Overhead Slides for

Chapter 8

of

Fundamentals of Atmospheric Modeling

by

Mark Z. Jacobson

Department of
Civil & Environmental Engineering
Stanford University
Stanford, CA 94305-4020

January 30, 2002

Reynolds Stress

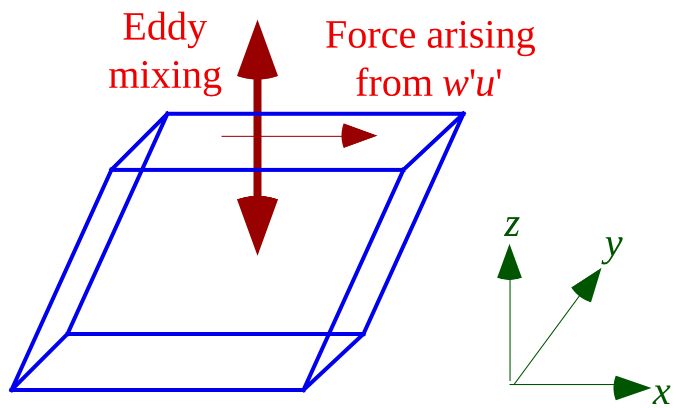
Stress

Force per unit area (*e.g.* dyne cm^{-2} or g cm^{-1} s^{-2})

Reynolds stress

Stress that causes a parcel of air to deform during turbulent motion of air

Fig. 8.1. Deformation by vertical momentum flux $\overline{w'u'}$



Stress from vertical transfer of turbulent *u*-momentum

$$\tau_{zx} = -\rho_a \overline{w'u'} \qquad (8.1)$$

zx = stress acting in *x*-direction, along a plane (*x*-*y*) normal to the *z*-direction

Momentum Fluxes

Magnitude of Reynolds stress at ground surface

$$|\tau_z| = \rho_a \left[\left(\overline{w'u'}\right)^2 + \left(\overline{w'v'}\right)^2 \right]^{1/2} \qquad (8.2)$$

Kinematic turbulent flux terms

$$\overline{w'u'} = -\frac{\tau_{zx}}{\rho_a}$$

$$\overline{w'v'} = -\frac{\tau_{zy}}{\rho_a}$$

Friction velocity

$$u_* = \left[\left(\overline{w'u'}\right)_s^2 + \left(\overline{w'v'}\right)_s^2 \right]^{1/4} = \left(|\tau_z| / \rho_a \right)_s^{1/2} \qquad (8.5)$$

Heat and Water Vapor Fluxes

Surface vertical turbulent sensible heat flux

$$H_f = \rho_a c_{p,d} \left(\overline{w' \theta'_v} \right)_s \tag{8.3}$$

Surface vertical turbulent water vapor flux

$$E_f = \rho_a \left(\overline{w' q'_v} \right)_s \tag{8.4}$$

Surface Roughness Length for Momentum

Height above surface at which mean wind extrapolates to zero

- Longer roughness length --> greater turbulence
- Exactly smooth surface, roughness length = 0
- Approximately 1/30 the height of the average roughness element protruding from the surface

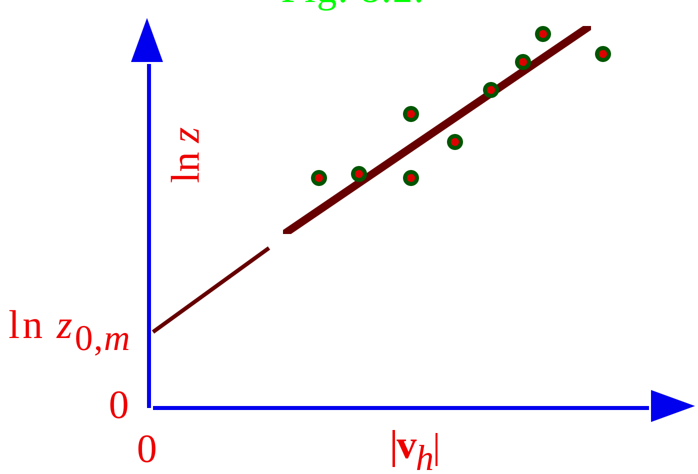
Method of calculating roughness length

1) Find wind speeds at many heights when wind is strong

2) Plot speeds on ln (height) vs. wind speed diagram

3) Extrapolate wind speed to altitude at which speed = 0

Fig. 8.2.



Parameterizations For Surface Roughness Length for Momentum

Over smooth ocean with low wind speed

$$z_{0,m} \approx 0.11\nu_a / u_* \quad (8.6)$$

Over a rough ocean with a high wind speed (Charnock relation)

$$z_{0,m} \approx \alpha_c u_*^2 / g \quad (8.7)$$

Over urban areas containing structures

$$z_{0,m} \approx 0.5 h_o S_o / A_o \quad (8.8)$$

Roughness Length for Momentum, Canopy Height, and Displacement Height

Table 8.1.

Surface Type	$z_{0,m}$ (m)	h_c (m)	d_c (m)
Smooth sea	0.00001		
Rough sea	0.000015-0.0015		
Ice	0.00001		
Snow	0.00005-0.0001		
Level desert	0.0003		
Short grass	0.03-0.01	0.02-0.1	
Long grass	0.04-0.1	0.25-1.0	
Savannah	0.4	8	4.8
Agricultural crops	0.04-0.2	0.4-2	0.27-1.3
Orchard	0.5-1.0	5-10	3.3-6.7
Coniferous forest	0.28-3.9	10.4-27.5	6.3-25.3
Tropical forest	2.2	35	29.8
2500 m^2 lot w/ a building 8-m high and 160 m^2 silhouette	0.26	8	
25000 m^2 lot w/ a building 80-m high and 3200 m^2 silhouette	5.1	80	

Roughness Lengths for Heat and Moisture

Surface roughness length for heat

$$z_{0,h} = \frac{D_h}{k u_*} \tag{8.9}$$

Surface roughness length for moisture

$$z_{0,v} = \frac{D_v}{k u_*} \tag{8.9}$$

Molecular diffusion coefficient of heat

$$D_h = \frac{\kappa_d}{\rho_a c_{p,d}} \tag{8.10}$$

Molecular diffusion coefficient of water vapor

$$D_v = \frac{3}{8 A d_v^2 \rho_a} \sqrt{\frac{R^* T m_a}{2\pi} \left(\frac{m_v + m_a}{m_v} \right)} \tag{8.10}$$

Fluxes From Bulk Aerodynamic Formulae

Vertical kinematic momentum fluxes

$$\left(\overline{w'u'}\right)_s = -C_D \left|\bar{\mathbf{v}}_h(z_r)\right| \left[\bar{u}(z_r) - \bar{u}(z_{0,m})\right] \qquad (8.11)$$

$$\left(\overline{w'v'}\right)_s = -C_D \left|\bar{\mathbf{v}}_h(z_r)\right| \left[\bar{v}(z_r) - \bar{v}(z_{0,m})\right] \qquad (8.11)$$

Vertical kinematic heat flux

$$\left(\overline{w'\theta'_v}\right)_s = -C_H \left|\bar{\mathbf{v}}_h(z_r)\right| \left[\bar{\theta}_v(z_r) - \bar{\theta}_v(z_{0,h})\right] \qquad (8.12)$$

Surface vertical turbulent sensible heat flux

$$H_f \approx \rho_a c_{p,d} C_H \left|\bar{\mathbf{v}}_h(z_r)\right| \left[\bar{\theta}_v(z_{0,h}) - \bar{\theta}_v(z_r)\right]$$

(8.14)

Vertical kinematic water vapor flux

$$\left(\overline{w'q'_v}\right)_s = -C_E \left|\bar{\mathbf{v}}_h(z_r)\right| \left[\bar{q}_v(z_r) - \bar{q}_v(z_{0,v})\right] \qquad (8.13)$$

Surface vertical turbulent water vapor flux

$$E_f \approx \rho_a C_E \left|\bar{\mathbf{v}}_h(z_r)\right| \left[\bar{q}_v(z_{0,v}) - \bar{q}_v(z_r)\right] \qquad (8.14)$$

Eddy Diffusion Coefficients From Bulk Aerodynamic Formulae

Eddy diffusion coefficients for momentum from *K*-theory

$$K_{m,zx} = -\frac{\left(\overline{w'u'}\right)_s}{\partial\bar{u}/\partial z} \approx C_D\left|\bar{\mathbf{v}}_h\left(z_r\right)\right|\left(z_r - z_{0,m}\right) \approx -\frac{\left(\overline{w'v'}\right)_s}{\partial\bar{v}/\partial z} = K_{m,zy}$$

(8.15)

Eddy diffusion coefficient for heat from *K*-theory

$$K_{h,zz} = -\frac{\left(\overline{w'\theta'_v}\right)_s}{\partial\bar{\theta}_v/\partial z} \approx C_H\left|\bar{\mathbf{v}}_h\left(z_r\right)\right|\left(z_r - z_{0,h}\right) \qquad (8.16)$$

Similarity Relationship

Dimensionless wind shear

$$\frac{\varphi_m}{k} = \frac{z}{u_*}\frac{\partial\left|\overline{\mathbf{v}}_h\right|}{\partial z} \tag{8.17}$$

Parameterization of dimensionless wind shear from field data

$$\varphi_m = \begin{cases} 1+\beta_m \dfrac{z}{L} & \dfrac{z}{L} > 0 \quad \text{stable} \\ \left(1-\gamma_m \dfrac{z}{L}\right)^{-1/4} & \dfrac{z}{L} < 0 \quad \text{unstable} \\ 1 & \dfrac{z}{L} = 0 \quad \text{neutral} \end{cases} \tag{8.18}$$

Integrate (8.17) from $z_{0,m}$ to z_r

$$u_* = \frac{k\left|\overline{\mathbf{v}}_h(z_r)\right|}{\int_{z_{0,m}}^{z_r} \varphi_m \frac{dz}{z}} \tag{8.19}$$

Integral of Dimensionless Wind Shear

Integral of the dimensionless wind shear

$$\int_{z_{0,m}}^{z_r} \varphi_m \frac{dz}{z} =$$

$$\begin{cases} \ln \dfrac{z_r}{z_{0,m}} + \dfrac{\beta_m}{L}\left(z_r - z_{0,m}\right) & \dfrac{z}{L} > 0 \quad \text{stable} \\ \ln \dfrac{\left(1-\gamma_m \dfrac{z_r}{L}\right)^{1/4} - 1}{\left(1-\gamma_m \dfrac{z_r}{L}\right)^{1/4} + 1} - \ln \dfrac{\left(1-\gamma_m \dfrac{z_{0,m}}{L}\right)^{1/4} - 1}{\left(1-\gamma_m \dfrac{z_{0,m}}{L}\right)^{1/4} + 1} & \\ +2\tan^{-1}\left(1-\gamma_m \dfrac{z_r}{L}\right)^{1/4} - 2\tan^{-1}\left(1-\gamma_m \dfrac{z_{0,m}}{L}\right)^{1/4} & \dfrac{z}{L} < 0 \quad \text{unstable} \\ \ln \dfrac{z_r}{z_{0,m}} & \dfrac{z}{L} = 0 \quad \text{neutral} \end{cases}$$

(8.20)

Monin-Obukhov Length

Height proportional to the height above the surface at which buoyant production of turbulence first equals mechanical (shear) production of turbulence.

$$L = -\frac{u_*^3 \bar{\theta}_v}{kg\left(\overline{w'\theta_v'}\right)_s} = \frac{u_*^2 \bar{\theta}_v}{kg\theta_*} \qquad (8.21)$$

Vertical kinematic heat flux

$$\left(\overline{w'\theta_v'}\right)_s \approx -u_*\theta_* \qquad (8.22)$$

Potential Temperature Scale

Dimensionless temperature gradient

$$\frac{\varphi_h}{k} \approx \frac{z}{\theta_*}\frac{\partial \bar{\theta}_v}{\partial z} \qquad (8.23)$$

Parameterization of dimensionless temperature gradient

$$\varphi_h = \begin{cases} \mathrm{Pr}_t + \beta_h \dfrac{z}{L} & \dfrac{z}{L} > 0 \quad \text{stable} \\ \mathrm{Pr}_t \left(1 - \gamma_h \dfrac{z}{L}\right)^{-1/2} & \dfrac{z}{L} < 0 \quad \text{unstable} \\ \mathrm{Pr}_t & \dfrac{z}{L} = 0 \quad \text{neutral} \end{cases} \qquad (8.24)$$

Integrate (8.23) from $z_{0,m}$ to z_r

$$\theta_* = \frac{k\left[\bar{\theta}_v(z_r) - \bar{\theta}_v(z_{0,h})\right]}{\int_{z_{0,h}}^{z_r} \varphi_h \dfrac{dz}{z}} \qquad (8.25)$$

Integral of Dimensionless Temperature Gradient

Integral of dimensionless temperature gradient

$$\int_{z_{0,h}}^{z_r} \varphi_h \frac{dz}{z} =$$

$$\begin{cases} \Pr_t \ln \dfrac{z_r}{z_{0,h}} + \dfrac{\beta_h}{L}\left(z_r - z_{0,h}\right) & \dfrac{z}{L} > 0 \quad \text{stable} \\[2ex] \Pr_t \left[\ln \dfrac{\left(1 - \gamma_h \dfrac{z_r}{L}\right)^{1/2} - 1}{\left(1 - \gamma_h \dfrac{z_r}{L}\right)^{1/2} + 1} - \ln \dfrac{\left(1 - \gamma_h \dfrac{z_{0,h}}{L}\right)^{1/2} - 1}{\left(1 - \gamma_h \dfrac{z_{0,h}}{L}\right)^{1/2} + 1} \right] & \dfrac{z}{L} < 0 \quad \text{unstable} \\[2ex] \Pr_t \ln \dfrac{z_r}{z_{0,h}} & \dfrac{z}{L} = 0 \quad \text{neutral} \end{cases}$$

(8.26)

Scale Parameterization

Friction wind velocity

$$u_* \approx \frac{k\left|\bar{\mathbf{v}}_h(z_r)\right|}{\ln\left(z_r / z_{0,m}\right)}\sqrt{G_m} \qquad (8.28)$$

Potential temperature scale

$$\theta_* \approx \frac{k^2\left|\bar{\mathbf{v}}_h(z_r)\right|\left[\theta_v(z_r) - \theta_v(z_{0,h})\right]}{u_* \Pr_t \ln^2\left(z_r / z_{0,m}\right)} G_h \qquad (8.28)$$

$$G_m = 1 - \frac{9.4\mathrm{Ri}_b}{1 + \dfrac{70k^2\left(\left|\mathrm{Ri}_b\right| z_r / z_{0,m}\right)^{0.5}}{\ln^2\left(z_r / z_{0,m}\right)}} \qquad \mathrm{Ri}_b \le 0$$

$$G_h = 1 - \frac{9.4\mathrm{Ri}_b}{1 + \dfrac{50k^2\left(\left|\mathrm{Ri}_b\right| z_r / z_{0,m}\right)^{0.5}}{\ln^2\left(z_r / z_{0,m}\right)}} \qquad \mathrm{Ri}_b \le 0$$

$$G_m, G_h = \frac{1}{\left(1 + 4.7\mathrm{Ri}_b\right)^2} \qquad \mathrm{Ri}_b > 0$$

(8.29)

Bulk Richardson number

Ratio of buoyancy to mechanical shear

$$\mathrm{Ri}_b = \frac{g\left[\bar{\theta}_v(z_r) - \bar{\theta}_v(z_{0,h})\right]\left(z_r - z_{0,m}\right)^2}{\bar{\theta}_v(z_{0,h})\left[\bar{u}(z_r)^2 + \bar{v}(z_r)^2\right]\left(z_r - z_{0,h}\right)} \qquad (8.27)$$

Example Problem

$z_{0,m}$	= 0.01 m	Pr_t	= 0.95
$z_{0,h}$	= 0.0001 m	k	= 0.4
$u(z_r)$	= 10 m s^{-1}	$v(z_r)$	= 5 m s^{-1}
$\theta_v(z_r)$	= 285 K	$\theta_v(z_{0,h})$	= 288 K

---> $\left|\bar{\mathbf{v}}_h(z_r)\right|$ = 11.18 m s^{-1}

---> Ri_b = -8.15 x 10^{-3}

---> G_m = 1.046

---> G_h = 1.052

---> u_* = 0.662 m s^{-1}

---> θ_* = -0.188 K

---> L = -169 m

---> $$K_{m,zx} = \frac{u_*^2 \bar{u}(z_r)}{\left|\bar{\mathbf{v}}(z_r)\right|(\partial \bar{u}/\partial z)}$$ = 0.39 m^2 s^{-1}

---> $$K_{h,zz} = \frac{\theta_* u_*}{(\partial \bar{\theta}_v/\partial z)}$$ = 0.41 m^2 s^{-1}

---> $K_{m,zx}/K_{h,zz}$ = 0.95

Gradient Richardson Number

$$\mathrm{Ri}_g = \frac{\dfrac{g}{\bar{\theta}_v}\dfrac{\partial \bar{\theta}_v}{\partial z}}{\left(\dfrac{\partial \bar{u}}{\partial z}\right)^2 + \left(\dfrac{\partial \bar{v}}{\partial z}\right)^2} \qquad (8.30)$$

Table 8.2. Characteristics of vertical flow of air for different values of the bulk or gradient Richardson numbers.

Value of Ri_b or Ri_g	Type of Flow	Level of Turbulence Due to Buoyancy	Level of Turbulence Due to Shear
Large, negative	Turbulent	Large	Small
Small, negative	Turbulent	Small	Large
Small positive	Turbulent	None (weakly stable)	Large
Large positive	Laminar	None (strongly stable)	Small

Laminar flow becomes turbulent when Ri_g decreases to less than the critical Richardson number (Ri_c) = 0.25

Turbulent flow becomes laminar when Ri_g increase to greater than the termination Richardson number (Ri_T) = 1.0

Heat and Water Vapor Fluxes From Similarity Theory

Vertical kinematic heat flux

$$\left(\overline{w'\theta_v'}\right)_s = -u_*\theta_*$$

Surface vertical turbulent sensible heat flux

$$H_f \approx \rho_a c_{p,d}\left(\overline{w'\theta_v'}\right)_s = -\rho_a c_{p,d} u_*\theta_* \qquad (8.31)$$

Vertical kinematic water vapor flux

$$\left(\overline{w'q_v'}\right)_s = -u_* q_*$$

Surface vertical turbulent water vapor flux

$$E_f = \rho_a\left(\overline{w'q_v'}\right)_s = -\rho_a u_* q_* \qquad (8.31)$$

Dimensionless specific humidity gradient

$$\frac{\varphi_q}{k} = \frac{z}{q_*}\frac{\partial \bar{q}_v}{\partial z} \qquad (8.32)$$

Specific humidity scale

$$q_* = \frac{k\left[\bar{q}_v(z_r) - \bar{q}_v(z_{0,v})\right]}{\int_{z_{0,v}}^{z_r} \varphi_h \frac{dz}{z}} \qquad (8.33)$$

Eddy Diffusion Coefficients From Similarity Theory

Substitute

$$\left(\overline{w'u'}\right)_s = -C_D\left|\bar{\mathbf{v}}_h(z_r)\right|\left[\bar{u}(z_r)-\bar{u}(z_{0,m})\right] \qquad (8.11)$$

$$\left(\overline{w'v'}\right)_s = -C_D\left|\bar{\mathbf{v}}_h(z_r)\right|\left[\bar{v}(z_r)-\bar{v}(z_{0,m})\right] \qquad (8.11)$$

into

$$u_* = \left[\left(\overline{w'u'}\right)_s^2 + \left(\overline{w'v'}\right)_s^2\right]^{1/4} \qquad (8.5)$$

Solve for drag coefficient

$$C_D = \left(\frac{u_*}{\left|\bar{\mathbf{v}}_h(z_r)\right|}\right)^2$$

Substitute drag coefficient into (8.11)

$$\left(\overline{w'u'}\right)_s = -\frac{u_*^2}{\left|\bar{\mathbf{v}}_h(z_r)\right|}\bar{u}(z_r) \qquad (8.35)$$

$$\left(\overline{w'v'}\right)_s = -\frac{u_*^2}{\left|\bar{\mathbf{v}}_h(z_r)\right|}\bar{v}(z_r) \qquad (8.35)$$

Eddy Diffusion Coefficients From Similarity Theory

Eddy diffusion coefficients as a function of friction velocity

$$K_{m,zx} = \frac{u_*^2}{\left|\bar{\mathbf{v}}_h(z_r)\right|}\frac{\bar{u}(z_r)}{\partial\bar{u}/\partial z} \approx \frac{u_*^2}{\left|\bar{\mathbf{v}}_h(z_r)\right|}\left(z_r - z_{0,m}\right) \approx \frac{u_*^2}{\left|\bar{\mathbf{v}}_h(z_r)\right|}\frac{\bar{v}(z_r)}{\partial\bar{v}/\partial z} = K_{m,zy}$$

(8.36)

Wind shear

$$\frac{\partial\left|\bar{\mathbf{v}}_h\right|}{\partial z} \approx \frac{\left|\bar{\mathbf{v}}_h\right|}{z_r - z_{0,m}}$$

Dimensionless wind shear

$$\frac{\varphi_m}{k} = \frac{z}{u_*}\frac{\partial\left|\bar{\mathbf{v}}_h\right|}{\partial z} \qquad (8.17)$$

Combine expressions above

$$K_{m,zx} = \frac{kzu_*}{\varphi_m} = K_{m,zy} \qquad (8.37)$$

kz = mixing length = average distance an eddy travels before exchanging momentum with surrounding eddies

Logarithmic Wind Profile

Dimensionless wind shear

$$\frac{\varphi_m}{k} = \frac{z}{u_*}\frac{\partial|\bar{\mathbf{v}}_h|}{\partial z} \qquad (8.17)$$

Rewrite

$$\frac{\partial|\bar{\mathbf{v}}_h(z)|}{\partial z} = \frac{u_*}{kz}\varphi_m = \frac{u_*}{kz}\left[1-\left(1-\varphi_m\right)\right] \qquad (8.39)$$

Integrate --> vertical profile of wind speed in surface layer

$$|\bar{\mathbf{v}}_h(z)| = \frac{u_*}{k}\left[\ln\left(\frac{z}{z_{0,m}}\right) - \psi_m\right] \qquad (8.40)$$

Influence function for momentum

$$\psi_m = \int_{z_{0,m}}^{z}\left(1-\varphi_m\right)\frac{dz}{z} \qquad (8.42,3)$$

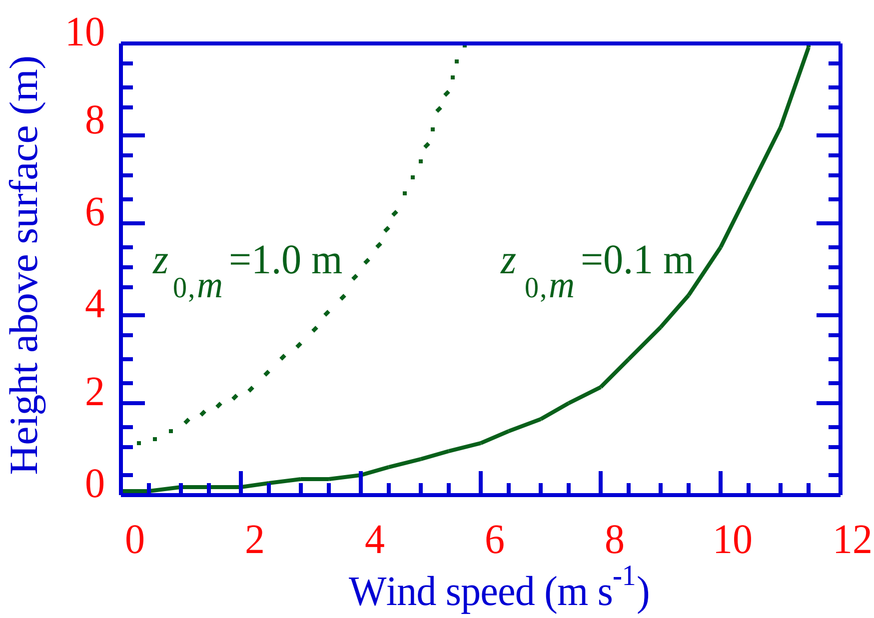
$$= \begin{cases} -\dfrac{\beta_m}{L}\left(z - z_{0,m}\right) & \dfrac{z}{L} > 0 \quad \text{stable} \\ \ln\dfrac{\left[1+\varphi_m(z)^{-2}\right]\left[1+\varphi_m(z)^{-1}\right]^2}{\left[1+\varphi_m\left(z_{0,m}\right)^{-2}\right]\left[1+\varphi_m\left(z_{0,m}\right)^{-1}\right]^2} & \\ -2\tan^{-1}\left[\varphi_m(z)\right]^{-1} + 2\tan^{-1}\left[\varphi_m\left(z_{0,m}\right)\right]^{-1} & \dfrac{z}{L} < 0 \quad \text{unstable} \\ 0 & \dfrac{z}{L} = 0 \quad \text{neutral} \end{cases}$$

Logarithmic Wind Profile

Neutral conditions --> logarithmic wind profile

$$|\bar{\mathbf{v}}_h(z)| = \frac{u_*}{k} \ln \frac{z}{z_{0,m}} \qquad (8.45)$$

Fig. 8.3. Logarithmic wind profiles when $u_* = 1$ m s^{-1}.



Vertical Profile of Potential Temperature

Dimensionless potential temperature gradient

$$\frac{\varphi_h}{k} \approx \frac{z}{\theta_*}\frac{\partial \bar{\theta}_v}{\partial z} \qquad (8.23)$$

Rewrite

$$\frac{\partial \bar{\theta}_v}{\partial z} = \frac{\theta_*}{kz}\varphi_h = \frac{\theta_*}{kz}\left[1-\left(1-\varphi_h\right)\right] \qquad (8.39)$$

Integrate --> vertical profile of potential temperature

$$\bar{\theta}_v(z) = \bar{\theta}_v\left(z_{0,h}\right) + \Pr_t \frac{\theta_*}{k}\left[\ln\left(\frac{z}{z_{0,h}}\right) - \psi_h\right] \qquad (8.41)$$

Influence function for momentum

$$\psi_h = \int_{z_{0,h}}^{z}\left(1-\varphi_h\right)\frac{dz}{z} \qquad (8.42,4)$$

$$= \begin{cases} -\dfrac{1}{\Pr_t}\dfrac{\beta_h}{L}\left(z - z_{0,h}\right) & \dfrac{z}{L} > 0 \quad \text{stable} \\ 2\ln \dfrac{1+\varphi_h(z)^{-1}}{1+\varphi_h\left(z_{0,h}\right)^{-1}} & \dfrac{z}{L} < 0 \quad \text{unstable} \\ 0 & \dfrac{z}{L} = 0 \quad \text{neutral} \end{cases}$$

Vertical Profiles in a Canopy

Momentum

$$|\bar{\mathbf{v}}_h(z)| = \frac{u_*}{k}\left[\ln\left(\frac{z-d_c}{z_{0,m}}\right) - \psi_m\left(\frac{z-d_c}{L}\right)\right] \quad (8.47)$$

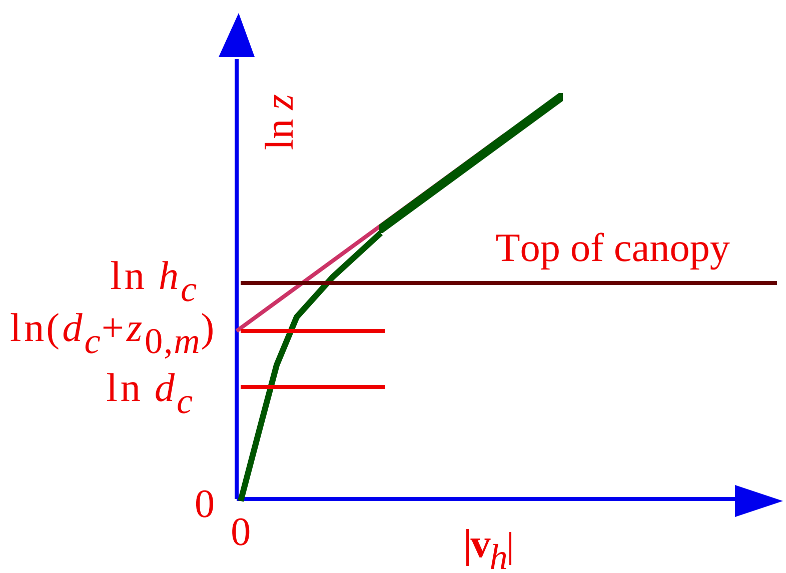
Potential temperature

$$\bar{\theta}_v(z) = \bar{\theta}_v\left(d_c + z_{0,h}\right) + \mathrm{Pr}_t\,\frac{\theta_*}{k}\left[\ln\left(\frac{z-d_c}{z_{0,h}}\right) - \psi_h\left(\frac{z-d_c}{L}\right)\right] \quad (8.48)$$

Specific humidity

$$\bar{q}_v(z) = \bar{q}_v\left(d_c + z_{0,v}\right) + \mathrm{Pr}_t\,\frac{q_*}{k}\left[\ln\left(\frac{z-d_c}{z_{0,v}}\right) - \psi_h\left(\frac{z-d_c}{L}\right)\right] \quad (8.49)$$

Fig. 8.4. Relationship among d_c, h_c, and $z_{0,m}$.



Diffusion Coefficient Above Surface

For momentum under stable / weakly unstable conditions

$$K_{m,zx} \approx K_{m,zy} \approx \left(\frac{kz}{1+kz/\lambda_m}\right)^2 \sqrt{\left(\frac{\Delta \bar{u}}{\Delta z}\right)^2+\left(\frac{\Delta \bar{v}}{\Delta z}\right)^2}\,\frac{\mathrm{Ri}_c-\mathrm{Ri}_b}{\mathrm{Ri}_c} \qquad (8.50)$$

For heat under stable / weakly unstable conditions

$$K_{m,zx} \approx K_{m,zy} \approx \mathrm{Pr}_t\, K_{h,zz}$$

Heat Conduction Equation

Heat conduction equation

$$\frac{\partial T_s}{\partial t} = \frac{1}{\rho_g c_G} \frac{\partial}{\partial z} \left(\kappa_s \frac{\partial T_s}{\partial z} \right) \quad (8.51)$$

Thermal conductivity of soil-water-air mixture

$$\kappa_s = \max\left(418 e^{-\log_{10}\left|\psi_p\right| - 2.7}, 0.172 \right) \quad (8.52)$$

Moisture potential

Potential energy required to extract water from capillary and adhesive forces in the soil

$$\psi_p = \psi_{p,s} \left(\frac{w_{g,s}}{w_g} \right)^b \quad (8.53)$$

Density x specific heat of soil-water-air mixture

$$\rho_g c_G = \left(1 - w_{g,s}\right) \rho_s c_S + w_g \rho_w c_W \quad (8.54)$$

Rate of change of soil water content

$$\frac{\partial w_g}{\partial t} = \frac{\partial}{\partial z} \left[K_g \left(\frac{\partial \psi_p}{\partial z} + 1 \right) \right] = \frac{\partial}{\partial z} \left(D_g \frac{\partial w_g}{\partial z} + K_g \right) \quad (8.55)$$

Heat Conduction Equation

Hydraulic conductivity of soil

Coefficient of permeability of liquid through soil

$$K_g = K_{g,s}\left(\frac{w_g}{w_{g,s}}\right)^{2b+3} \qquad (8.56)$$

Diffusion coefficient of water in soil

$$D_g = K_g \frac{\partial \psi_p}{\partial w_g} = -\frac{bK_{g,s}\psi_{p,s}}{w_g}\left(\frac{w_g}{w_{g,s}}\right)^{b+3} = -\frac{bK_{g,s}\psi_{p,s}}{w_{g,s}}\left(\frac{w_g}{w_{g,s}}\right)^{b+2} \qquad (8.57)$$

Rate of change of ground surface temperature

$$\frac{\partial T_s}{\partial t} = \frac{1}{\rho_g c_G}\frac{\partial}{\partial z}\left(\kappa_s \frac{\partial T_s}{\partial z} + F_{n,g} - H_f - L_e E_f\right) \qquad (8.58)$$

Rate of change of moisture content at the surface

$$\frac{\partial w_g}{\partial t} = \frac{\partial}{\partial z}\left(D_g \frac{\partial w_g}{\partial z} + K_g + \frac{E_f - P_g}{\rho_w}\right) \qquad (8.59)$$