/2}-P_k\right)+\sigma_{k-1/2}\left(P_k-P_{k-1/2}\right)}{\Delta\sigma_k}\right]_{i+1,j}\end{array}\right)\end{array}\right]_{t-h}$$

Eddy diffusion term (7.32)

$$+\left\{\left(\pi_{a,t-h}\Delta A\right)_{i+1/2,j}\left[\frac{\left(\nabla_z\cdot\rho_a\mathbf{K}_m\nabla_z\right)u}{\rho_a}\right]_{i+1/2,j,k,t-h}\right\}$$

The *u*-Momentum Equation

Column pressure multiplied by grid-cell area at a *u*-point

$$\left(\pi_a \Delta A\right)_{i+1/2,j} = \frac{1}{8}\left\{\begin{array}{l}\left(\pi_a \Delta A\right)_{i,j+1} + \left(\pi_a \Delta A\right)_{i+1,j+1} \\ +2\left[\left(\pi_a \Delta A\right)_{i,j} + \left(\pi_a \Delta A\right)_{i+1,j}\right] \\ +\left(\pi_a \Delta A\right)_{i,j-1} + \left(\pi_a \Delta A\right)_{i+1,j-1}\end{array}\right\} \qquad (7.33)$$

Grid-cell area

$$\Delta A = R_e^2 \cos\phi \Delta\lambda_e \Delta\phi \qquad (7.34)$$

Interpolation for vertical velocity term (7.35)

$$\left(\pi_{a,t}\Delta A\dot{\sigma}_{k-1/2,t}\right)_{i+1/2,j} = \frac{1}{8}\left[\begin{array}{l}\left(\pi_{a,t}\Delta A\dot{\sigma}_{k-1/2,t}\right)_{i,j+1} + \left(\pi_{a,t}\Delta A\dot{\sigma}_{k-1/2,t}\right)_{i+1,j+1} \\ +2\left\{\left(\pi_{a,t}\Delta A\dot{\sigma}_{k-1/2,t}\right)_{i,j} + \left(\pi_{a,t}\Delta A\dot{\sigma}_{k-1/2,t}\right)_{i+1,j}\right\} \\ +\left(\pi_{a,t}\Delta A\dot{\sigma}_{k-1/2,t}\right)_{i,j-1} + \left(\pi_{a,t}\Delta A\dot{\sigma}_{k-1/2,t}\right)_{i+1,j-1}\end{array}\right]$$

The *u*-Momentum Equation

Interpolation for fluxes (7.36-9)

$$B_{i,j}=\frac{1}{12}\left[F_{i-1/2,j-1}+F_{i+1/2,j-1}+2\left(F_{i-1/2,j}+F_{i+1/2,j}\right)+F_{i-1/2,j+1}+F_{i+1/2,j+1}\right]$$

$$C_{i+1/2,j-1/2}=\frac{1}{12}\Big[G_{i,j-3/2}+G_{i+1,j-3/2}+2\left(G_{i,j-1/2}+G_{i+1,j-1/2}\right)+G_{i,j+1/2}+G_{i+1,j+1/2}$$

$$D_{i,j+1/2}=\frac{1}{24}\Big(G_{i,j-1/2}+2G_{i,j+1/2}+G_{i,j+3/2}+F_{i-1/2,j}+F_{i-1/2,j+1}+F_{i+1/2,j}+F_{i+1/2,j}$$

$$E_{i,j+1/2}=\frac{1}{24}\Big(G_{i,j-1/2}+2G_{i,j+1/2}+G_{i,j+3/2}-F_{i-1/2,j}-F_{i-1/2,j+1}-F_{i+1/2,j}-F_{i+1/2,j}$$

u-values at the bottom of a layer (7.40)

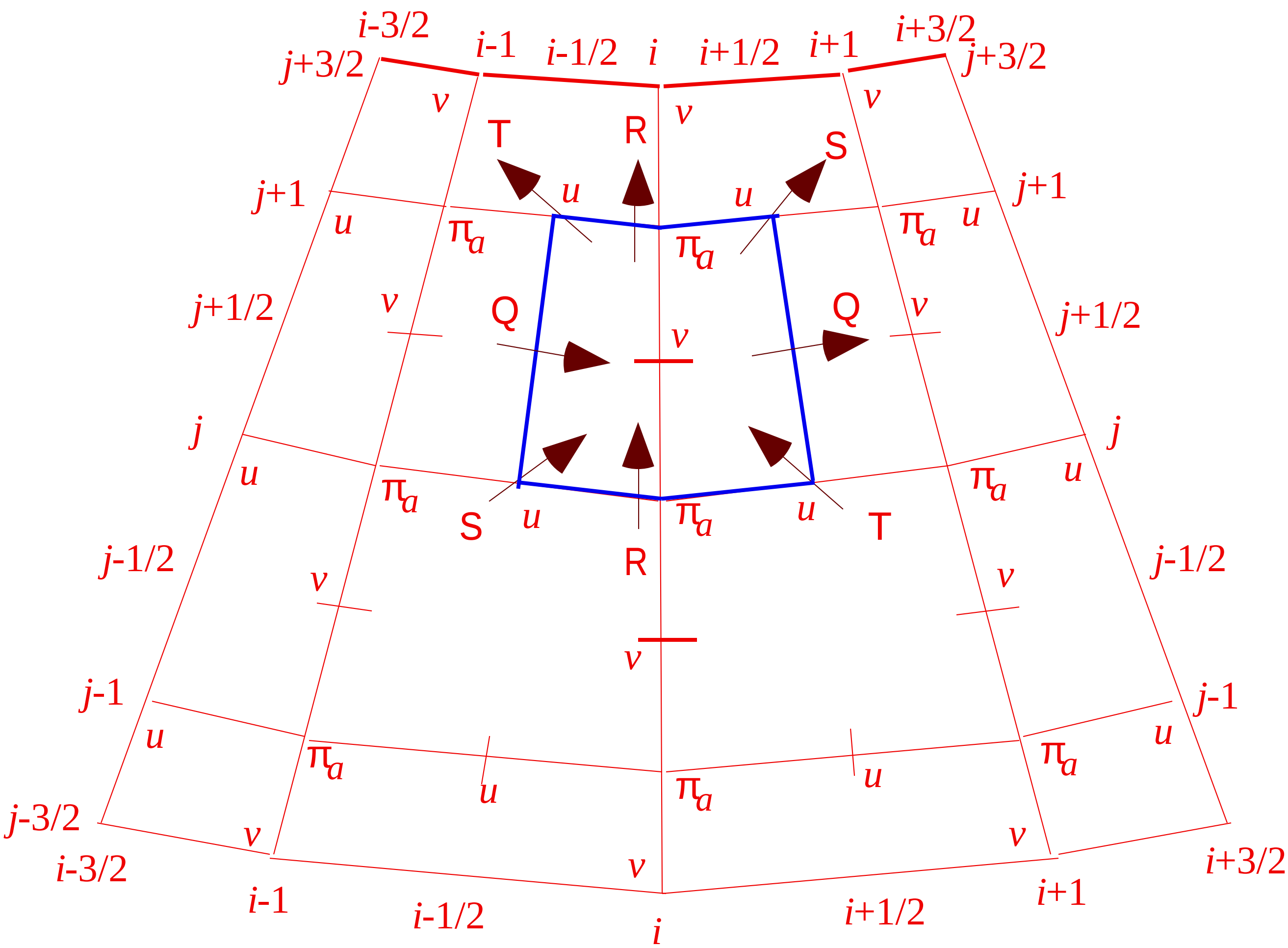
$$u_{i+1/2,j,k+1/2,t-h}=\frac{\Delta\sigma_{k+1}u_{i+1/2,j,k,t-h}+\Delta\sigma_k u_{i+1/2,j,k+1,t-h}}{\Delta\sigma_k+\Delta\sigma_{k+1}}$$

Boundary conditions for pressure-gradient term (7.41)

$$-R_e\Delta\phi_{I+1/2,j}\left[\begin{array}{l}\left(\Phi_{I,j,k,t-2h}-\Phi_{I,j,k,t-h}\right)\pi_{a,I,j,t-h}+\left(\pi_{a,I,j,t-2h}-\pi_{a,I,j,t-h}\right)\\ \times c_{p,d}\left[\theta_{v,k}\dfrac{\sigma_{k+1/2}\left(P_{k+1/2}-P_k\right)+\sigma_{k-1/2}\left(P_k-P_{k-1/2}\right)}{\Delta\sigma_k}\right]_{I,j,t-h}\end{array}\right]$$

Grid for Differencing the *v*-Momentum Equation

Fig. 7.5.



Vertical Momentum Equation

Hydrostatic equation

$$\mathrm{d}\Phi = -c_{p,d}\theta_v \mathrm{d}P$$

Geopotential at vertical center of bottom layer (7.56)

$$\Phi_{i,j,N_L,t-h} = \Phi_{i,j,N_L+1/2} - c_{p,d}\left[\theta_{v,N_L}\left(P_{N_L} - P_{N_L+1/2}\right)\right]_{i,j,t-h}$$

Geopotential at bottom of each subsequent layer

$$\Phi_{i,j,k+1/2,t-h} = \Phi_{i,j,k+1,t-h} - c_{p,d}\left[\theta_{v,k+1}\left(P_{k+1/2} - P_{k+1}\right)\right]_{i,j,t-h} \tag{7.57}$$

Geopotential at vertical midpoint of each subsequent layer

$$\Phi_{i,j,k,t-h} = \Phi_{i,j,k+1/2,t-h} - c_{p,d}\left[\theta_{v,k}\left(P_k - P_{k+1/2}\right)\right]_{i,j,t-h} \tag{7.58}$$

Time Stepping Schemes

Time derivative of an advected species

$$\frac{\partial q}{\partial t} = f(q) \qquad (7.59)$$

Matsuno step

Explicit forward difference in time

$$q_{est} = q_{t-h} + hf(q_{t-h})$$

Solve for final value with another forward difference

$$q_t = q_{t-h} + hf(q_{est})$$

Leapfrog step

Finite difference form

$$q_{t+h} = q_{t-h} + 2hf(q_t)$$

Fig. 7.5. Example dynamical time-stepping scheme