**Overhead Slides for**

# Chapter 9

**of**

# Fundamentals of Atmospheric Modeling

by

Mark Z. Jacobson

Department of
Civil & Environmental Engineering
Stanford University
Stanford, CA 94305-4020

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# Cloud Formation

Table 9.1. Altitude range of different étages

| Étage | Polar Regions | Temperature Regions | Tropical Regions |
|---|---|---|---|
| High | 3-8 km | 5-13 km | 6-18 km |
| Middle | 2-4 km | 2-7 km | 2-8 km |
| Low | 0-2 km | 0-2 km | 0-2 km |

# Cloud Classification

Table 9.2.

| Genera | Étage |
| --- | --- |
| Cirrus (Ci) | High |
| Cirrocumulus (Cc) | High |
| Cirrostratus (Cs) | High |
| Altocumulus (Ac) | Middle |
| Altostratus (As) | Middle<br>High |
| Nimbostratus (Ns) | Low<br>Middle<br>High |
| Stratocumulus (Sc) | Low |
| Stratus (St) | Low |
| Cumulus (Cu) | Low<br>Middle |
| Cumulonimbus (Cb) | Low<br>Middle<br>High |

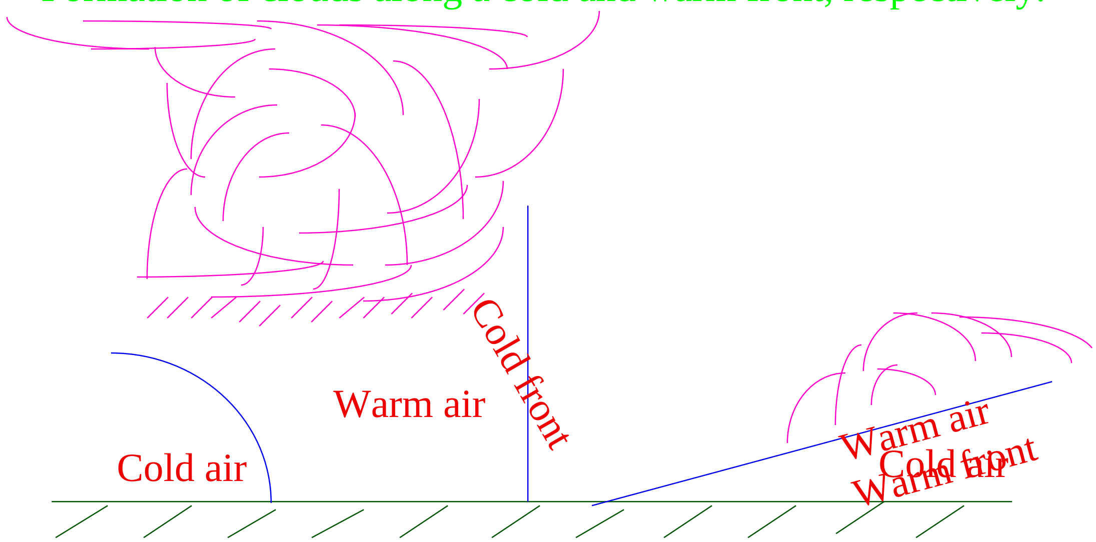
# Cloud Formation

Cloud Formation Mechanisms

- free convection
- forced convection
- orography
- frontal lifting

Fig. 9.1.

Formation of clouds along a cold and warm front, respectively.




Fog

- radiation fog
- advection fog
- upslope fog
- evaporation fog
  - steam fog
  - frontal fog

# Pseudoadiabatic Process

Condensation, latent heat release occurs during adiabatic ascent

Adiabatic process

$dQ = 0$

Pseudoadiabatic process

$$dQ = -L_e d\omega_{v,s} \tag{9.1}$$

Saturation mass mixing ratio of water over liquid

$$\omega_{v,s} = \frac{\varepsilon p_{v,s}}{p_d}$$

Combine (9.1) with first law of thermodynamics

$$-L_e d\omega_{v,s} = c_{p,d} dT_v - \alpha_a dp_a \tag{9.2}$$

Substitute $\alpha_a = 1/\rho_a = R'T_v/p_a$ into (9.2)

$$dT_v = \frac{R'T_v}{c_{p,d}p_a} dp_a - \frac{L_e}{c_{p,d}} d\omega_{v,s} \tag{9.3}$$

Differentiate (9.3), substitute $\partial p_a/\partial z = -\rho_a g$ & $p_a = \rho_a R'T_v$

$$\left(\frac{\partial T_v}{\partial z}\right)_w = \frac{R'T_v}{c_{p,d}\,p_a}\frac{\partial p_a}{\partial z} - \frac{L_e}{c_{p,d}}\frac{\partial \omega_{v,s}}{\partial z} = -\frac{g}{c_{p,d}} - \frac{L_e}{c_{p,d}}\frac{\partial \omega_{v,s}}{\partial z} \tag{9.4}$$

Simplifies to the dry adiabatic equation when $\omega_{v,s} = 0$

# Pseudoadiabatic Process

Differentiate $\omega_{v,s} = \varepsilon p_{v,s}/p_d$ with respect to altitude, substitute

$$dp_{v,s} = L_e p_{v,s} dT \big/ R_v T^2 \qquad \omega_{v,s} = \varepsilon p_{v,s}/p_d$$

$$dT = R' dT_v / R_m = T dT_v / T_v \qquad R' = \varepsilon R_v$$

$$\partial p_d / \partial z = -p_d g / R'T$$

$$\frac{\partial \omega_{v,s}}{\partial z} = \frac{\varepsilon}{p_d}\left(\frac{\partial p_{v,s}}{\partial z} - \frac{p_{v,s}}{p_d}\frac{\partial p_d}{\partial z}\right) = \frac{L_e \varepsilon \omega_{v,s}}{R' T T_v}\frac{\partial T_v}{\partial z} + \frac{\omega_{v,s} g}{R'T} \qquad (9.5)$$

Substitute (9.5) and $\Gamma_d = g/c_{p,d}$ into (9.4)

$$\left(\frac{\partial T_v}{\partial z}\right)_w = -\Gamma_w = -\Gamma_d \left(1 + \frac{L_e \omega_{v,s}}{R'T}\right) \Bigg/ \left(1 + \frac{L_e^2 \varepsilon \omega_{v,s}}{R' c_{p,d} T T_v}\right) \qquad (9.6)$$

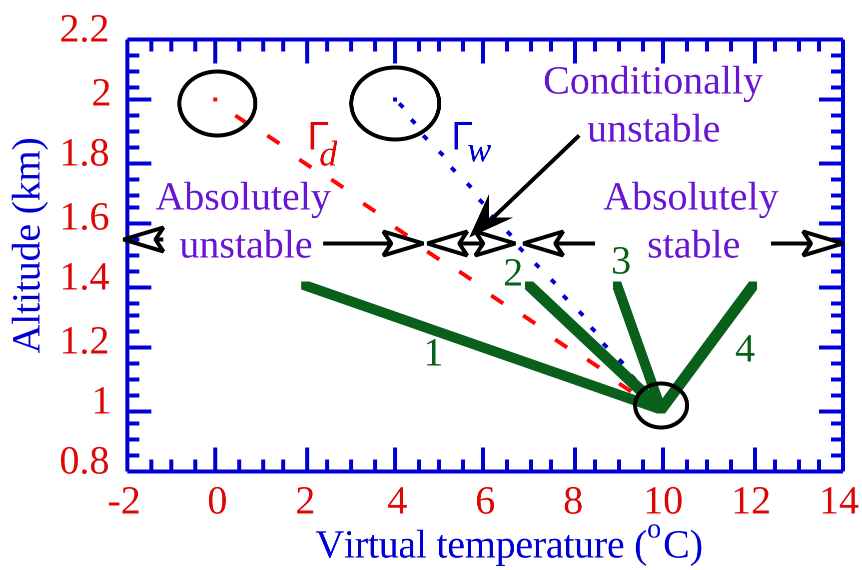
Example 9.1.

| | | |
|---|---|---|
| | $p_d$ | = 950 mb |
| | $T$ | = 283 K |
| ---> | $p_{v,s}$ | = 12.27 mb |
| ---> | $\omega_{v,s}$ | = 0.00803 kg kg$^{-1}$ |
| ---> | $T_v$ | = 284.4 K |
| ---> | $\Gamma_w$ | = 5.26 K km$^{-1}$ |
| | $T$ | = 293 K |
| ---> | $\Gamma_w$ | = 4.26 K km$^{-1}$ |

# Stability Criteria

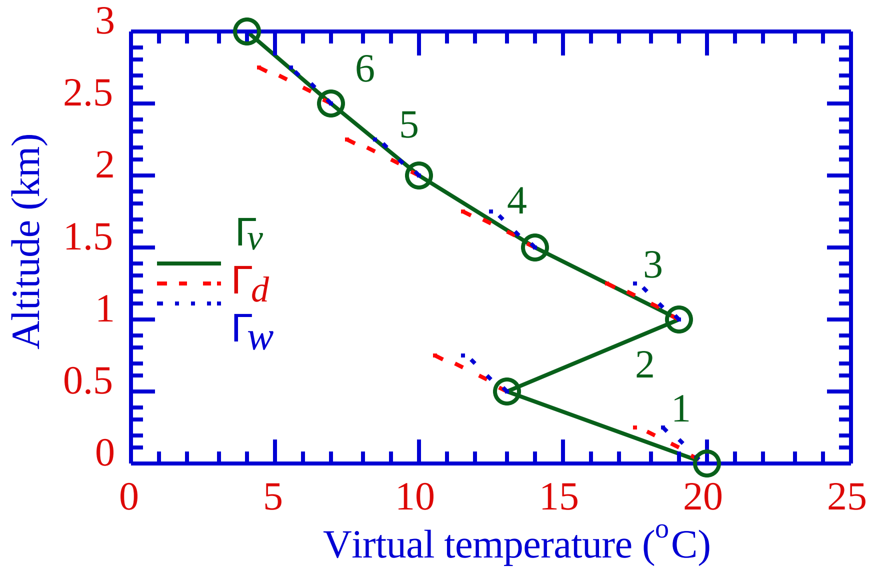
$$\begin{cases} \Gamma_v > \Gamma_d & \text{absolutely unstable} \\ \Gamma_v = \Gamma_d & \text{unsaturated neutral} \\ \Gamma_d > \Gamma_v > \Gamma_w & \text{conditionally unstable} \\ \Gamma_v = \Gamma_w & \text{saturated neutral} \\ \Gamma_v < \Gamma_w & \text{absolutely stable} \end{cases} \tag{9.7}$$

Fig. 9.2. Stability criteria for dry or moist air.



# Determination of Stability in Multiple Layers of Air

Fig. 9.3.



# Equivalent Potential Temperature

Potential temperature a parcel of air would have if all its water vapor were condensed and the resulting latent heat were released and used to heat the parcel

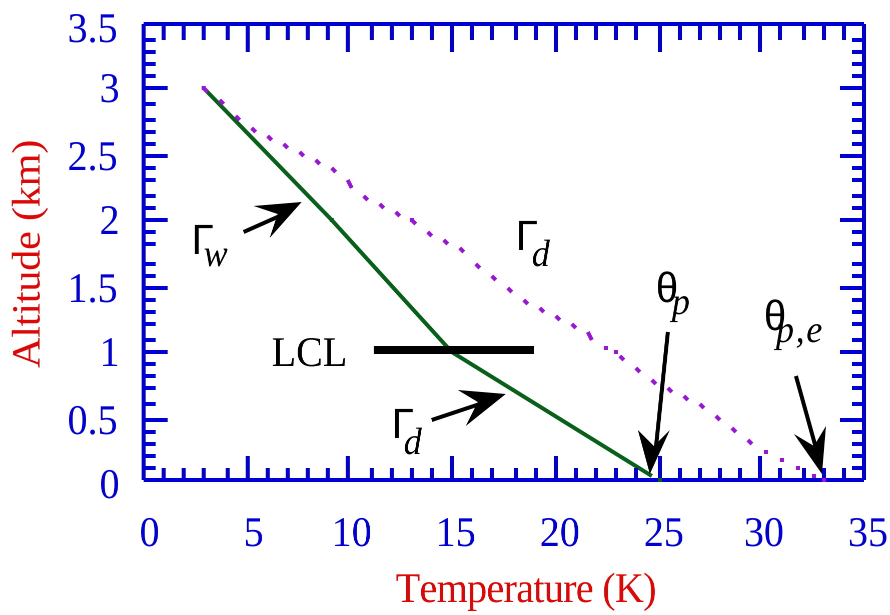
Equivalent potential temperature in unsaturated air

$$\theta_{p,e} \approx \theta_p \exp\left(\frac{L_e}{c_{p,d}T}\omega_{v,s}\right) \qquad (9.8)$$

Equivalent potential temperature in unsaturated air

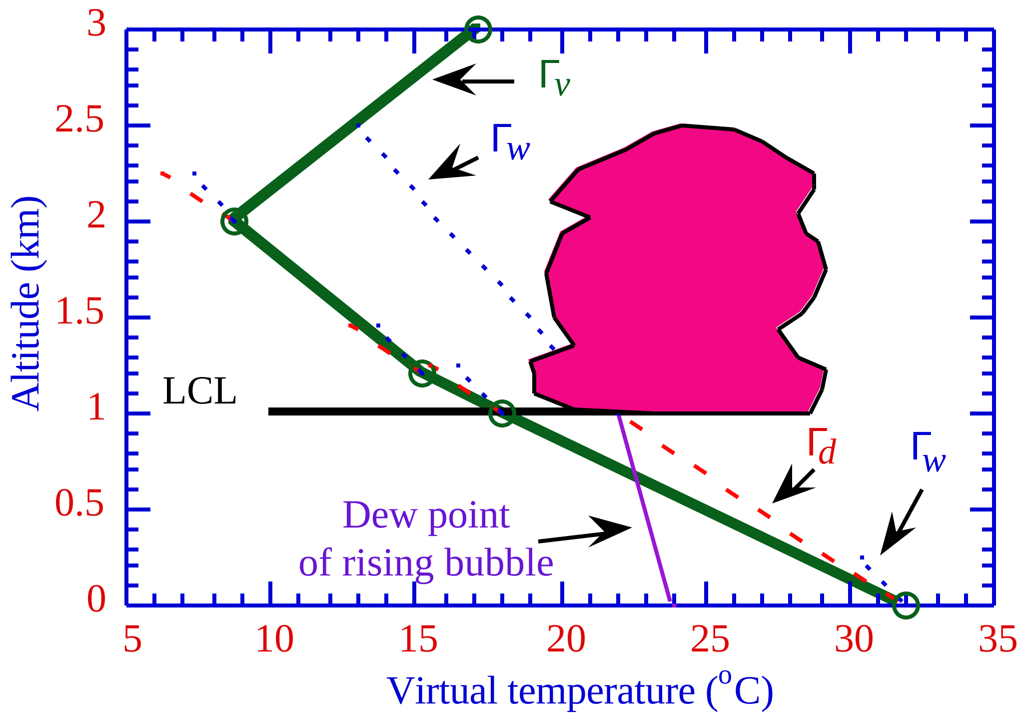
$$\theta_{p,e} \approx \theta_p \exp\left(\frac{L_e}{c_{p,d}T_{\mathrm{LCL}}}\omega_{v}\right) \qquad (9.9)$$

Fig. 9.4. Relationship between potential temperature and equivalent potential temperature.



# Cumulus Cloud Formation

Fig. 9.5.



# Isentropic Condensation Temperature

Temperature at the base of a cumulus cloud

Occurs at the lifting condensation level (LCL), which is that altitude at which the dew point meets parcel temperature.

Isentropic condensation temperature

$$T_{v,L} \approx \frac{4880.357 - 29.66 \ln\left[\frac{\omega_v p_{a,0}}{\varepsilon}\left(\frac{T_{v,L}}{T_{v,0}}\right)^{1/\kappa}\right]}{19.48 - \ln\left[\frac{\omega_v p_{a,0}}{\varepsilon}\left(\frac{T_{v,L}}{T_{v,0}}\right)^{1/\kappa}\right]} \qquad (9.11)$$

# Entrainment

Mixing of relatively cool, dry air from outside the cloud with warm, moist air inside the cloud

Factors affecting the temperature inside a cloud

1) Energy loss from cloud due to warming of entrained, ambient air by the cloud

$$dQ_1^* = -c_{p,d}\left(T_v - \hat{T}_v\right)dM_c \qquad (9.12)$$

2) Energy loss from cloud due to evaporation of liquid water in the cloud to ensure entrained, ambient air is saturated

$$dQ_2^* = -L_e\left(\omega_{v,s} - \hat{\omega}_v\right)dM_c \qquad (9.13)$$

3) Energy gained by cloud during condensation of rising air

$$dQ_3^* = -M_c L_e d\omega_{v,s} \qquad (9.14)$$

# Entrainment

Sum the three sources and sinks of energy

$$dQ^{*} = -c_{p,d}\left(T_v - \hat{T}_v\right)dM_c - L_e\left(\omega_{v,s} - \hat{\omega}_v\right)dM_c - M_c L_e d\omega_{v,s} \qquad (9.15)$$

First law of thermodynamics

$$dQ^{*} = M_c\left(c_{p,d}\,dT_v - \alpha_a\,dp_a\right) \qquad (9.16)$$

Subtract (9.16) from (9.15) and rearrange

$$c_{p,d}dT_v - \alpha_a dp_a = -\left[c_{p,d}\left(T_v - \hat{T}_v\right) + L_e\left(\omega_{v,s} - \hat{\omega}_v\right)\right]\frac{dM_c}{M_c} - L_e d\omega_{v,s} \qquad (9.17)$$

Divide by $c_{p,d}T_v$ and substitute $\alpha_a = R'T_v / p_a$

$$\frac{dT_v}{T_v} - \frac{R'}{c_{p,d}}\frac{dp_a}{p_a} = -\left[\frac{T_v - \hat{T}_v}{T_v} + \frac{L_e\left(\omega_{v,s} - \omega'_v\right)}{c_{p,d}T_v}\right]\frac{dM_c}{M_c} - \frac{L_e d\omega_{v,s}}{c_{p,d}T_v} \qquad (9.18)$$

Rearrange and differentiate with respect to height

$$\frac{\partial T_v}{\partial z} = -\frac{g}{c_{p,d}} - \left[\left(T_v - \hat{T}_v\right) + \frac{L_e}{c_{p,d}}\left(\omega_{v,s} - \hat{\omega}_v\right)\right]\frac{1}{M_c}\frac{\partial M_c}{\partial z} - \frac{L_e}{c_{p,d}}\frac{\partial \omega_{v,s}}{\partial z} \qquad (9.19)$$

No entrainment ($dM_c = 0$) --> pseudoadiabatic temp. change

# Vertical Temperature Change in a Cloud

Change of potential temperature with altitude

$$\frac{\partial \theta_v}{\partial z} = \frac{\theta_v}{T_v}\frac{\partial T_v}{\partial z} - \kappa \frac{\theta_v}{p_a}\frac{\partial p_a}{\partial z} \qquad (2.97)$$

Rearrange

$$\frac{\partial T_v}{\partial z} = \frac{T_v}{\theta_v}\frac{\partial \theta_v}{\partial z} + \frac{R' T_v}{c_{p,d}\, p_a}\frac{\partial p_a}{\partial z} = \frac{T_v}{\theta_v}\frac{\partial \theta_v}{\partial z} - \frac{g}{c_{p,d}} \qquad (9.20)$$

Substitute into (9.19)

--> change of potential temperature in entrainment region

$$\frac{\partial \theta_v}{\partial z} = -\frac{\theta_v}{T_v}\left[\left(T_v - \hat{T}_v\right) + \frac{L_e}{c_{p,d}}\left(\omega_{v,s} - \hat{\omega}_v\right)\right]\frac{1}{M_c}\frac{\partial M_c}{\partial z} - \frac{\theta_v}{T_v}\frac{L_e}{c_{p,d}}\frac{\partial \omega_{v,s}}{\mathrm{d}z} \qquad (9.21)$$

Multiply through by $\mathrm{d}z$ and dividing through by $\mathrm{d}t$

$$\frac{\mathrm{d}\theta_v}{\mathrm{d}t} = -\frac{\theta_v}{T_v}\left[\left(T_v - \hat{T}_v\right) + \frac{L_e}{c_{p,d}}\left(\omega_{v,s} - \hat{\omega}_v\right)\right]\mathrm{E} - \frac{\theta_v L_e}{c_{p,d} T_v}\frac{\mathrm{d}\omega_{v,s}}{\mathrm{d}t} \qquad (9.22)$$

Entrainment rate

$$\mathrm{E} = \frac{1}{M_c}\frac{\mathrm{d}M_c}{\mathrm{d}t} \approx \frac{3}{4\pi r_t^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4\pi r_t^3}{3}\right) \qquad (9.23)$$

# Thermodynamic Energy Equation in a Cloud

Add terms to (9.22)

--> thermodynamic energy equation in a cloud

$$\frac{d\theta_v}{dt} = -\frac{\theta_v}{T_v}\left[\left(T_v - \hat{T}_v\right) + \frac{L_e}{c_{p,d}}\left(\omega_{v,s} - \hat{\omega}_v\right)\right]E + \frac{1}{\rho_a}\left(\nabla \cdot \rho_a \mathbf{K}_h \nabla\right)\theta_v$$

$$+ \frac{\theta_v}{c_{p,d}T_v}\left(-L_e\frac{d\omega_{v,s}}{dt} - L_m\frac{d\omega_L}{dt} - L_s\frac{d\omega_{v,I}}{dt} + \frac{dQ_{solar}}{dt} + \frac{dQ_{ir}}{dt}\right)$$

(9.24)

# Vertical Momentum Equation in a Cloud

Vertical momentum equation in Cartesian / altitude coordinates

$$\frac{dw}{dt} = -g - \frac{1}{\rho_a}\frac{\partial p_a}{\partial z} + \frac{1}{\rho_a}\left( \quad \cdot \rho_a \mathbf{K}_m \quad \right) w \qquad (9.25)$$

Add hydrostatic equation, $\partial \hat{p}_a / \partial z = -\hat{\rho}_a g$ for air outside cloud

$$\frac{dw}{dt} = -g\frac{\rho_a - \hat{\rho}_a}{\rho_a} - \frac{1}{\rho_a}\frac{\partial\left(p_a - \hat{p}_a\right)}{\partial z} + \frac{1}{\rho_a}\left( \quad \cdot \rho_a \mathbf{K}_m \quad \right) w$$

$$(9.26)$$

Buoyancy factor

$$B = -\frac{\rho_a - \hat{\rho}_a}{\rho_a} = -\frac{p_a \hat{T}_v - \hat{p}_a T_v}{p_a \hat{T}_v} = -\frac{\hat{T}_v - T_v}{\hat{T}_v} + \left(\frac{T_v}{\hat{T}_v}\right)\frac{\hat{p}_a - p_a}{p_a} \approx -\frac{\hat{\theta}_v - \theta_v}{\hat{\theta}_v}$$

$$(9.27)$$

Adjust buoyancy factor for condensate

$$B = -\frac{\rho_a - \hat{\rho}_a}{\rho_a} = -\frac{\hat{\theta}_v\left(1 + \omega_L\right) - \theta_v\left(1 + \hat{\omega}_L\right)}{\hat{\theta}_v} \approx \frac{\theta_v - \hat{\theta}_v}{\hat{\theta}_v} - \omega_L$$

$$(9.28)$$

# Vertical Momentum Equation in a Cloud

Substitute (9.28) into (9.26)

$$\frac{dw}{dt} = g\left(\frac{\theta_v - \hat{\theta}_v}{\hat{\theta}_v} - \omega_L\right) - \frac{1}{\rho_a}\frac{\partial\left(p_a - \hat{p}_a\right)}{\partial z} + \frac{1}{\rho_a}\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)w \qquad (9.29)$$

Rewrite pressure gradient term

$$\frac{1}{\rho_a}\frac{\partial p_a}{\partial z} = -g = -\frac{\partial\Phi}{\partial z} = c_{p,d}\theta_v\frac{\partial P}{\partial z} \qquad (9.30)$$

Substitute (9.30) and (9.29)

--> vertical momentum equation in a cloud

$$\frac{dw}{dt} = g\left(\frac{\theta_v - \hat{\theta}_v}{\hat{\theta}_v} - \omega_L\right) - c_{p,d}\theta_v\frac{\partial\left(P - \hat{P}\right)}{\partial z} + \frac{1}{\rho_a}\left(\nabla\cdot\rho_a\mathbf{K}_m\nabla\right)w \qquad (9.31)$$

# Simplified Expression for Vertical Velocity

Simplify (9.31) for basic calculations

Ignore pressure perturbation and the eddy diffusion term

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{dz}{dt} = \frac{dw}{dz}w = g\left(\frac{\theta_v - \hat{\theta}_v}{\hat{\theta}_v} - \omega_L\right) = gB \qquad (9.32)$$

where

$$w = \frac{dz}{dt}$$

Rearrange (9.32)

$$w\,dw = gB\,dz$$

Integrate over altitude --> estimate of vertical velocity in a cloud

$$w^2 = w_a^2 + 2g\int_{z_a}^{z}\left(\frac{\theta_v - \hat{\theta}_v}{\hat{\theta}_v} - \omega_L\right)dz = w_a^2 + 2g\int_{z_a}^{z}B\,dz \qquad (9.33)$$

# Convective Available Potential Energy

$$\text{CAPE} = g\int_{z_{\text{LFC}}}^{z_{\text{LNB}}} B\,\mathrm{d}z \approx g\int_{z_{\text{LFC}}}^{z_{\text{LNB}}} \left(\frac{\theta_v - \hat{\theta}_v}{\hat{\theta}_v}\right)\mathrm{d}z \qquad (9.34)$$