COMPUTATION OF ELECTRIC POWER PRODUCTION COST WITH TRANSMISSION CONSTRAINTS

EMF SR 6

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December 1996

Energy Modeling Forum
Terman Engineering Center
Stanford University
Stanford, California

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A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF OPERATIONS RESEARCH

AND THE COMMITTEE ON GRADUATE STUDIES

OF STANFORD UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Robert Leonard Earle

December 1996

I certify that I have read this dissertation and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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B. Curtis Eaves

Approved for the University Committee on Graduate Studies:

gradient of cost with respect to transmission capacities. Third, we give an algorithm for generating the non-redundant constraints from a Gale-Hoffman type region. The Gale-Hoffman conditions characterize feasibility of flow in a network. This is useful not only in calculating reliability, but it turns out that in order to calculate the system cost we integrate over Gale-Hoffman type regions as well. As a result, for many broad classes of networks, enormous computational effort is saved. We also gather together some existing and new results on Gale-Hoffman regions and put them in a unified framework. Fourth, in order to derive the multi-area production cost curves and also to perform the integration of the multivariate Edgeworth series, an asymptotic series used to represent probability densities, we need wedge shaped regions (a wedge is the affine image of an orthant). We give an algorithm for decomposing any polyhedral set into wedges. Fifth, multivariate integration of the normal distribution is a problem with importance in many areas and central to calculation of the production cost. This thesis gives a new method for one dimensional numerical integration of the trivariate normal. The best methods previously known were only able to reduce the problem to a two dimensional numerical integration.

Abstract

The production cost in operating an electric power system is the cost of generation to meet the customer load or demand. Production costing models are used in analysis of electric power systems to estimate this cost for various purposes such as evaluating long term investments in generating capacity, contracts for sales, purchases, or trades of power. A multi-area production costing model includes the effects of transmission constraints in calculating costs. Including transmission constraints in production costing models is important because the electric power industry is interconnected and trades or sales of power amongst systems can lower costs. Moreover, the ongoing deregulation of the power industry and growth of a more open market for power make the need to explicitly account for the effects of transmission on costs more vital.

This thesis develops an analytical model for multi-area production costing. The advantage of this approach is that it explicitly examines the underlying structure of the problem. The major contributions of our research are as follows. First, we develop the multivariate model not just for transportation type models of electric power network flows, but also for the direct current power flow model. This overcomes the objection that power flows are unrealistically modeled by a transportation network model. Most of the competing approaches suffer from this problem. In fact, with the approach developed here, other exogenous restrictions could be placed on the system subject to some conditions. Second, this thesis derives the multi-area production cost curve in the general case. This new result gives a simple formula for determination of system cost and the

Acknowledgments

A couple of years ago, I was thinking that a perhaps fitting subtitle to this "Acknowledgments" section would be "It takes a village to write a dissertation", but our current First Lady, Hillary Clinton grabbed that cliché before I could finish. Many people have contributed in a great variety of ways to my research and if I, by chance, do not mention some crucial aspect of this, I trust their understanding. I want to begin this section in a nontraditional manner and hope for the indulgence of those who would want otherwise.

First and foremost, I want to thank my wife, Sri Susilowati, for her support and encouragement through this process. She put up with my *alon-alon asal kelakon, kebat kliwat*[1] approach and supported my intellectual curiosity as being a good thing, and not just a necessary evil. Sri also gave me her skills as a computer graphist by drawing most of the figures in this dissertation. The degree to which they lack aesthetic appeal is my own fault in scribbling vague pictures and poorly describing the idea behind the figure.

My advisor, John P. Weyant, has been relentlessly encouraging and supportive. He financially supported my research in the early stages. His comments were very helpful in organizing this thesis. It has been a pleasure to get to know him and get his broad, global view.

Jeremy Bloom at the Electric Power Research Institute provided the idea and starting point for this research. As the reader of this dissertation will see, many of the key

[1] Roughly, "slowly, but surely, haste makes waste", though I think in my case it has been more surely slow than sure.

insights herein are due to him. Through Dr. Bloom, the Electric Power Research Institute provided funding for the bulk of this research for which I am grateful. I have enjoyed working with him.

B. Curtis Eaves has over a number of years provided an ear for my research ideas. In my search for a research topic, he was a useful sounding board and encouraged my interest in electric power industry topics. I appreciate the time he has spent listening to my ideas and giving freely of his own perspective.

I want to thank Ross Shachter and Warren Hausmann for serving on my orals committee and asking questions that gave another perspective on my research. I also want to thank "Pete" Veinott, my program advisor, and Richard Cottle for encouraging me throughout my time at Stanford.

Karl Schmedders has been a good friend to me the past few years. He listened to many wildly improbable brainstorms I had about various aspects of my research, kept me going when I was down, and actually appreciated my humor.

Many other classmates and colleagues at Stanford have encouraged me and been a source of joy in my life. Thanks you to you all.

I would also like to thank Paul McEntire for his friendship, lessons in finance, and use of extra CPU cycles at Skye Investment Advisors.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

The production cost in operating an electric power system is the cost of generation to meet the customer load or demand. Production costing models are used in analysis of electric power systems to estimate this cost for various purposes such as evaluating long term investments in generating capacity, contracts for sales, purchases, or trades of power. A multi-area production costing model includes the effects of transmission constraints in calculating costs. Including transmission constraints in production costing models is important because the electric power industry is interconnected and trades or sales of power amongst systems can lower costs. Moreover, the ongoing deregulation of the power industry and growth of a more open market for power make the need to explicitly account for the effects of transmission on costs more vital.

Problems related to production costing are those of calculating indices related to the reliability of an electric power system. A principle motivation for including transmission in modeling power production in the past has been the need to examine issues of reliability in the operation of power systems. We shall see, however, that while reliability and costing models have many elements in common they differ in important, fundamental ways. The principle contribution of this thesis is to give a method for analytic calculation of the expected electric power production cost with transmission constraints.

While the level of power trades amongst producers has always been important, they have tended to take place in the context of power pools or other non-market

arrangements. There has been a growing trend in the United States and abroad towards deregulation of the power production part of the electric power industry. Since this trend is just reaching the point of implementation, it is not clear what form the market will take, but it is clear that more explicit ways to value the service of transmission of electrical power are needed. While there are many proposed ways to price power transmission, (Hogan (1992) provides a discussion of some along with his own proposal), underlying issues of how to model the effects of transmission on production cost still remain. As discussed below, existing models work in only limited ways so that larger public policy and business strategy questions such as the benefits of increased amounts of available transmission, siting of new generation, and the price for transmission service are difficult to address. Hayashi (1988) in examining the Southwest Market Experiment and Blair (1991) amongst others have cited the difficulties in addressing these questions without improved underlying models. A recent issue of *The Electricity Journal* (June 1996) was devoted to various parts of the question of how the new, deregulated market for transmission should be organized. What is striking, however, is that the fundamental issue of actually analyzing the underlying tradeoffs amongst generation and transmission is not addressed anywhere in the special issue, perhaps due to the fact the currently available approaches are unsatisfactory.

This thesis develops a new approach to multi-area production costing that has several advantages over the existing approaches. Because it is analytic, not simulative, it focuses explicitly on the underlying structure of the problem. Second, we do not suffer the problems that other heuristic approaches do in the uncertainty of whether our solution actually solves the model set forward. While our method involves an approximation step, we give the exact solution of an approximate model compared with others approximate solution of an exact model, which is itself an approximation. Third, one of the complaints of Hogan (1992) is that most costing approaches model the transmission problem as a

transportation problem and so do not reflect some of the important physical features of the problem. We can overcome this limitation by solving the problem for the D.C. power flow model (see, for example, Schweppe, Caraminis, Tabors, and Bohn (1988)) which is recognized as being sufficiently close to underlying physical reality for all but very short term problems. The principle difference between the transportation model and the linearized D.C. power flow model is that the latter includes linear constraints in addition to those of the transportation model that represent the additional constraints imposed by direct current power flow in a network.

The remainder of this introduction is organized as follows. We begin by reviewing single area reliability and production costing models with emphasis on the Gram-Charlier approach to single area production costing. Then we discuss the two area analytical approach, other approaches to multi-area production costing, and finally our own approach. The chapter concludes with an outline of this thesis.

Single Area Production Costing Models

Because of the complexity of electric power generation, different types of production costing models focus on various features of the systems and so are useful for answering different questions about the production cost. Two main categories of production costing models are usually referred to as *chronological simulation* and *probabilistic simulation* (originally due to Baleriaux, Jamoulle, and de Guertechin (1967) and Booth (1972)). The word *simulation* as used here does not refer to either Monte-Carlo or discrete event simulation, but rather is used in the sense of the word *model.*

In chronological simulation, one follows the evolving state of the system through time as the level of demand and availability of supply changes. A typical starting point would be with a sequence of customer demands, perhaps randomized somehow, for each unit of time over the period analyzed. Then the model would go through each time unit

constructing an operating schedule. This gives the modeler great flexibility in modeling various operating policies and constraints, so as a result, the chronological method is useful in answering questions about detailed operating policies. However, two problems are inherent in the chronological method. First, it is computationally expensive, especially as the time period examined increases and more accurate estimates are required of the model. Second, when no longer dealing with short term problems, the significance of operating details, which cannot be predicted accurately anyway, comes into question.

In contrast, probabilistic simulation concentrates not on the specific time evolutionary aspects of the system state, but rather on its statistical behavior. This model treats the system state as a random process and calculates specific statistics of the state analytically from its probability structure. The first step in this formulation is the construction of a *load duration curve*. This is simply one minus the probability distribution of load or demand, or in reliability theory terms, the *survival distribution*. It is usually constructed as a histogram from a chronological load curve, so can be considered an empirical distribution of demand. This is a reasonable approach if we assume that *dispatch*, the method by which electric power generators are chosen to meet demand, depends only on the instantaneous state of demand. As a result, if we treat load as a random process, we only care about its steady state distribution. This steady state distribution exists under a variety of reasonable and general conditions and permits direct calculation of the expected value of production cost and related statistics. Mazumdar and Bloom (1990) discuss this issue in detail. A question that arises naturally is why do we not then calculate the variance as well as the mean of production cost with our model in order to better estimate future costs. It turns out (Ryan and Mazumdar (1989)) that while the mean of the probabilistic simulation model reflects that of the true, chronological situation, the variance does not. This is because while the mean does not depend on the duration of

generator outages, the variance does, and we can not capture the duration of generator outages within a probabilistic simulation framework.

The second basic part of probabilistic simulation involves the electric power generators. An important aspect to production costing is the occurrence of generator failures, or outages. In probabilistic simulation a generator is often modeled as having a given steady state probability of outage, say p, so that with probability p none of the generator's rated capacity is available and with probability $1 - p$ all of the rated capacity is available. Generator outage probabilities are usually considered to be independent of each other and of the level of demand.

The third part is the assumption that generators can be dispatched in *economic merit order*. That is, each generator has a linear cost of operation in the amount of production and that generators can be dispatched in order of increasing cost. This assumption is reasonably close to practice for many situations and is useful in capturing one of the central tradeoffs that dominate many power systems: that less expensive generators are used to satisfy base demand and more expensive ones satisfy peak demand. As a result of the foregoing assumptions it is easy to see that the optimal dispatch order is the same as the economic merit order. The last concept required to round out the picture of probabilistic simulation is that of the *equivalent load duration curve*. For each unit, this is simply the load duration curve net of the probable capacity of previously dispatched units. So, the equivalent load duration for a generator represents the residual demand after dispatching the previous generators in the merit order. In other words, if we consider load as a random variable (with positive value) and supply as a random variable (with positive value) then the equivalent load duration curve is one minus the probability distribution of the load minus the supply. Units are dispatched in turn, in economic merit order, giving rise to successive equivalent load duration curves.

Because our interest here lies in questions of the effects of transmission capacity on production cost and because we wish to avoid the problems associated with chronological simulation which are exacerbated in multi-area systems, we focus here on the probabilistic simulation model. Our method extends an analytic approach to solving the probabilistic simulation model to networks of three or more nodes. Before exploring our analytic approach further we will first examine the development of the probabilistic simulation model in the single area context. In order to make these ideas more precise, let us first define some notation (in general we follow Bloom (1992)):

Q is the system load, $Q \geq 0$ (MW);

$g(Q)$ is the probability density of the system load;

I is the number of generating plants;

u_i, $i = 1, ..., I$ is the available capacity of the i^{th} plant, (MW);

$p_i(u_i)$ is the probability of that generator i has u_i available;

F_i is the cost per unit of generation for the i^{th} generator (\$/MWH);

F_{I+1} is the cost per unit of unserved energy (\$/MWH).

We assume without loss of generality that the generators are indexed in economic merit order. That is, $F_i \leq F_{i+1}$ for all i.

So, the load duration curve at Q equals $1 - \int_0^Q g(x)dx$, where the integral is of the appropriate type. If we let $g_0(Q) = g(Q)$, we can derive the density of net demand, that is demand minus supply, through the convolution

$$g_i(Q) = \int_{-\infty}^{\infty} g_{i-1}(Q - x)p_i(x)dx. \quad (1)$$

We can do this because of the independence of demand and the generators. Thus, $1 - \int_0^Q g_i(x)dx$ is the value of the equivalent load duration curve after loading the first i generators. Note that these definitions define the sign convention for net demand that we will adhere to throughout this thesis: demand is positive and supply is negative. So, the

loss of load probability (LOLP), or the probability that not all demand is supplied, after loading the first i generators,

$$LOLP_i = \int_0^\infty g_i(x)dx, \tag{2}$$

and the expected unserved energy (EUE) after loading the first i generators,

$$EUE_i = \int_0^\infty x\, g_i(x)dx. \tag{3}$$

(1), (2), and (3) adumbrate a leitmotif of this thesis: the operations of convolution and integration. The convolution operation is used to generate new densities from combinations of previous and integration is used to calculate the desired indices.

The expected energy served by a particular generator is given by the expected unserved energy before loading that generator minus the expected unserved energy after loading that generator. So, the expected cost from the i^{th} generator is $F_i*(EUE_{i-1} - EUE_i)$, where EUE_0 is just the system demand, and we can calculate the expected cost of operating the system by

$$C = \sum_{i=1}^{I+1} F_i*(EUE_{i-1} - EUE_i). \tag{4}$$

The probabilistic simulation model began with the work of Baleriaux and de Guertechin (1967) and Booth (1972) in which they formulated the basic model outlined above. Their main interest was in the one area problem and production costing. They represented the load duration curve through a table and calculated the production cost by numerical convolution of the supply provided by each generator with the load duration curve to obtain successive equivalent load duration curves. This procedure was both computationally expensive and subject to numerical errors. As a result various ways of representing and calculating the $g_i(\cdot)$ and their integrals were explored (Mazumdar (1983) gives an overview of various approaches). Rau and Schenk (1979), Rau, Toy, and Schenk (1980), and Stremel, Jenkins, Babb, and Bayless (1980) introduced the most

robust of these, that is the idea of using a truncated Gram-Charlier series representation. The Gram-Charlier series is an asymptotic expansion based on the gaussian or normal density and its derivatives. The coefficients in the series are linear combinations of the moments or cumulants of approximated function. Although there are no known error bounds for the series, it works fairly well in practice with just the first four to six terms of the series. From a modeling point of view, while using the series results in a further approximation of the modeled system, this is more than offset by its flexibility and the ability to develop results in closed form. Cramer (1974) and Kendall and Stuart (1977) develop the series in the statistical context while Beckmann (1973) puts the series in the more general context of orthogonal polynomials. In Chapter Two we develop the Gram-Charlier series in the multivariate context. Since, based on the data for the problem, the empirical LDC and outage rates and capacities for generators, we can calculate moments or cumulants to any order we wish, the Gram-Charlier series becomes a convenient tool for analysis. In particular the problem of calculating LOLP and EUE become simply a matter of calculating the required moments and an integral of the univariate normal density.

It should be noted that the Gram-Charlier series approach is better suited for cost calculations than for calculation of LOLP. The reason for this is that in the LOLP calculation we need to calculate a tail probability, (2), that is usually very small. As a result the precision of the estimate is very important. On the other hand, since most of the cost of operating a power system occurs with the base units for which the energy served is high, precision becomes less important.

A number of models have been based on using the Gram-Charlier series approach to probabilistic simulation. Bloom (1982) developed a capacity expansion planning model based on it, Bloom (1984) incorporated energy storage units, and Bloom (1992) developed a single Gram-Charlier series to represent the expected cost curve of a system.

The importance of this latter result is several fold. First, it means that in order to calculate the expected cost only one truncated series must be integrated rather than having to calculate the expected unserved energy after each unit is loaded as in (4). Second, the calculation of the gradient of production cost with respect to plant capacities and changes in the load shape is similarly simplified.

An Analytic Approach to Two Area Production Costing

All the models discussed so far assume that all demand and generation occurs at one location, so that the effects of transmission are not taken into account. When more than one area is modeled, then transmission constraints cause a departure from economic merit order since the cheapest generator cannot always be used to satisfy remaining demand. Noyes (1983) points out that the expected energy served by a generator is still given by the difference of the expected unserved energy before it is loaded and the expected unserved energy after it is loaded. However, the calculation for the expected unserved energy is now quite different.

In Noyes (1983) and Bloom (1990) the probabilistic simulation model using the Gram-Charlier series was extended to two areas. Lee (1990) also documents some of these results. Bloom also extended the approach to have a single production cost curve in analogy to Bloom (1992)[1]. A bivariate Gram-Charlier series was used to represent the bivariate probability density of net demand, and transmission constraints were modeled as a transportation network. So, power flow from area one to two or vice versa is limited by some amount, say t. By defining constraints for the region over which demand is satisfied, one can integrate the density of net demand over the region defined by the constraints to get the system reliability. Since the unserved energy is equal to the smallest amount of

[1]Private communication.

energy needed to make the system feasible, these inequalities also become useful in defining the integrals for unserved energy and production cost and figure prominently in the development of our model. Let x_i be the net demand in area i. So x_i is negative when there is more supply than demand. Then the inequalities

$$\begin{aligned} x_1 &\leq t \\ x_2 &\leq t \\ x_1 + x_2 &\leq 0 \end{aligned}$$

define the region in which there is sufficient supply to satisfy all the demand. These inequalities are the Gale-Hoffman (Gale (1957) and Hoffman (1960)) inequalities that define the existence of a feasible flow in a network. Bloom (1990) and Lee (1990) describe this analytic approach to the two area problem in detail.

Extending the Two Area Analytical Approach

This thesis addresses the problem of extending this analytical approach for the two-dimensional model to three or more areas. As yet there has been no success in extending this method or by some other method exactly calculating the production cost in the multi-area context. Our work develops such a method. In order to do this we need to be able to characterize the region of integration for the various indices, derive a multi-area cost curve based on the regions of integration, and implement an integration algorithm. It turns out that in order to derive the multi-area cost curve it is necessary to have regions of integration of a certain shape. The shape necessary is called a *wedge* which is the affine image of an orthant. A wedge in n dimensions is the intersection of n or fewer half-spaces such that there is a boundary point common to all the half-spaces. Why this shape is necessary has to do with the expression of the integral as an iterated integral. So, the first step in the method is to decompose the region of integration into wedges. This done, we can derive the multi-area cost curve. A further complication in the method is that because

we rely on the Gale-Hoffman constraints to characterize the region, we potentially generate many redundant constraints when we have sparse networks. Therefore, we need to find a method to overcome this difficulty.

Other Approaches to Multi-Area Production Costing

A number of other approaches have been taken in attempting to solve the probabilistic simulation model of multi-area production costing. Because the problem is related, we discuss the multi-area reliability problem here as well. As in the case of production costing without taking into account multi-area demand and production, when we try to model multi-area production and demand there is a tradeoff between the questions we wish to answer and the level of detail possible and meaningful in the model. At the very detailed end are AC power flow models which take into account many of the detailed physical characteristics and constraints of a system (for example, the reliability model of Bhavaraju (1988)). Because of the level of detail, however, these models cannot be run to examine varying demand and production conditions needed for anything more than very short term planning.

The linearized DC power flow model simplifies the problem, but retains enough of the complexity of the power flow that it is widely considered to be close enough to the physical model for anything but short term problems (see, for example, Pereira, Pinto, Oliveira, and Cunha (1987)). The only successful approach to this model so far has been that of simulation in Pereira, Pinto, Cunha, Oliveira, Mazumdar, and Yengar (1990). Their focus is on calculating reliability indices and uses Monte-Carlo sampling and variance reduction techniques such as importance sampling and antithetic variables to achieve its solution. While this model has had some success it has been criticized on two counts. First, it is still limited in the number of system production and demand states which can be modeled (Hobbs and Ji (1994)). Second, it does not provide any insight into the

underlying structure of the problem, nor are sensitivity analyses and gradients easily obtainable.

Approaches that take a simpler approach in treating the transmission network as a transportation problem take various forms. Clancy, Gross, and Wu (1983), Singh and Lago-Gonzalez (1989), Deng and Singh (1993), and Gubbala and Singh (1993) address the reliability problem and all use a state-space decomposition approach with various refinements. The basic approach is to divide the multivariate state space into successive partitions until the probability of the area remaining is low. While some new successes have been achieved through this approach, the number of areas and units able to be modeled this way is limited. The largest network reported solved in Gubbala and Singh (1993) has only ten areas with thirteen arcs.

Lee (1990) presents a heuristic approach for the production costing problem which has shown good results for problems with over three areas using two-area approximations via load duration curves. Unfortunately, because it is a heuristic both the characteristics and accuracy of any solution is in doubt.

The last approach, developed in Hobbs and Ji (1994), is a bounding method that makes use of Jensen's and Edmond's type inequalities along with generalized Bender's decomposition to get upper and lower bounds to the expected production cost. While they prove convergence and show fast convergence, the problem they solve has fixed demand and the largest problem solved is five areas with only ten generating units per area.

A word is in order here about another class of models exemplified by Hogan (1992) and Hogan (1993). Hogan is concerned with the institutional arrangements necessary to make a competitive electric power sector operate efficiently. He analyzes this question principally through consideration of short-term efficiency issues such as optimal spot-pricing. In doing this one of his main contributions is accurately reflecting the effect

of flow on system capacity. The model in this thesis, on the other hand, is concerned with longer term effects and as a result is forced to put the model in a probabilistic framework that Hogan avoids.

Thesis Overview

Because of the problems in other approaches to the multi-area production costing problem such as only being heuristic approaches or not being able to handle more than a few areas with limited variation in the probability space, we pursue the analytic approach that was discussed above for the two area problem. Moreover, in contrast to the simulation type approaches, our approach has the advantage that we have looked at the underlying structure of the problem. The major contributions of our research are as follows. First, we develop the multivariate model not just for transportation type models of electric power network flows, but also for the direct current power flow model. This overcomes the objection that power flows are unrealistically modeled by a transportation network model. Most of the competing approaches suffer from this problem. In fact, with the approach developed here other exogenous restrictions could be placed on the region subject to some conditions. Second, this thesis derives the multi-area production cost curve in the general case. This new result gives a simple formula for determination of system cost and the gradient of cost with respect to transmission capacities. Third, we give an algorithm for generating the non-redundant constraints from a Gale-Hoffman type region. This is useful not only in calculating LOLP, but it turns out that in order to calculate the system cost or EUE we integrate over Gale-Hoffman type regions as well. As a result, for many broad classes of networks, enormous computational effort is saved. We also gather together some existing and new results on Gale-Hoffman regions and put them in a unified framework. Fourth, in order to derive the multi-area production cost curves and also perform the integration of the multivariate Edgeworth series we need wedge

shaped regions. We give an algorithm for decomposing any polyhedral set into wedges. Fifth, in order to calculate the production cost, we need to integrate the Edgeworth series representation. This involves integration of the multivariate normal density. Also, the multivariate integration of the normal distribution is a problem with importance in many areas. This thesis gives a new method for one-dimensional numerical integration of the tri-variate normal. The best methods previously known were only able to reduce the problem to a two-dimensional numerical integration.

The remainder of this dissertation is organized as follows. Chapter Two develops our model in mathematical detail and derives the multi-area production cost curve. In Chapter Three we give some results for the structure of regions defined by the Gale-Hoffman inequalities, develop the algorithm for generating non-redundant constraints, and give the wedge decomposition. Chapter Four discusses the integration of the multivariate normal and multivariate Edgeworth series along with our new result for one dimensional numerical integration of the tri-variate normal density. In Chapter Five some computational results are discussed for cases with three areas. The thesis concludes with an examination of extensions and directions for further research. Appendix A gives some results that have to do with analysis of the problem of integration of the multivariate normal. Some of these results are used in establishing the trivariate result in Chapter Four and others provide the basis for some of the discussion in Chapter Six.

Chapter 2

Multi-Area Production Costing

This chapter develops our model in mathematical detail and derives the multi-area production cost curve. We begin by defining the basic mathematical structure and notation for the problem and our algorithm. After that, we discuss the Edgeworth or Gram-Charlier series representation for our probability distribution. The chapter concludes with the derivation of the formula for the multi-area production cost. Although the model and examples we use here are from the context of modeling the problem as a transportation network, the difference with the D.C. power flow model lies in the different constraint set defining feasible flow. In Chapter Three we show that the model developed here can be applied to the D.C. power flow model as well.

Model Structure

One can think of the electric power system as an undirected graph that consists of a set of N nodes and capacitated arcs between them. For two nodes, n and m, the arc between them, a_{nm}, has capacity $c_{nm} = c_{mn} \geq 0$. The system demand is a non-negative random vector Q, where Q_n is the demand at node n. Let $g(Q)$ be the density function for Q. At each node n in the network there are I_n generators with capacities $u_{n,i}$ and per unit costs of operation $F_{n,i}$, $i = 1, ..., I_n$. We order the generators in merit order without regard to location to get the costs $F_1, F_2, ..., F_I, F_{I+1}$ (i.e., $F_i \leq F_{i+1}$), so that $I = \sum_{n=1}^{N} I_n$.

F_{I+1} is the per unit cost of unserved energy. Each generator fails with probability $p_{n,i}$, in which case no capacity is available, and all the capacity, $u_{n,i}$ is available with probability $1 - p_{n,i}$. We define the net demand at a node for the first i generators to be the local demand minus the local supply after dispatching the first i generators. The vector of net demands x after loading the first i generators in the merit order in the system is the vector of local net demands. It is the system state for a particular outcome of our random variables and the set of all possible x is the state space for our system. Let $g_i(x)$ be the density function of net demand after the i^{th} generator in the system wide loading order is dispatched. The second section of this chapter addresses the representation of $g(Q)$ and $g_i(x)$ that we use. All that we need to assume for now is that they are integrable.

Let us further define the following notation:

$\Omega \equiv \{1, 2, ..., N\}$ is the set of N nodes;

small Greek letters such as θ and ψ represent subsets of Ω;

$\theta^c \equiv \Omega \backslash \theta$, the complement of θ in Ω;

$t_\theta \equiv \sum_{n \in \theta} \sum_{m \in \theta^c} c_{nm}$, the flow capacity from nodes in θ. $t_\emptyset = 0$;

e is the N dimensional real vector whose entries are all ones;

e_i is the N dimensional real vector whose entries are all zeroes except for a one in the i^{th} position;

$e_\theta \equiv \sum_{i \in \theta} e_i$.

We can define our problem finding the expected production cost as finding the solution to the stochastic linear program (SLP) which we write out in partially in words and partially with our already defined notation in order to avoid introducing new notation that we would only use temporarily.

C = Expected Value of

$$\min \sum_{n=1}^{N} \sum_{i=1}^{I_n} F_{ni} \times (\text{generation by generator } i \text{ at node } n) \qquad \text{(SLP)}$$

subject to

$$\text{generation by generator } i \text{ at node } n \leq \bar{u_{ni}},\ n = 1, ..., N;\ i = 1, ..., I_n$$

$$\textit{demand} \text{ at node } n - \sum_{i=1}^{I_n} \text{generator } i \text{ at node } n - \text{inflow} + \text{outflow} = 0$$

$$\text{flow on arc between nodes } n \text{ and } m \leq c_{nm} \text{ (arc capacity)}$$

The random variables in this problem are, of course, the demand and the generator capacities, $\bar{u_{ni}}$, which implicitly define each of the vectors net demands after loading the first i generators. For a particular value of demand and generator capacities one can solve this problem via linear programming methods or more specialized network algorithms. Doing this however requires that one determine what the optimal flows on the arcs are. A reformulation of the problem allows us to avoid this.

In solving (SLP) we still want to use the cheaper generators before the more expensive ones, but we are limited by the arc capacities. For a particular point in the state space (that is, demands and generator capacities), let W_i be the amount of energy left unserved after dispatching the first i generators and $\overline{W}_i$ be the expected value of W_i. As in Chapter One, we can express the expected production cost as

$$C = \sum_{i=1}^{I} F_i \times (\overline{W}_{i-1} - \overline{W}_i).$$

So, the term in parentheses is just the expected amount of energy produced by generator i. For a particular point in the state space (generator capacities and demand), we can think of W_i as the amount of energy (fictitious) that would be necessary to add to the system in order to make it feasible. It turns out (and this important insight by Noyes (1983)) that

there are a set of conditions that define when there is a feasible flow in a network. So, if we can find the $\overline{W}_i$ we can solve (SLP) by simply calculating $\sum_{i=1}^{I} F_i \times (\overline{W}_{i-1} - \overline{W}_i)$.

It turns out that we can solve for the W_i by solving a linear program that has a particularly nice solution. As a result, we can characterize the W_i as piecewise linear functions over the state space of net demand. So, we can get $\overline{W}_i$ by integrating the piecewise linear function that characterizes W_i over the state space. It turns out that using this characterization of unserved energy we can make a further simplification so that we avoid having to recompute unserved energy after each generator in the loading order. This is the result we call the "multi-area cost curve", and gives the expected production cost and thus solution to (SLP) with a single closed form solution. Now, the W_i are linear over polyhedral regions which are not always convenient for integration, so we need to decompose the linear regions in W_i into regions that we can express as iterated integrals. These are the previously described wedges. Finally, given a single formula and regions convenient for integration, we need a method to perform the integration. These steps give an outline the process necessary for multi-area production costing and is illustrated in figure 2.1.

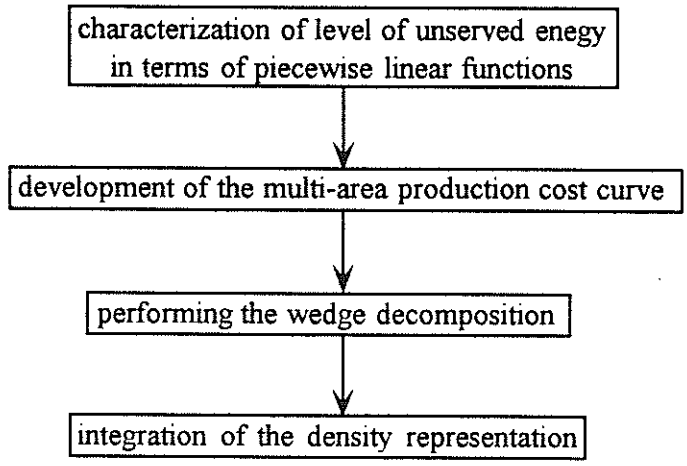

Figure 2.1. : Major Steps in Multi-Area Production Costing

This chapter covers the first two of the steps, characterization of the unserved energy and development of the multi-area production cost curve. The third step is covered in Chapter Three. Chapter Four covers the integration problem.

Characterization of Unserved Energy

As discussed above we can think of the unserved energy as the minimum amount of energy necessary to add to a network in order that there be feasible flow. The following theorem from Gale (1957) and Hoffman (1960) characterizes when there is a feasible flow in a network.

Theorem 2.1 Gale-Hoffman Feasible Flow. *There is a feasible flow if and only if* $\forall\ \theta \subseteq \Omega$,

$$e_\theta^T\ x \leq t_\theta. \tag{1}$$

In words, there is a feasible flow if and only if for every subset of nodes in the network the sums of the local excess demands is less than or equal to the total flow capacity into that subset of nodes.

The N dimensional region defined by the system (1) we will refer to as the feasible region or (FR). For a given x, we would like to characterize the unserved demand in the system. To do this we will first need the following "standard" lemma:

Lemma 2.2 Submodularity of flow capacity. *The flow capacity function* t_θ *is submodular. That is,*

$$t_\theta + t_\psi \geq t_{\theta\cup\psi} + t_{\psi\cap\theta}. \tag{2}$$

Proof: By the definition of t_θ,

$$t_\theta + t_\psi = \sum_{n\in\theta}\sum_{m\in\theta^c} c_{nm} + \sum_{n\in\psi}\sum_{m\in\psi^c} c_{nm}$$

$$= \sum_{n\in\theta\setminus\psi}\sum_{m\in\theta^c} c_{nm} + \sum_{n\in\theta\cap\psi}\sum_{m\in\theta^c} c_{nm} + \sum_{n\in\psi\setminus\theta}\sum_{m\in\psi^c} c_{nm} + \sum_{n\in\psi\cap\theta}\sum_{m\in\psi^c} c_{nm}$$

$$= \sum_{n\in(\theta\cup\psi)\setminus(\theta\cap\psi)} \sum_{m\in\theta^c\cap\psi^c} c_{nm} + \sum_{n\in\theta\setminus\psi} \sum_{m\in\theta^c\cap\psi} c_{nm} + \sum_{n\in\psi\setminus\theta} \sum_{m\in\psi^c\cap\theta} c_{nm}$$
$$+ \sum_{n\in\psi\cap\theta} \Big(\sum_{m\in(\theta\cap\psi)^c} c_{nm} + \sum_{m\in\theta^c\cap\psi^c} c_{nm} \Big)$$

combining the first term with the second part of the fourth term, then rewriting the first part of the fourth term, and combining with the second and third terms yields

$$= \sum_{n\in(\theta\cup\psi)\setminus(\theta\cup\psi)} \sum_{m\in(\theta\cup\psi)^c} c_{nm} + \sum_{n\in\theta\cap\psi} \sum_{m\in(\theta\cup\psi)^c} c_{nm} +$$
$$+ \sum_{n\in\theta\cap\psi} \sum_{m\in(\theta\cap\psi)^c} c_{nm} + 2 \sum_{n\in\theta\cap\psi^c} \sum_{m\in\theta^c\cap\psi} c_{nm}$$

adding the first two terms together and applying the definition of the flow capacity function yields

$$= t_{\theta\cup\psi} + t_{\theta\cap\psi} + 2 \sum_{n\in\theta\cap\psi^c} \sum_{m\in\theta^c\cap\psi} c_{nm} \geq t_{\theta\cup\psi} + t_{\theta\cap\psi}$$

since $c_{nm} \geq 0$. □

The proof is made a bit clearer by figure 2.2. Let γ and η be two subsets of nodes. Referring the figure we can see that

$t(\gamma) = a + e + f + c,$

$t(\eta) = b + e + f + d,$

$t(\gamma \cup \eta) = e + d + c,$ and

$t(\gamma \cap \eta) = a + e + b.$

So, $t(\gamma) + t(\eta) - (t(\gamma \cup \eta) + t(\gamma \cap \eta)) =$

$a + e + f + c + b + e + f + d - (e + d + c + a + e + b) =$

$a + b + c + d + 2e + 2f - (a + b + c + d + 2e) =$

$2f \geq 0.$

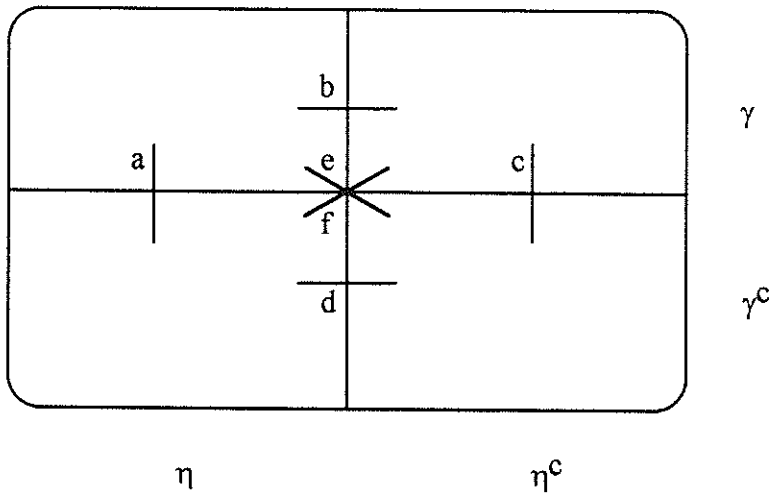

Figure 2.2. : Submodularity of Capacity Function

The unserved energy in the system is the minimum amount of power that needs to be injected into the system so that there is a feasible flow in the system. That is, the unserved energy,

$$z \equiv \min_{w} e^T w \text{ subject to } e_\theta^T\ (x - w)\ \leq\ t_\theta, \forall\ \theta \subseteq \Omega,\ w \geq 0. \qquad \text{(P)}$$

The following, "standard" theorem establishes z is the value of the most violated constraint. That is,

$$z\ = \max\{0,\ \{e_\theta^T\ x - t_\theta\ :\ \forall\ \theta \subseteq \Omega\}\}\ .$$

Theorem 2.3 Characterization of Amount of Unserved Energy. *The optimal value of (P) is the value of the largest constraint violation or zero, that is,* $z\ = \max\{0,\ \{e_\theta^T\ x - t_\theta\ :\ \forall\ \theta \subseteq \Omega\}\}$.

Proof: Let $j_\theta\ = \sum_{i \in \theta} 2^{i-1}$ and A be a 2^N by N matrix such that the i^{th} row of A is e_θ^T such that $j_\theta = i$. The dual of (P) is

$$\max_{u} \sum_{\theta \in \Omega} (e_\theta^T\ x - t_\theta) u_{j_\theta} \text{ subject to } A^T u\ \leq e,\ u\ \geq 0. \qquad \text{(D)}$$

Let $\psi = \arg\max_{\theta \in \Omega} (e_\theta^T\ x - t_\theta)$. That is, $e_\psi^T\ x - t_\psi$ is the most violated constraint in *(P)*. As a result, for $\eta \cap \psi = \emptyset$, $e_\eta^T\ x - t_\eta \leq 0$, since otherwise

$$e_{\eta\cup\psi}^T\ x - t_{\eta\cup\psi} = e_\eta^T\ x + e_\psi^T\ x - t_{\eta\cup\psi}$$
$$\geq e_\eta^T\ x + e_\psi^T\ x - t_\eta - t_\psi \geq e_\psi^T\ x - t_\psi,$$

where the first inequality follows because of submodularity. Let u^* be such that $u_{j_\psi}^* = 1$ and all other entries of u^* are zero. Clearly, $u^*$is a basic solution to (D). To see that it is both feasible and optimal, consider the simplex tableau for (D). Pivot on any non-zero row of the j_ψ column of A^T, say row k. So, $k\ \in \psi$. Remembering that the rows A^T represent nodes of the network, we see that the right-hand-side of the tableau has zeroes in rows $l \in \psi$, $l \neq k$ and ones for $l \notin \psi$. Thus, u^* is feasible. To show that it is optimal we need to price out the cost row in the simplex tableau and show that it is non-positive or that the min-ratio test is zero for any columns with positive prices. There are three cases:

1. If the column represents a set η disjoint with ψ, then as shown above $e_\eta^T\ x - t_\eta \leq 0$, and since $A \geq 0$, the pricing out operation only decreases the value of entries in the cost row.
2. If the column represents a set of that includes k, then it is priced out to be negative since by assumption $e_\psi^T\ x - t_\psi \geq e_\eta^T\ x - t_\eta\ \forall\ \eta$.
3. If the column represents a set that is neither disjoint with ψ nor includes k, then the minimum ratio test for that column will be zero since the column will have a one as its entry for a node in common with ψ and the right hand side will be zero. □

As a result of this theorem our approach is as follows. For every constraint $e_\theta^T\ x \leq t_\theta$ in the feasible demand constraint set there corresponds a region in the state space,

$$\omega_\theta = \{x : e_\theta^T\ x - t_\theta \geq 0;\ e_\theta^T\ x - t_\theta \geq e_\eta^T\ x - t_\eta, \eta \neq \theta\}$$

where the unserved demand is equal to $e_\theta^T\ x - t_\theta$. The expected unserved demand after loading the i^{th} generator is

$$\overline{W}_i \equiv \sum_{\theta \subset \Omega} \int_{\omega_\theta} (e_\theta^T\ x - t_\theta) g_i(x) dx_1 ... dx_N.$$

So as before, the expected production cost is given by

$$C \equiv \sum_{i=1}^{I} F_i * (\overline{W}_{i-1} - \overline{W}_i) + F_{I+1} \overline{W}_{I+1},$$

where $\overline{W}_0$ is the total network wide expected demand, that is,

$$\overline{W}_0 \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^T x\ g(x) dx_1 ... dx_N.$$

The last section of this chapter derives a production cost curve that simplifies this calculation. First, however, let us examine a two area example.

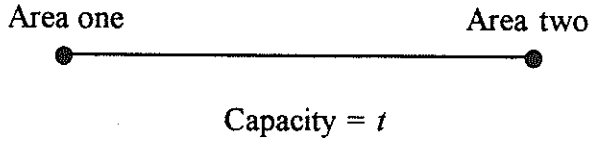

Figure 2.3 : Two Area Network

Consider the simple network depicted in figure 2.3. It has two nodes or areas and a transmission link between them of capacity t. Let the net demand at area one be denoted by $x_1$and the net demand at area two be denoted by x_2. If we just consider the network as a transportation network, all demand in the network can be satisfied if and only if the following Gale-Hoffman inequalities hold:

$$\begin{aligned} x_1 &\leq t \\ x_2 &\leq t \\ x_1 + x_2 &\leq 0 \end{aligned}$$

For the two area case, the first two conditions state that the demand at a node cannot be more than the transmission capacity into the node. The last condition means that

system wide net demand must be non-positive, that is, there is net supply on a system-wide basis. The conditions give rise to the picture shown in figure 2.4.

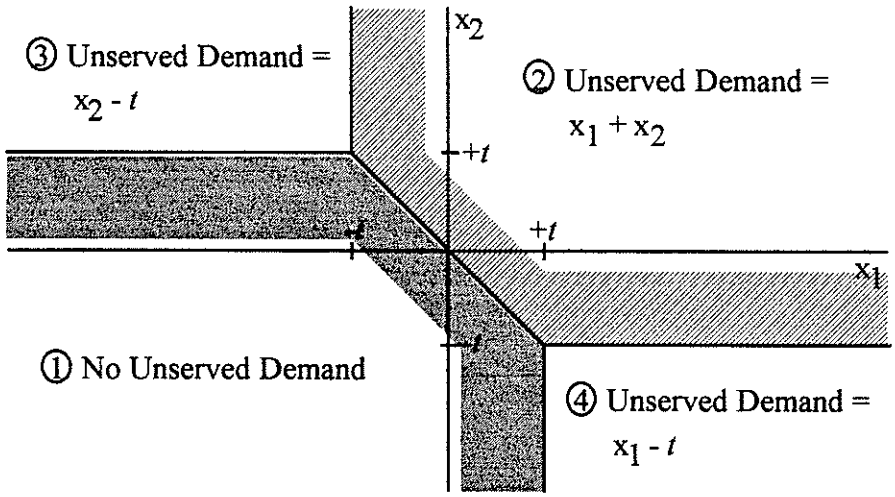

Figure 2.4. : Regions of Integration for a Two Area Network

The plane is divided into four regions. In region one, the Gale-Hoffman conditions are satisfied and there is no unserved demand. Region two has unserved demand equal to $x_1 + x_2$. The boundaries for this region can be generated by the formula $\omega_\theta = \{x : e_\theta^T\ x - t_\theta \geq 0;\ e_\theta^T\ x - t_\theta \geq e_\eta^T\ x - t_\eta, \eta \neq \theta\}$ for $\theta = \{1, 2\}$ to get

$$\omega_{\{1,2\}} = \{x : x_1 + x_2 \geq 0,\ x_1 + x_2 \geq x_1 - t,\ x_1 + x_2 \geq x_2 - t\} =$$

$$\{x : x_1 + x_2 \geq 0,\ x_2 \geq -t,\ x_1 \geq -t\}.$$

Regions three and four have unserved demand of $x_2 - t$ and $x_1 - t$, respectively. In region three the amount of energy supplied by area one to area two is limited by the transmission capacity, t.

To calculate the expected unserved demand after the first i generators have been loaded we need to integrate over the four regions in the plane. Clearly, the integral for region one is zero. Thus,

$$\overline{W}_i = \int_{\omega_{\{1\}}}(x_1 - t)g_i(x)dx_1dx_2 + \int_{\omega_{\{2\}}}(x_2 - t)g_i(x)dx_1dx_2 + \int_{\omega_{\{1,2\}}}(x_1 + x_2)g_i(x)dx_1dx_2,$$

where

$$\omega_{\{1\}} = \{x : x_1 \geq t,\ x_2 \leq -t\},$$
$$\omega_{\{2\}} = \{x : x_2 \geq t,\ x_1 \leq -t\},$$
$$\omega_{\{1,2\}} = \{x : x_1 + x_2 \geq 0,\ x_2 \geq -t,\ x_1 \geq -t\}.$$

This example illustrates several important points. Our algorithm starts with a set of constraints that we use to generate regions over which the unserved energy is a linear function. We perform integration of a probability density times that linear function over those regions to compute the expected cost for the system. The type of region for which integration is practicable is that of an orthant or equivalently no more than n constraints for n dimensional integration. We call such region *wedges*. As a result regions such as $\omega_{\{1,2\}}$ which has three constraints in two dimensions must be decomposed into equivalent wedges. For example,

$$\int_{\omega_{\{1,2\}}}(x_1 + x_2)g_i(x)dx_1dx_2 =$$
$$\int_{\alpha}(x_1 + x_2)g_i(x)dx_1dx_2 - \int_{\beta}(x_1 + x_2)g_i(x)dx_1dx_2,$$

where $\alpha = \{x :\ x_1 + x_2 \geq 0,\ x_2 \geq -t\}$ and $\beta = \{x :\ x_1 + x_2 \geq 0,\ x_1 \leq -t\}$. Chapter Three gives a method for generating these wedge decompositions.

Representation of Probability Distributions

In this section we develop our representation of probability densities. We use the standard tool of a truncated multivariate Edgeworth series to represent our densities. Our purpose in this section is only to give a brief introduction to some of the important concepts in Edgeworth series. Further justification and proofs can be found in the sources

cited. First, the multivariate Edgeworth or Gram-Charlier series is developed in general. Second, we describe how we apply the concepts to the data of our problem.

Multivariate Edgeworth or Gram-Charlier Series

Probability densities can be represented using asymptotic expansions based on another probability density and its derivatives with linear combinations of the moments of the density used as coefficients in the series. One such series is called the Edgeworth expansion in general and the Gram-Charlier series for the Edgeworth series using the normal as the approximating density and ordered in a particular way. Our development here follows McCullagh (1987) closely.

Because the use of index notation greatly simplifies and clarifies the concepts developed here, we introduce and use it in this section. The basic use of index notation is to facilitate the manipulation of multidimensional arrays. Following McCullagh (1987) we call the elements of these arrays *components*. So given a p-dimensional array Z, it has p, $p-1$ dimensional components, $Z^1, Z^2, ..., Z^p$. Subscripts and superscripts are written to indicate not necessarily a component of the array, but rather the object for which an operation takes place. So, if X is a random vector with p components, we can write

$$\kappa^i = E(X^i)$$

for the expected value vector of X and

$$\kappa^{ij} = E(X^i X^j)$$

for the matrix of second moments of X. The summation convention is that when any index is repeated once as a subscript and once as a superscript the objects are summed over that index. For example,

$$a_i X^i = a^1 X^1 + a^2 X^2 + ... + a^p X^p.$$

Given this notation we can write the multidimensional array of the d^{th} moments of a random variable as

$$\kappa^{i_1 i_2 \ldots i_d} = E(X^{i_1} X^{i_2} \ldots X^{i_d}).$$

The moment generating function of a random variable (vector) is given by

$$M_X(\xi) = E\{\exp(\xi_i X^i)\}$$

and may be expanded in the infinite series

$$M_X(\xi) = 1 + \xi_i \kappa^i + \xi_i \xi_j \kappa^{ij}/2! + \xi_i \xi_j \xi_k \kappa^{ijk}/3! + \xi_i \xi_j \xi_k \xi_l \kappa^{ijkl}/4! + \ldots$$

The moments of X are the partial derivatives of $M_X(\xi)$ evaluated at $\xi = 0$. Issues of when $M_X(\xi)$ is divergent or is not unique for a density are very technical in nature and beyond the scope of our discussion here. Suffice it to say that, though Feller (1971) gives an example of non-identical univariate density functions with identical moments, our concern here is whether we can flexibly and robustly represent density functions via their moments. Cramer (1974), Kendall and Stuart (1977), McCullagh (1987), Judd (forthcoming), the literature on the subject of power production costing, options pricing literature (for example, Ritchen and Vijh (1993)) and our own experience find this to be true.

The cumulant generating function is defined as

$$K_X(\xi) = \log M_X(\xi),$$

and can be expanded as

$$K_X(\xi) = \xi_i \kappa^i + \xi_i \xi_j \kappa^{i,j}/2! + \xi_i \xi_j \xi_k \kappa^{i,j,k}/3! + \xi_i \xi_j \xi_k \xi_l \kappa^{i,j,k,l}/4! + \ldots$$

The *cumulants* of X are the partial derivatives of $K_X(\xi)$ evaluated at $\xi = 0$. Note that we write the cumulants with a set of indices that is fully partitioned (and so separated by commas) whereas the moments have unpartitioned indices. In the case of the first moments and first cumulants, the coincidence of notation is fine since the first moments and cumulants are identical. Given these two expansions of moments and cumulants, we

can define one set in terms of the other. The univariate case can be found in Cramer (1974) and Kendall and Stuart (1977). McCullagh (1987) does this for the multivariate case.

We now give a purely formal derivation of the Edgeworth expansion following McCullagh (1987). Technical details such as regularity conditions can be found in McCullagh (1987), Skovgaard (1986) , and Bhattacharya and Ghosh (1978). As before, our attitude is that while these details are interesting for understanding the mechanics of how the Edgeworth expansion works, in practice, we are concerned with the robustness and flexibility of the technique. A wide variety of literature in varying fields as cited earlier indicates the technique meets these criteria well.

Suppose $f_X(x)$ is the joint density of the random vector X with cumulants κ^i, $\kappa^{i,j}$ $\kappa^{i,j,k}$... and $f_0(x)$ is the approximating density with cumulants $\lambda^i, \lambda^{i,j}, \lambda^{i,j,k}$. Define

$$\eta^i = \kappa^i - \lambda^i, \ \eta^{i,j} = \kappa^{i,j} - \lambda^{i,j}, \eta^{i,j,k} = \kappa^{i,j,k} - \lambda^{i,j,k}, \ldots$$

If we subtract the cumulant generating functions for $f_X(x)$ and $f_0(x)$, we get

$$K_X(\xi) = K_0(\xi) + \xi_i\eta^i + \xi_i\xi_j\eta^{i,j}/2! + \xi_i\xi_j\xi_k\eta^{i,j,k}/3! + \ldots$$

Exponentiation yields

$$M_0(\xi)\{1 + \xi_i\eta^i + \xi_i\xi_j\eta^{ij}/2! + \xi_i\xi_j\xi_k\eta^{ijk}/3! + \ldots\},$$

where the formal 'moments', η^i, $\eta^{ij}, \eta^{ijk}, \ldots$, have the usual relationship with the formal 'cumulants', η^i, $\eta^{i,j}, \eta^{i,j,k}, \ldots$.

By inverting this last expression we obtain the Edgeworth expansion,

$$f_X(x) = f_0(x) + \eta^i f_i(x) + \eta^{ij} f_{ij}(x)/2! + \eta^{ijk} f_{ijk}/3! + \ldots,$$

where $f_i(x) = (-1)^1 \partial f_0(x)/\partial x^i$, $f_{ij}(x) = (-1)^2 \partial^2 f_0(x)/\partial x^i \partial x^j, \ldots$.

Application of Edgeworth Series to the Multi-Area Production Costing Model

In practice, Edgeworth series are truncated after a few terms to perform computation. So, in order to use this truncated Edgeworth series we need to be able to

compute the first few sets of moments or cumulants of the densities. In the context of production costing, the usual approach to the demand side of the problem is to develop tables of demand for different areas by the hour and day of week. As described in the introduction, these can be used as load duration curves. Or, as in our approach here, empirical moments can be calculated from them. Moments for the supply distribution are easily calculated based on the capacities and outage probabilities. Combining demand and supply is straightforward and formulas can be found in McCullagh (1987) who points out that under linear transformations of the random variables both moments and cumulants transform like contravariant tensors. In our case, the independence of supply simplifies the calculation, though it is not necessary for our calculations.

Multiarea Production Cost Curves

Now we give the main result of this chapter, the derivation of the multiarea production cost curve. By deriving a single cost curve we not only save work in computation we facilitate calculation of derivatives of production cost as well. The basic idea here is to break the probability expression into a convolution and then bring the limits of integration into the integral itself. Recall that the expected unserved demand after loading the i^{th} generator is

$$\overline{W}_i \equiv \sum_{\theta \subset \Omega} \int_{\omega_\theta} (e_\theta^T \; x - t_\theta) g_i(x) dx_1 ... dx_N$$

and the expected production cost is given by

$$C \equiv \sum_{i=1}^{I} F_i * (\overline{W}_{i-1} - \overline{W}_i) + F_{I+1} \overline{W}_{I+1}.$$

Consider the calculation of each part of $\overline{\overline{W}}_k$, $\int_{\omega_\theta} (e_\theta^T \; x - t_\theta) g_k(x) dx_1 ... dx_N$. Define the following functions (see, for example, Bracewell (1986) for the first three):

$$\delta(y) \quad = \begin{cases} 0, \, x \neq 0 \\ \int_{-\infty}^{\infty} \delta(z) dz \, = \, 1 \end{cases} \text{, the Dirac impulse symbol,}$$

$$R^0(y) = \begin{cases} 0, \, y < \, 0 \\ 1, \, y \geq 0 \end{cases} \text{,} \qquad \text{the unit step function,}$$

$$R^1(y) = \begin{cases} 0, \, y < \, 0 \\ y, \, y \geq 0 \end{cases} \text{,} \qquad \text{the unit ramp,}$$

$r_{ki}(x)$ is the density of supply at the i^{th} node after the first k generators have been loaded,

$h_\eta(x) = e_\eta^T \, x - t_\eta$, the excess demand function for the set of nodes η, and

$\hat{h}_{\theta,\eta}(x) = h_\theta(x) - h_\eta(x)$, the difference in excess demand for sets of nodes θ and η.

Further, let Ψ be an set of $N-1$ non-empty subsets of Ω such that $\theta \notin \Psi$ which we have indexed. For the following derivation the exact contents of $\Psi$are not important except that the set $\{e_\eta : \eta \in \Psi\} \cup \{\theta\}$ be linearly independent. This condition is equivalent in this context to the necessity of the wedge decomposition.

Given these definitions we can write

$\int_{\omega_\theta} (e_\theta^T \, x - t_\theta) g_k(x) dx_1 ... dx_N \, = \int_{\omega_\theta} h_\theta(x) g_k(x) dx_1 ... dx_N \, =$

Next we use indicator functions to rewrite the integral as an integral over the whole state space.

$$(\prod_{i=1}^{N} \int_{-\infty}^{\infty}) \; (\prod_{\eta \subset \Omega, \eta \neq \theta} R^0(\hat{h}_{\theta,\eta}(x)) \; R^1(h_\theta(x)) \; g_k(x) \prod_{i=1}^{N} dx_i \, =$$

We can split up the density function g into its component parts, that is, write it as the convolution between the demand and supply densities.

$$(\prod_{i=1}^{N}\int_{-\infty}^{\infty})\ (\prod_{\eta\subset\Omega,\eta\neq\theta} R^0(\hat{h}_{\theta,\eta}(x))\ R^1(h_\theta(x))*$$
$$(\prod_{i=1}^{N}\int_{-\infty}^{\infty})(\prod_{i=1}^{N} r_{ki}(x_i-y_i))g(y)\prod_{i=1}^{N}dy_i\prod_{i=1}^{N}dx_i\ =$$

We bring the linear replace $R^1(h_\theta(x))$ with $\int_{-\infty}^{0}R^0(z+h_\theta(x))dz$, so that we can treat it as we do the other indicator functions. Note that if we wanted to calculate higher moments of (SLP) we could apply this technique repeatedly to obtain a solution similar to that below with one more integral.

$$(\prod_{i=1}^{N}\int_{-\infty}^{\infty})(\prod_{i=1}^{N}\int_{-\infty}^{\infty})\int_{-\infty}^{0}\ (\prod_{\eta\subset\Omega,\eta\neq\theta} R^0(\hat{h}_{\theta,\eta}(x)))\ R^0(z+h_\theta(x))*$$
$$(\prod_{i=1}^{N} r_{ki}(x_i-y_i))g(y)\,dz\prod_{i=1}^{N}dx_i\prod_{i=1}^{N}dy_i\ =$$

Now, split the indicator functions into two sets, those involving the set Ψ or θ and those outside it. That is, we replace $(\prod_{\eta\subset\Omega,\eta\neq\theta} R^0(\hat{h}_{\theta,\eta}(x)))$
with $(\prod_{\eta\in\Psi^c\setminus\{\theta\}} R^0(\hat{h}_{\theta,\eta}(x)))\quad(\prod_{\eta\in\Psi} R^0(\hat{h}_{\theta,\eta}(x))),$

$$(\prod_{i=1}^{N}\int_{-\infty}^{\infty})(\prod_{i=1}^{N}\int_{-\infty}^{\infty})\int_{-\infty}^{0}\ (\prod_{\eta\in\Psi^c\setminus\{\theta\}} R^0(\hat{h}_{\theta,\eta}(x)))\quad(\prod_{\eta\in\Psi} R^0(\hat{h}_{\theta,\eta}(x)))*$$
$$R^0(z+h_\theta(x))\ (\prod_{i=1}^{N} r_{ki}(x_i-y_i))g(y)\,dz\prod_{i=1}^{N}dx_i\prod_{i=1}^{N}dy_i\ =$$

Now replace the $R^0(\hat{h}_{\theta,\eta}(x))$ in the first set with $\int_{-\infty}^{0}\delta(w_i+\hat{h}_{\theta,\eta}(x))dw_i$. We do this to bring in the terms inside the indicator functions into the convolution.

$$(\prod_{i=1}^{N}\int_{-\infty}^{\infty})(\prod_{i=1}^{N}\int_{-\infty}^{\infty})\int_{-\infty}^{0}(\prod_{i=1}^{N}\int_{-\infty}^{0})\ (\prod_{\eta\in\Psi^c\setminus\{\theta\}} R^0(\hat{h}_{\theta,\eta}(x)))\ *$$
$$(\prod_{\eta\in\Psi}\delta(w_i+\hat{h}_{\theta,\eta}(x)))\,\delta(w_N+z+h_\theta(x))*$$

$$(\prod_{i=1}^{N} r_{ki}(x_i - y_i))g(y)\prod_{i=1}^{N} dw_i \; dz \prod_{i=1}^{N} dx_i \prod_{i=1}^{N} dy_i =$$

Let $A^{-T} = ((e_\theta - e_\eta : \eta \in \Psi), e_\theta)^{-1}$ and $\mathrm{t} = (t_\theta - t_{\eta_1}, ..., t_\theta - t_{\eta_{N-1}}, t_\theta)$. We replace the univariate version of the Dirac impulse symbol with the multivariate version and then using the definition of A^{-T} we replace it with an equivalent version.

$$(\prod_{i=1}^{N} \int_{-\infty}^{\infty})(\prod_{i=1}^{N} \int_{-\infty}^{\infty})\int_{-\infty}^{0}(\prod_{i=1}^{N} \int_{-\infty}^{0}) \; (\prod_{\eta \in \Psi^c \setminus \{\theta\}} R^0(\hat{h}_{\theta,\eta}(x)))) \; *$$
$$(\prod_{i=1}^{N} \delta(x_i + [A^{-1}(\mathrm{t} + w + e_N z)]_i))*$$
$$(\prod_{i=1}^{N} r_{ki}(x_i - y_i))g(y)\prod_{i=1}^{N} dw_i \; dz \prod_{i=1}^{N} dx_i \prod_{i=1}^{N} dy_i =$$

Now, switch the order of integration in order to integrate over the x_i first.

$$(\prod_{i=1}^{N} \int_{-\infty}^{\infty})\int_{-\infty}^{0}(\prod_{i=1}^{N} \int_{-\infty}^{0}) \; (\prod_{i=1}^{N} \int_{-\infty}^{\infty}) \; (\prod_{\eta \in \Psi^c \setminus \{\theta\}} R^0(\hat{h}_{\theta,\eta}(x)))) \; *$$
$$(\prod_{i=1}^{N} \delta(x_i + [A^{-1}(\mathrm{t} + w + e_N z)]_i))*$$
$$(\prod_{i=1}^{N} r_{ki}(x_i - y_i))g(y)\prod_{i=1}^{N} dx_i \prod_{i=1}^{N} dw_i \; dz \prod_{i=1}^{N} dy_i =$$

We integrate with respect to x and use the sifting property of δ , that is, $\int_{-\infty}^{\infty} f(x + c)\delta(x)dx = f(c)$.

$$(\prod_{i=1}^{N} \int_{-\infty}^{\infty})\int_{-\infty}^{0}(\prod_{i=1}^{N} \int_{-\infty}^{0}) \; (\prod_{\eta \in \Psi^c \setminus \{\theta\}} R^0(\hat{h}_{\theta,\eta}(A^{-1}(\mathrm{t} + w + e_N z)]))) \; *$$
$$(\prod_{i=1}^{N} r_{ki}(-[A^{-1}(\mathrm{t} + w + e_N z)]_i - y_i))g(y) \prod_{i=1}^{N} dw_i \; dz \prod_{i=1}^{N} dy_i =$$

Change order of integration so we can use the whole expression in our result for the expected cost.

$$\int_{-\infty}^{0}(\prod_{i=1}^{N}\int_{-\infty}^{0})(\prod_{i=1}^{N}\int_{-\infty}^{\infty})\ (\prod_{\eta\in\Psi^c\setminus\{\theta\}} R^0(\hat{h}_{\theta,\eta}(A^{-1}(\mathrm{t}+w+e_N z)])))\ *$$
$$(\prod_{i=1}^{N} r_{ki}(-[A^{-1}(\mathrm{t}+w+e_N z)]_i - y_i))g(y)\prod_{i=1}^{N}dy_i\prod_{i=1}^{N}dw_i\ dz.$$

Now, define the N-area cost curve as

$$l_1(x) \equiv \sum_{i=1}^{I} f_i\{\prod_{k=1}^{N} r_{i-1,k}(x_k) - \prod_{k=1}^{N} r_{i,k}(x_k)\} + f_{I+1}\prod_{k=1}^{N} r_{I,k}(x_k),$$

where $f_i = F_i/F_{I+1}$.

So we can write the expected system cost in the form of a multiarea production cost curve:

$$C = F_{I+1}\sum_{\theta\subset\Omega}\int_{-\infty}^{0}(\prod_{i=1}^{N}\int_{-\infty}^{0})(\prod_{i=1}^{N}\int_{-\infty}^{\infty})$$
$$(\prod_{\eta\in\Psi^c\setminus\{\theta\}} R^0(\hat{h}_{\theta,\eta}(A^{-1}(\mathrm{t}+w+e_N z)])))\ *$$
$$l_1(-[A^{-1}(\mathrm{t}+w+e_N z)]_i - y_i)g(y)\prod_{i=1}^{N}dy_i\prod_{i=1}^{N}dw_i\ dz.$$

Notice that the innermost set of integrals, those with respect to y, represent a convolution between $l_1$and g. We can perform this convolution analytically through the moments of l_1 and g. Writing $h = l_1 * g$ where * is the convolution operator the cost calculation becomes

$$C = F_{I+1}\sum_{\theta\subset\Omega}\int_{-\infty}^{0}(\prod_{i=1}^{N}\int_{-\infty}^{0})\ (\prod_{\eta\in\Psi^c\setminus\{\theta\}} R^0(\hat{h}_{\theta,\eta}(A^{-1}(\mathrm{t}+w+e_N z)])))\ *$$
$$h(-[A^{-1}(\mathrm{t}+w+e_N z)]_i)\prod_{i=1}^{N}dw_i\ dz.$$

The terms $R^0(\hat{h}_{\theta,\eta}(A^{-1}(\mathrm{t}+w+e_N z)]))$ can be decomposed into wedges as shown in the next chapter, so we can write this integral as an iterated integral with clear upper and lower bounds. Or, we could decompose the regions of integration into wedges first and then make an almost identical derivation for the cost curve. Furthermore, as long as there are at least N linearly independent constraints in the constraint set, the argument above applies. Thus, the multi-area production cost curve derived above applies to arbitrary constraint sets with this one condition.

Chapter 3

Analysis of the Integration Polyhedra

This chapter pulls together a variety of different results on the regions over which integration must be performed for the computation of production cost. We start off by developing an algorithm for generating the non-redundant constraints from a Gale-Hoffman type region. With this algorithm we greatly enhance the utility of our approach and those of others such as Prekopa and Boros (1991) by avoiding having to deal (at least at the outset) with an exponential number of constraints. After this we present a section containing some results on Gale-Hoffman type regions. These results will be used in Chapter Six and show that Gale-Hoffman regions have many interesting properties. The last section presents an algorithm for decomposing any polyhedral set into wedges and discusses the application of our method to the D.C. power flow model.

Elimination of Redundancies from the Gale-Hoffman Constraints

The N dimensional region defined by the Gale-Hoffman constraints has up to 2^{N-1} inequalities. Except for small networks, this number becomes quickly too large to handle. However, it has been pointed out that for many practical networks, many of the constraints will be redundant. Prekopa and Boros (1991) amongst others provide various ways for eliminating redundant constraints from such systems. These methods include

specialized algorithms for elimination and elimination via linear programming. What is lacking is a complete characterization of redundancies and a unified and fast way of eliminating them. Prekopa and Boros (1991) use five different methods for eliminating redundancies, and in the end resort to linear programming which does not take advantage of the special structure of the constraints. Moreover, one of the particular problems is that each of the constraints corresponding to a subset of $\{1, ..., N\}$ must be examined to see if it is redundant, so that no matter how efficient an elimination technique is, the process is still, at best, exponential in the number of nodes. In contrast, this section provides a characterization of redundancies in a Gale-Hoffman system and an algorithm for creating the set of non-redundant constraints starting from the null set, instead of elimination from an exponentially large set. As a result, when there are only a polynomial in N number of non-redundant constraints (and there is a large class of networks for which this is true), we can in polynomial time find the constraints defining the Gale-Hoffman region. Though our characterization of redundancy in Gale-Hoffman regions bears similarity to results in matroid theory (see, for example, Nemhauser and Wolsey (1988)), it is important to note that the structure we are dealing with is not a polymatroid because of the lack of non-negativity constraints and non-monoticity of the flow function, t_θ. It is not clear to what extent matroid theory applies to the results presented below. Suffice it to say that we have not seen this redundancy characterization applied to Gale-Hoffman regions elsewhere.

We start by defining a number of concepts for graphs:

A *path* is a finite sequence of arcs with positive capacities $< a_{n_1 m_1}, a_{n_2 m_2}, ..., a_{n_p m_p} >$ such that $m_i = n_{i+1}$ for $i = 1, ..., p-1$.

A graph is *connected* if there is a path between every pair of nodes in the graph.

A *subgraph* induced by a set of nodes in a graph is itself a graph consisting of a set of nodes equal to the nodes that induce the subgraph and any arcs in the original graph whose nodes are in the set that induces the subgraph.

Two nodes in a graph are *s-connected* (short for subgraph connected) if the subgraph induced by them is connected. Two disjoint s-connected subsets of nodes in a graph are s-connected if the subgraph induced by their union is connected. If two nodes, a subset, or two disjoint s-connected subsets are not s-connected, we say they are *s-unconnected*. Note that two subsets θ and ψ are s-unconnected if and only if $c_{nm} = 0$ $\forall n \in \theta, m \in \psi$ and any subset of nodes of a complete graph is s-connected.

Let $\theta \subseteq \{1, ..., N\}$ and $\psi \subseteq \theta$. Then ψ is a *maximally s-connected* subset of θ if it is s-connected and there is no s-connected $\eta \subseteq \theta$ such that $\psi \subset \eta$ and $\psi \neq \eta$.

For $\theta \subset \{1, ..., N\}$ we say that the associated constraint in (FR) $e_\theta^T x \leq t_\theta$ is of *order* k where k is the cardinality of θ.

As a corollary to Theorem 2.1 establishing the submodularity of t_θ we get a useful corollary characterizing s-connectedness.

Corollary 3.1 S-connectedness, part 1. *Two disjoint, s-connected subsets θ and ψ are s-unconnected if and only if*

$$t_\theta + t_\psi = t_{\theta \cup \psi}.$$

Proof: From the proof of the submodularity of t_θ,

$$t_\theta + t_\psi = t_{\theta \cup \psi} + t_{\theta \cap \psi} + 2 \sum_{n \in \theta \cap \psi^c} \sum_{m \in \theta^c \cap \psi} c_{nm}.$$

Since $\theta \cap \psi = \emptyset$, $t_{\theta \cap \psi} = 0$, and the last term in the equation above can be rewritten so we have

$$t_\theta + t_\psi = t_{\theta\cup\psi} + 2 \sum_{n\in\theta} \sum_{m\in\psi} c_{nm}.$$

If θ and ψ are s-unconnected then the second term in the equation is zero and $t_\theta + t_\psi = t_{\theta\cup\psi}$.

On the other hand, if

$$t_\theta + t_\psi = t_{\theta\cup\psi}$$

then

$$\sum_{n\in\theta} \sum_{m\in\psi} c_{nm} = 0.$$

Since arc flow capacities are non-negative, then

$$c_{nm} = 0 \,\forall n \in \theta,\, m \in \psi,$$

that is the two subsets are s-unconnected. □

A second, more general corollary follows:

Corollary 3.2 S-connectedness, part 2. *Given* $\theta,\ \psi \subset \Omega$,

$$t_\theta + t_\psi = t_{\theta\cup\psi} + t_{\theta\cap\psi} \tag{1}$$

if and only if

$$c_{nm} = 0 \,\forall n \in \theta\backslash\psi,\, m \in \psi\backslash\theta.$$

Proof: If $\theta \cap \psi = \emptyset$, then it holds by the previous corollary. Now,

$$t_\theta = \sum_{n\in\theta\cap\psi} \sum_{m\in\theta^c\cap\psi} c_{nm} + \sum_{n\in\theta\cap\psi} \sum_{m\in\theta^c\cap\psi^c} c_{nm} + \sum_{n\in\theta\cap\psi^c} \sum_{m\in\theta^c\cap\psi} c_{nm} + \sum_{n\in\theta\cap\psi^c} \sum_{m\in\theta^c\cap\psi^c} c_{nm},$$

$$t_\psi = \sum_{n\in\psi\cap\theta} \sum_{m\in\psi^c\cap\theta} c_{nm} + \sum_{n\in\psi\cap\theta} \sum_{m\in\psi^c\cap\theta^c} c_{nm} + \sum_{n\in\psi\cap\theta^c} \sum_{m\in\psi^c\cap\theta} c_{nm} + \sum_{n\in\psi\cap\theta^c} \sum_{m\in\psi^c\cap\theta^c} c_{nm},$$

$$t_{\theta\cup\psi} = \sum_{n\in\theta\cap\psi} \sum_{m\in\theta^c\cap\psi^c} c_{nm} + \sum_{n\in\psi\cap\theta^c} \sum_{m\in\psi^c\cap\theta^c} c_{nm} + \sum_{n\in\theta\cap\psi^c} \sum_{m\in\theta^c\cap\psi^c} c_{nm}, \text{ and}$$

$$t_{\theta\cap\psi} = \sum_{n\in\theta\cap\psi} \sum_{m\in\theta^c\cap\psi} c_{nm} + \sum_{n\in\theta\cap\psi} \sum_{m\in\theta\cap\psi^c} c_{nm} + \sum_{n\in\theta\cap\psi} \sum_{m\in\theta^c\cap\psi^c} c_{nm}.$$

So, $t_\theta + t_\psi - (t_{\theta\cup\psi} + t_{\theta\cap\psi}) = 2 \sum_{n\in\psi\cap\theta^c} \sum_{m\in\psi^c\cap\theta} c_{nm}.$ □

Let $b_1, b_2, ..., b_p \in \mathbb{R}^N,\ s_1, s_2, ..., s_p \in \mathbb{R}$,

$$\Gamma \equiv \{y \in \mathbb{R}^N : b_i^T y \leq s_i,\ i = 1, ..., p\},$$

and

$$\Gamma_j \equiv \{y \in \mathbb{R}^N : b_i^T y \leq s_i,\ i = 1, ...j-1, j+1, ..., p\}.$$

In words, Γ_j is the system of constraints defined by Γ with constraint j removed. We say constraint j, $b_j^T y \leq s_j$, is *redundant* in system Γ if $\Gamma = \Gamma_j$.

It is a basic result in linear algebra that redundancy is equivalent to the existence of redundancy multipliers, scalars $\pi_i \geq 0$, $i = 1, ..., j-1, j+1, ...p$, $\pi_k > 0$ for some $k \in \{1, ..., j-1, j+1, ...p\}$ such that

$$\sum_{i=1,...,j-1,j+1,...p} \pi_i b_i = b_j$$

and

$$\sum_{i=1,...,j-1,j+1,...p} \pi_i s_i \leq s_j.$$

Based on this fact, we show that for a system in which the entries in the b_i are zero or one and the right hand side is the capacity flow function if redundancy multipliers exist, then there are redundancy multipliers that are either zero or one. First we need some intermediate results for s-unconnected subsets.

Lemma 3.3 Redundancy of s-unconnected sets. *If θ is s-unconnected then the associated constraint $e_\theta^T x \leq t_\theta$ is redundant in (FR). Moreover, the redundancy multipliers are either zero or one.*

Proof: θ is s-unconnected only if there are two disjoint, s-unconnected subsets θ_1 and θ_2 such that $\theta = \theta_1 \cup \theta_2$. Then by Corollary 3.1 $t_{\theta_1} + t_{\theta_2} = t_\theta$. So, multipliers of one for the constraints corresponding to θ_1 and θ_2 and zero otherwise give the desired result.□

Lemma 3.4 0-1 Redundancy Multipliers. *Given* Γ *as defined above, if the entries in the* b_i *are all zero or one and the* s_i *are capacity flow functions constraint* j *is redundant if and only if there exist redundancy multipliers that are zero or one.*

Proof:

$\Leftarrow$ Clearly, if there exist redundancy multipliers that are zero or one, then the constraint j is redundant.

$\Rightarrow$ If constraint j is redundant, then there exist redundancy multipliers as above. Let

$$\beta_k = \{i : (b_k)_i = 1,\ i = 1, ..., N\},\ k = 1, ..., p.$$

So if

$$\beta_j^c \cap \beta_n \neq \emptyset$$

then

$$\pi_n = 0$$

since

$$\sum_{i=1,...,j-1,j+1,...p} \pi_i b_i = b_j.$$

Let $\gamma = \{n : \pi_n > 0,\ n = 1, ...j-1, j+1, ..., N\}$. Therefore, $\beta_n \subset \beta_j$ for $n \in \gamma$. Also, for $r \in \beta_j$, $\sum_{n:r\in\beta_n} \pi_n = 1$. So, $s_j \geq \sum_{i\in\gamma} \pi_i s_i$ or

$$\sum_{n\in\beta_j} \sum_{m\in\beta_j^c} c_{nm} \geq \sum_{i\in\gamma} \pi_i \sum_{n\in\beta_i} \sum_{m\in\beta_i^c} c_{nm}$$
$$= \sum_{i\in\gamma} \pi_i \sum_{n\in\beta_i} \Big(\sum_{m\in\beta_j^c} c_{nm} + \sum_{m\in\beta_n^c\setminus\beta_j^c} c_{nm} \Big)$$

since $\beta_n \subset \beta_j$,

$$= \sum_{l\in\beta_j} \sum_{i\in\gamma: l\in\beta_j} \pi_i \Big(\sum_{m\in\beta_j^c} c_{nm} + \sum_{m\in\beta_n^c\setminus\beta_j^c} c_{nm} \Big)$$

since for $r \in \beta_j$, $\sum_{n:r\in\beta_n} \pi_n = 1$,

$$= \sum_{l\in\beta_j} \Big(\sum_{m\in\beta_j^c} c_{lm} + \sum_{i\in\gamma: l\in\beta_j} \pi_i \sum_{m\in\beta_n^c\setminus\beta_j^c} c_{nm} \Big).$$

So,

$$\sum_{l\in\beta_j} \sum_{i\in\gamma: l\in\beta_j} \pi_i \sum_{m\in\beta_n^c\setminus\beta_j^c} c_{nm} = 0$$

and thus $c_{nm} = 0$ for $l \in \beta_j$, $m \in \beta_n^c \setminus \beta_j^c \; \forall\, n \in \gamma$.

Thus the θ_n, $n \in \gamma$, form a disjoint s-unconnected partition of θ. Hence, by Lemma 3.3 there are zero-one redundancy multipliers for θ. □

We are now ready to state the main theoretical result of this section, the equivalence of s-connectedness and non-redundancy in (FR).

Theorem 3.5 S-connectedness = Non-redundancy. *A constraint $e_\theta^T x \leq t_\theta$ is redundant in (FR) if and only if θ is s-unconnected.*

Proof: From Lemmas 3.3 and 3.4. □

Based on this theorem we now present an algorithm to find all the s-connected subsets of a graph which then at the same time generates all the non-redundant constraints of (FR).

Algorithm for Finding Connected Subsets of a Graph.

Let $G(N, A)$ be a connected graph. Let C be a structure for holding all the s-connected subsets of G and H be the node-incidence matrix.

$C := \{\{1\}, \{2\}, ..., \{N\}\}$.

For $i = 1$ *to* $N - 1$

For $j = i + 1$ *to* N

If $H_{ij} = 1$

$C := C \cup \{i, j\}$

End If

End For

End For

For $i = 2,\ N - 1$

For every set θ of cardinality i in C

For every set ψ of cardinality 2 in C

If $\theta \cap \psi \neq \emptyset$ and $\theta \cap \psi \neq \psi$

$C := C \cup \{\psi \cup \theta\}$

End If

End For

End For

End For

$C := C \cup \Omega$

End Program.

In this section we have developed a characterization of the non-redundant constraints of the feasible region (FR) and an algorithm for finding them. Before leaving this topic it is worth noting that many network topologies of interest have only a polynomial number of non-redundant constraints. For instance, a ring shaped network (example of which are solved in Gubbala and Singh (1993)) with N nodes has $N(N-1)+1$ non-redundant constraints since it has N order j s-connected subgraphs for $j = 1, ..., N-1$ and 1 order N s-connected subgraph. A line has $(N+1)N/2$ non-redundant constraints since it has $N - (n-1)$ order n s-connected subgraphs for $n = 1, ..., N$.

We conclude this section with a theorem giving an upperbound for the number of s-connected subgraphs in a graph. This theorem may be useful in quickly determining the amount of work involved in generating the non-redundant constraints for (FR).

Theorem 3.6. Upperbound on S-unconnected Sets. *Given a connected graph G and two s-connected node disjoint subgraphs of G, θ_1 and θ_2 with only one arc between them and $\theta_1 \cup \theta_2 = \Omega$. Let $c(\cdot)$ be the number of s-connected subgraphs of its argument. Then*

$$c(G) \leq c(\theta_1) + c(\theta_2) + c(\theta_1)c(\theta_2).$$

Proof: The number of s-connected subgraphs in G will be given by the number in θ_1 plus the number in $\theta_2$plus those subgraphs involving nodes from both θ_1 and θ_2. There are then at most $c(\theta_1)c(\theta_2)$ such subgraphs involving nodes from both of the subsets. □

Properties of from the Gale-Hoffman Regions

In this section we develop some properties of the feasible region (FR). While the extreme points of (FR) have some characteristics similar to that of a polymatroid (see, for example, Bixby, Cunningham, and Topkis (1985)), the region we work with is not a polymatroid because there are no non-negativity constraints and the capacity flow function is not monotonic. The extreme points of (FR) do share with polymatroids the property that they are completely characterized by a partial order on the set $\{1, ..., N\}$. Further, it turns out that (FR) has a particularly 'nice' structure consisting of a base region that is a bounded polytope of $N - 1$ dimensions and extreme directions of a simple form. Finally, we conclude this section with a theorem that shows that the base of the Gale-Hoffman polyhedron has some super-symmetrical properties that become important in Chapter Six.

We start with some basic definitions and added notation:

A graph is said to be *complete* if there is an arc of positive capacity between every pair of nodes in the graph. Thus, a complete graph with N nodes has $N(N-1)/2$ arcs.

A set $\theta \subset \Omega$ (s-unconnected or not) is said to be *tight* at point $x \in$ (FR) if $e_\theta^T x = t_\theta$. This definition is from Bixby, Cunningham, and Topkis (1985).

Given the set $\Omega = \{1, ..., N\}$ we write $b_{<\Omega>}$ for some ordering of Ω. We write $(b_{<\Omega>})_n$ for the n^{th} in order element of $b_{<\Omega>}$. So, if the ordering applied to Ω is the usual ordering of the natural numbers then $(b_{<\Omega>})_n = n$. We further write the unordered set $\theta \sqsubseteq b_{<\Omega>}$ if $\theta = \{(b_{<\Omega>})_1, (b_{<\Omega>})_2, ..., (b_{<\Omega>})_{|\theta|}\}$ where $|\theta|$ is the number of elements of θ and write $\theta(b_{<\Omega>}, n)$ for the first n elements of $b_{<\Omega>}$ taken as an unordered set. $\theta(b_{<\Omega>}, 0)$ is understood to be the empty set, $\emptyset$. So, if we let $m = |\theta|$, then $\theta \sqsubseteq b_{<\Omega>}$ means $\theta = \theta(b_{<\Omega>}, m)$. Obviously, $\theta(b_{<\Omega>}, n-1) \subset \theta(b_{<\Omega>}, n)$.

Following Bixby, Cunningham, and Topkis (1985) (but with different notation) we define a vector value function $v(b_{<\Omega>})$ on orderings of Ω for some graph G to be

$$(v(b_{<\Omega>}))_{(b_{<\Omega>})_n} \equiv t_{\theta(b_{<\Omega>},n)} - t_{\theta(b_{<\Omega>},n-1)}.$$

The following is a version of a "standard" lemma stated in Bixby, Cunningham, and Topkis (1985).

Lemma 3.7. *For $x \in$ (FR) the family of all sets tight at x is closed under intersection and unions.*

Proof: Let θ and ψ be tight at x, then $t_\theta = e_\theta^T x$ and $t_\psi = e_\psi^T x$. So,

$$\begin{aligned} t_{\theta\cup\psi} + t_{\theta\cap\psi} &\geq e_{\theta\cup\psi}^T x + e_{\theta\cap\psi}^T x \\ &= e_\theta^T x + e_\psi^T x = t_\theta + t_\psi \\ &\geq t_{\theta\cup\psi} + t_{\theta\cap\psi}. \end{aligned}$$

The first inequality follows because of feasibility and the second because of submodularity. □

The next theorem gives the result that all the extreme points of (FR) are located on the plane $e^T x = 0$.

Theorem 3.8. Location of extreme points of (FR). *If x is an extreme point of (FR), then $e^T x = 0$. In other words, Ω is tight at x.*

Proof: Suppose x is an extreme point of (FR) but $e^T x \neq 0$. Since $x \in$ (FR), $e^T x \leq 0$. So, $e^T x < 0$. Then, because of Lemma 3.7, the union of all sets tight at x is a proper subset of Ω. As a result, there is an $n \in \Omega$ such that for θ such that $n \in \theta$, θ is not tight at x. This means we can perturb x by a small amount $\epsilon > 0$ in the $\pm e_n$ directions and still have a feasible point. Then $x + e_n \in$ (FR) and $x - e_n \in$ (FR) and $x = \frac{1}{2}(x + e_n) + \frac{1}{2}(x - e_n)$. Thus, x is not an extreme point of (FR). Therefore, x is not an extreme point of (FR) if $e^T x < 0$. □

We can now determine the extreme directions of the polytope (FR). Clearly, $-e_n$ is a feasible direction for (FR) for every $n \in \Omega$ since all the coefficients of the variables in the inequalities are non-negative. Suppose d is a feasible direction for (FR) with $d_n > 0$ for some n. Then we must have $e_n^T \lambda d \leq t_{\{n\}}$ for all $\lambda > 0$. But if arc capacities are finite and positive this cannot hold. Thus, any feasible direction must be non-positive. We state the following theorem that depends on Theorem 3.11 for its proof.

Theorem 3.9. The extreme directions of (FR). *The extreme directions of (FR) are given by taking, for each s-connected subset $\psi \subset \Omega$, the $N - 2$ s-connected subsets θ of the form $(\psi \subset \theta)$ or $(\theta \subset \psi)$ or $(\theta \backslash \psi)$ and $(\psi \backslash \theta)$ are not s-connected.*

With these two theorems the picture of (FR) begins to emerge. There is a base region on the hyperplane $e^T x = 0$ that is a bounded polytope and a cone-like extension of the polytope in the extreme directions.

Now we show that for any ordering of Ω, $b_{<\Omega>}$, $v(b_{<\Omega>})$ is an extreme point of (FR). Then we show that all the extreme points of (FR) are given by the values of $v(\cdot)$ on orderings of Ω. In the case of a complete graph, they are unique and so we have $N!$

extreme points for a graph of N nodes. For a non-complete graph we will show that there are always s-unconnected subsets of the graph so that the value of some of the orderings will be the same. This property is connected with the redundancy of a constraint in (FR). The first two theorems are based on, though slightly different from, those in Bixby, Cunningham, and Topkis (1985) that deal with a different context.

Theorem 3.10. Extreme Points from Orderings of Ω. *Given any ordering of* Ω, $b_{<\Omega>}$, $v(b_{<\Omega>})$ *is an extreme point of (FR).*

Proof: First, show that $v(b_{<\Omega>}) \in$ (FR). We need to show that $\forall\ \psi \subseteq \Omega$, $\sum_{n\in\psi}(v(b_{<\Omega>}))_n \leq t_\psi$.

If $\psi \sqsubseteq b_{<\Omega>}$ then

$$\sum_{n\in\psi}(v(b_{<\Omega>}))_n = \sum_{n=1}^{|\psi|}(t_{\theta(b_{<\Omega>},n)} - t_{\theta(b_{<\Omega>},n-1)}) = t_\psi.$$

So, the constraint holds.

If $\psi \not\sqsubseteq b_{<\Omega>}$, then we show that the corresponding constraint must hold by induction on the cardinality of ψ.

Suppose $|\psi| = 1$. Let θ be the smallest cardinality set such that $\psi \subset \theta$ and $\theta \sqsubseteq b_{<\Omega>}$. Then

$$\sum_{n\in\psi}(v(b_{<\Omega>}))_n = t_{\theta\cup\psi} - t_\theta.$$

By submodularity $t_{\theta\cup\psi} - t_\theta \leq t_\psi$, so the constraint holds.

Suppose for all sets with $|\psi| = k$, the corresponding constraint holds. Let $|\psi| = k+1$. Label the elements of ψ in the order in which they occur in the ordering $b_{<\Omega>}$ by $i_1, i_2, ..., i_{k+1}$. Let θ_j be the smallest cardinality such that $i_j \in \theta_j$ and $\theta_j \sqsubseteq b_{<\Omega>}$ for $j = 1, ..., k+1$. Then

$$\sum_{n\in\psi}(v(b_{<\Omega>}))_n = \sum_{n=1}^{k+1}(t_{\theta_n\cup\{i_n\}} - t_{\theta_n})$$

$$\leq t_{\psi\setminus\{i_{k+1}\}} + t_{\theta_{k+1}\cup\{i_{k+1}\}} - t_{\theta_{k+1}} \leq t_\psi,$$

where the first inequality follows from the induction assumption and the second from submodularity of t.

Thus, $v(b_{<\Omega>}) \in$ (FR). Now show that $v(b_{<\Omega>})$ is extremal. Suppose not. Then there are w and z both in (FR) such that $v(b_{<\Omega>}) = \frac{1}{2}(w + z)$. For every $\theta \sqsubseteq b_{<\Omega>}$, $e_\theta^T v(b_{<\Omega>}) = t_\theta$. Thus for such θ,

$$e_\theta^T v(b_{<\Omega>}) = e_\theta^T(\tfrac{1}{2}(w + z)) = \tfrac{1}{2}(e_\theta^T w + e_\theta^T z) = t_\theta.$$

Since w and z are both feasible, $e_\theta^T w \leq t_\theta$ and $e_\theta^T z \leq t_\theta$. Therefore for every $\theta \sqsubseteq b_{<\Omega>}$,

$$e_\theta^T w = e_\theta^T z = t_\theta. \tag{2}$$

But then $w = z = v(b_{<\Omega>})$ since (2) completely determines the values of w and z, just as it does $v(b_{<\Omega>})$.

Therefore $v(b_{<\Omega>})$ must be an extreme point of (FR). □

Now we wish to show that all the extreme points of (FR) are given by the $v(b_{<\Omega>})$. In order to do his we need to first recall a few results for polytopes. First, given an $m \times n$ non-redundant inequality system with $m > n$, an extreme point of the polyhedron so defined will satisfy exactly m of the inequalities exactly. Second, given the set of constraints that an extreme point satisfies exactly, the set of constraints satisfied exactly by an adjacent extreme point will differ by at most one constraint (see, for example, Murty (1971)). Third, a result in Balinski (1961) shows that if all the adjacent extreme points of a class of extreme points for a polyhedron are themselves in that class then that class of extreme points includes all the extreme points of the polyhedron. As a result, in order to show that all the extreme points of (FR) are generated by orderings of Ω via $v(b_{<\Omega>})$, we just need to show that the extreme points adjacent to the $v(b_{<\Omega>})$ are themselves generated by another ordering of Ω.

Before we prove this we need a number of an intermediate results. For the following we employ this definition: let us say that one (not necessarily non-redundant) constraint θ *touches* another constraint ψ in (FR) when

$$\{x : e_\theta^T x = t_\theta, e_\psi^T x = t_\psi, x \in (FR)\} \neq \emptyset.$$

Theorem 3.11. Characterization of touching constraints. *Given a constraint θ of (FR) another constraint ψ of (FR) touches θ in (FR) if and only if $\psi \subset \theta$ or $\theta \subset \psi$ or $\theta \backslash \psi$ and $\psi \backslash \theta$ are not s-connected.*

Proof:

Case 1: $\theta \cap \psi = \emptyset$ and not s-connected. Then $x = v(< \theta, \psi, (\theta \cup \psi)^c >) \in$ (FR) with $e_\theta^T x = t_\theta$ and $e_\psi^T x = t_\psi$, so θ and ψ touch.

Case 2: $\theta \cap \psi = \emptyset$ and s-connected. Then

$$t_{\theta \cup \psi} < t_\theta + t_\psi \tag{3}$$

since θ and ψ are s-connected and

$$e_{\theta \cup \psi}^T x \leq e_\theta^T x + e_\psi^T x \tag{4}$$

since $\theta \cap \psi = \emptyset$. If $e_\theta^T x = t_\theta$ and $e_\psi^T x = t_\psi$, then by Lemma 3.7 $e_{\theta \cup \psi}^T x = t_{\theta \cup \psi}$. But then (3) and (4) imply a contradiction, so θ and ψ cannot touch.

Case 3: $\psi \subset \theta$. $x = v(< \psi, \theta \backslash \psi, \theta^c >) \in$ (FR) with $e_\theta^T x = t_\theta$ and $e_\psi^T x = t_\psi$, so θ and ψ touch. By symmetry, we also have this for $\theta \subset \psi$.

Case 4: $\theta \not\subseteq \psi$, $\psi \not\subseteq \theta$, and $\theta \cap \psi \neq \emptyset$. If $\theta \backslash \psi$ and $\psi \backslash \theta$ then $x = v(< \theta \backslash \psi, \psi \backslash \theta, \theta \cap \psi, (\theta \cup \psi)^c >) \in$ (FR) with $e_\theta^T x = t_\theta$ and $e_\psi^T x = t_\psi$, so θ and ψ touch. On the other hand, by Lemma 3.7 $e_\theta^T x = t_\theta$ and $e_\psi^T x = t_\psi$ imply $t_{\theta \cup \psi} + t_{\theta \cap \psi} = t_\theta + t_\psi$. So by Corollary 3.2, $\theta \backslash \psi$ and $\psi \backslash \theta$ are s-unconnected. □

Lemma 3.12. *Let $x = v(b_{<\Omega>})$ be an extreme point of (FR) generated by ordering $b_{<\Omega>}$. If there is only one tight order one constraint, there are no tight redundant constraints.*

Proof: First, we note that for every ordering $b_{<\Omega>}$, $v(b_{<\Omega>})$ is tight on at least one constraint of each order each order $1, ..., N$ (take $\theta(b_{<\Omega>}, n)$, $n = 1, ..., N$). Though some of these constraints may be redundant in (FR), taken as a system:

$$\begin{aligned} e^T_{\theta(b_{<\Omega>},1)} x &= t_{\theta(b_{<\Omega>},1)} \\ e^T_{\theta(b_{<\Omega>},2)} x &= t_{\theta(b_{<\Omega>},2)} \\ &\vdots \\ e^T_{\theta(b_{<\Omega>},N)} x &= t_{\theta(b_{<\Omega>},N)} \end{aligned} \tag{5}$$

the equalities completely determine the point $v(b_{<\Omega>})$. Now we show that each $\theta(b_{<\Omega>}, n)$ must be non-redundant for $n = 1, ..., N$. Of course any order one constraint is not redundant. Assuming that $\theta(b_{<\Omega>}, k)$ is not redundant we need to show that $\theta(b_{<\Omega>}, k+1)$ is not redundant. Suppose to the contrary that $\theta(b_{<\Omega>}, k+1)$ is redundant. Then $\{(b_{<\Omega>})_{k+1}\}$ must be tight since $\theta(b_{<\Omega>}, k)$ and $\theta(b_{<\Omega>}, k+1)$ are s-unconnected. So there are two tight constraints that are order one, a contradiction to the lemma hypothesis. Thus, each $\theta(b_{<\Omega>}, n)$ must be non-redundant for $n = 1, ..., N$. If ψ were a redundant constraint for which x is tight, then x would be tight on the maximally s-connected subsets of ψ. But not more than one these subsets can be one of the $\theta(b_{<\Omega>}, n)$, since $\theta(b_{<\Omega>}, n-1) \subset \theta(b_{<\Omega>}, n)$, $n = 2, ..., N$. Thus we have more than N tight non-redundant constraints, a contradiction. Therefore, no redundant constraint is tight at x. □

Lemma 3.13. *Every extreme point of (FR) adjacent to an extreme point generated by an ordering has a tight constraint of order one.*

Proof: As in the previous lemma we note that for every ordering $b_{<\Omega>}$, $v(b_{<\Omega>})$ is tight on at least one constraint of each order each order $1, ..., N$ (take $\theta(b_{<\Omega>}, n)$,

$n = 1, ..., N$). In particular, (5) continues to hold. We note also that if we have any consistent system of N equalities with one of each order from (FR) that system defines an extreme point. Clearly, any extreme point generated by an ordering of $\{1, ..., N\}$ has a tight constraint of order one (just take $\theta(b_{<\Omega>}, 1)$). For some $b_{<\Omega>}$ let ω_i, $i = 1, ..., N$ be the set of non-redundant constraints of (FR) tight at $v(b_{<\Omega>})$. Further, let y be an extreme point adjacent to $v(b_{<\Omega>})$. Then by the principle cited in Murty (1971) all but one of the ω_i must hold tightly for y. Call that constraint ω_j. Moreover, there must be another constraint ψ such that y is tight at ψ and

$\psi \notin \{ \omega_1, \omega_2, ..., \omega_N \}$. In other words, y is completely determined by the system

$$\begin{aligned} e_{\omega_i}^T y &= t_{\omega_i},\ i = 1, ..., j-1, j+1, ..., N \\ e_{\psi}^T y &= t_{\psi} \end{aligned} \tag{6}$$

Suppose y is not tight for any order one constraint. Then clearly, $|\omega_j| = 1$, $|\omega_i| > 1$ for $i \neq j$ and $|\psi| > 1$. By Lemma 3.12, $\theta(b_{<\Omega>}, n)$, $n = 2, ..., N$ are non-redundant and so must be tight for y as well (since ω_j is the differing non-redundant constraint). We must have $\omega_j \subset \psi$ since otherwise we could take the first $\theta(b_{<\Omega>}, n)$ with a non-empty intersection with ψ, say $\theta(b_{<\Omega>}, k)$, to get an order one constraint $\psi \cap \theta(b_{<\Omega>}, k)$ that is tight at y by Lemma 3.7. Assume by induction that $\theta(b_{<\Omega>}, k) \subset \psi$. We show that $\theta(b_{<\Omega>}, k) \subset \psi$ implies $\theta(b_{<\Omega>}, k+1) \subset \psi$ which is a contradiction, and so the lemma is proven.

Let $\theta(b_{<\Omega>}, k) \subset \psi$. If $|\psi| = k+1$, then $\psi \backslash \theta(b_{<\Omega>}, k+1) = (b_{<\Omega>})_{k+2}$, since otherwise $\psi \cup \theta(b_{<\Omega>}, k+1)$ would be non-redundant (since the union of two non-disjoint, s-connected subsets is s-connected) and tight for y by Lemma 3.7, so that as a result, y would be tight at $N+1$ non-redundant constraints. But then $(b_{<\Omega>})_k$ and $(b_{<\Omega>})_{k+1}$ are s-unconnected since $\theta(b_{<\Omega>}, k) = \theta(b_{<\Omega>}, k-1) \cup \{(b_{<\Omega>})_k\}$ and $\psi = \theta(b_{<\Omega>}, k-1) \cup \{(b_{<\Omega>})_{k+2}\}$ are both tight implying

$$t_{\theta(b_{<\Omega>}, k+2)} + t_{\theta(b_{<\Omega>}, k-1)} = t_{\theta(b_{<\Omega>}, k)} + t_{\psi}.$$

Then we have

$$\begin{aligned} e^T_{\theta(b_{<\Omega>},k-1)} y &= t_{\theta(b_{<\Omega>},k-1)} \\ e^T_{\theta(b_{<\Omega>},k)} y &= t_{\theta(b_{<\Omega>},k)} \\ e^T_{\psi} y &= t_{\psi} \\ e^T_{\theta(b_{<\Omega>},k+1)} y &= t_{\theta(b_{<\Omega>},k+1)} \cdot \end{aligned}$$

Now the first and the second equations imply $y_{(b_{<\Omega>})_k} = t_{\theta(b_{<\Omega>},k)} - t_{\theta(b_{<\Omega>},k-1)}$; the first and third equations imply $y_{(b_{<\Omega>})_{k+1}} = t_{\psi} - t_{\theta(b_{<\Omega>},k-1)}$. But then the first equation with these two new ones imply

$$e^T_{\theta(b_{<\Omega>},k+2)} y = t_{\theta(b_{<\Omega>},k)} + t_{\psi} - t_{\theta(b_{<\Omega>},k-1)}.$$

This means that constraint $e^T_{\theta(b_{<\Omega>},k+2)} y \leq t_{\theta(b_{<\Omega>},k+2)}$ is redundant, a contradiction.

Therefore, ψ cannot be of order $k+1$. Furthermore, we must have $\theta(b_{<\Omega>}, k+1) \subset \psi$, since otherwise there would be a smallest cardinality ω_i, $i \geq k+1$ such that $\theta(b_{<\Omega>}, k) \subset \omega_i \cap \psi$ giving another non-redundant constraint of order k, a contradiction. □

Theorem 3.14. All extreme points from orderings. *Let z be an extreme point of (FR). Then there is an ordering $\hat{b}_{<\Omega>}$ such that $v(\hat{b}_{<\Omega>}) = z$.*

Proof: The argument is that for any extreme point adjacent to one generated by an ordering we can find constraints ν_n, $n = 1, ..., N$ where n gives the order of the constraint and $\nu_{n-1} \subset \nu_n$, $n = 1, ..., N$ and $\nu_0 \equiv \emptyset$. We take the ordering to be $(\hat{b}_{<\Omega>})_n = \nu_n \backslash \nu_{n-1}$. The proof is by induction on the order of the constraint. In Lemma 3.13 we provided the case for order one. Assume that it is true that for an extreme point z adjacent to the extreme point $v(b_{<\Omega>})$ we have tight constraints ν_n , $n = 1, ..., k$ such that $\nu_{n-1} \subset \nu_n$. As before let ω_i , $i = 1, ..., N$ be the set of non-redundant constraints of (FR)

tight at $v(b_{<\Omega>})$, ω_j be the non-redundant constraint that is tight at $v(b_{<\Omega>})$, but not at z, and ψ be the non-redundant constraint tight at z, but not at $v(b_{<\Omega>})$.

Suppose to the contrary that there is no tight constraint of order $k+1$. Then ν_n, $n = 1, ..., k$ must be tight for $v(b_{<\Omega>})$ as well since if $\theta(b_{<\Omega>}, k+1)$ is non-redundant, $\omega_j = \theta(b_{<\Omega>}, k+1)$ and we can take $\nu_n = \theta(b_{<\Omega>}, n)$, $n = 1, ..., k$. If $\theta(b_{<\Omega>}, k+1)$ is redundant then $\omega_j \subset \theta(b_{<\Omega>}, k+1)$, but $\omega_j \cap \theta(b_{<\Omega>}, n)$, $n = 1, ..., k$ is tight for $v(b_{<\Omega>})$, so we can again take $\nu_n = \theta(b_{<\Omega>}, n)$, $n = 1, ..., k$. Furthermore, the $\theta(b_{<\Omega>}, n)$, $n = k+2, ..., N$, are non-redundant since otherwise for such $\theta(b_{<\Omega>}, n)$ we would be able to construct a $k+1$ order constraint that is tight as follows.

Let $M(l)$ be the set of maximally s-connected subsets of $\theta(b_{<\Omega>}, l)$. Now $\omega_j \in M(k+1)$ since otherwise we would be able to construct $\theta(b_{<\Omega>}, k+1)$. Also, $\omega \notin M(n)$ for $n \leq k$. So $M(k+1) \backslash M(k) = \{\omega_j\}$. Suppose $\theta(b_{<\Omega>}, k+2)$ is redundant. If $\omega_j \in M(k+1)$, then we can construct the order $k+1$ constraint $(\bigcup_{\theta \in M(k+2)\backslash\{\omega_j\}} \theta) \cup (\omega_j \cap \theta(b_{<\Omega>}, k))$ that is tight at z. But then we can construct an order $k+1$ tight constraint by taking the union of all the elements of $M(k+2)$ except for some element of order one. Or, if there is no element of order one we choose some element of $M(k+2)$ other than that which contains ω_j and take the intersection of it and the $\theta(b_{<\Omega>}, n)$, $n \leq k$ which gives us a set of one less cardinality. Then we take the union of this intersection with all the other elements in $M(k+2)$. In general, the process is similar for any $n > k+2$, so that the $\theta(b_{<\Omega>}, n)$, $n = k+2, ..., N$ are non-redundant.

Thus, we have for z tight constraints $\theta(b_{<\Omega>}, n)$, $n = 1, ..., k, k+2, ..., N$. As a result $|\psi| \leq k$ since otherwise the intersection of ψ and some $\theta(b_{<\Omega>}, n)$, $n > k+1$ would have order $k+1$. Moreover, by the same reasoning for $n = 1, ..., N$

$$|\theta(b_{<\Omega>}, n) \cup \psi| \leq k.$$

Therefore $\psi \subset \theta(b_{<\Omega>}, n)$. As a result for some $n \leq k$, $\psi \in M(n)$. But then this is a contradiction since ψ is not tight for $v(b_{<\Omega>})$. □

So, orderings determine the extreme points. In the case of complete graphs the extreme points determined by each ordering will be unique and so the number of extreme points will be $N!$ It is clear from the foregoing (as with polymatroids) that adjacent extreme points differ in their corresponding orderings only by the interchange of two succeeding elements in the ordering. In other words, if $b_{<\Omega>} = < o_1, o_2, ..., o_N >$, the ordering for the adjacent extreme point will be the same except that for some $1 \leq k \leq N-1$, o_k and o_{k-1} will be switched in the ordering. In the case of incomplete graphs switching the order of nodes that are successive in the ordering but s-unconnected does not change the resulting extreme point. The character of this degeneracy becomes clear in the next theorem.

Theorem 3.15. Distance between adjacent extreme points. *Given two adjacent extreme points y and z, let nodes i and j be the nodes whose place in the order is interchanged from the ordering from one extreme point to the next. Then, the Euclidean distance between the extreme points is $2^{\frac{3}{2}} c_{ij}$, where c_{ij} is the capacity of the arc between nodes i and j.*

Proof: Let $b_{<\Omega>}$ be the ordering for y and $\hat{b}_{<\Omega>}$ be the ordering for z. Without loss of generality, let $(b_{<\Omega>})_k = i$ and $(b_{<\Omega>})_{k+1} = j$. So, $\theta(b_{<\Omega>}, n) = \theta(\hat{b}_{<\Omega>}, n)$ for $1 \leq n \leq N$, $n \neq k$. Thus, $(y)_n = (z)_n$ for $1 \leq n \leq N$, $n \neq i, j$. Therefore,

$$\|y - z\|_2 =$$

$$[((y)_i - (z)_i)^2 + ((y)_j - (z)_j)^2]^{\frac{1}{2}} =$$

$$[((t_{\theta(b_{<\Omega>},k-1)\cup\{i\}} - t_{\theta(b_{<\Omega>},k-1)}) - (t_{\theta(b_{<\Omega>},k-1)\cup\{i,j\}} - t_{\theta(b_{<\Omega>},k-1)\cup j}))^2 +$$

$$((t_{\theta(b_{<\Omega>},k-1)\cup\{i,j\}} - t_{\theta(b_{<\Omega>},k-1)\cup\{i\}}) - (t_{\theta(b_{<\Omega>},k-1)\cup\{j\}} - t_{\theta(b_{<\Omega>},k-1)}))^2]^{\frac{1}{2}}$$

$$= 2^{\frac{1}{2}} \left| t_{\theta(b_{<\Omega>},k-1)\cup\{i,j\}} - t_{\theta(b_{<\Omega>},k-1)\cup\{i\}} - t_{\theta(b_{<\Omega>},k-1)\cup\{j\}} + t_{\theta(b_{<\Omega>},k-1)} \right|$$

Now,

$$t_{\theta(b_{<\Omega>},k-1)\cup\{i,j\}} - t_{\theta(b_{<\Omega>},k-1)\cup\{i\}} - t_{\theta(b_{<\Omega>},k-1)\cup\{j\}} + t_{\theta(b_{<\Omega>},k-1)} =$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{i,j\}} \sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{i,j\})^c} c_{nm} -$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{i\}} \sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{i\})^c} c_{nm} -$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{j\}} \sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{j\})^c} c_{nm} +$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)} \sum_{m\in(\theta(b_{<\Omega>},k-1))^c} c_{nm} =$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{i\}} \sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{i,j\})^c} c_{nm} -$$

$$\sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{i,j\})^c} c_{jm} -$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{i\}} \sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{i,j\})^c} c_{nm} -$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{i\}} c_{nj} -$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{j\}} \sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{j\})^c} c_{nm} -$$

$$\sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{j\})^c} c_{jm} +$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{j\}} \sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{j\})^c} c_{nm} +$$

$$\sum_{n\in\theta(b_{<\Omega>},k-1)} c_{nj} =$$

$$\sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{i,j\})^c} c_{jm} - \sum_{n\in\theta(b_{<\Omega>},k-1)\cup\{i\}} c_{nj} -$$

$$\sum_{m\in(\theta(b_{<\Omega>},k-1)\cup\{j\})^c} c_{jm} + \sum_{n\in\theta(b_{<\Omega>},k-1)} c_{nj} =$$

$$-2c_{ij}.$$

So,

$$\|y - z\|_2 = 2^{\frac{3}{2}} c_{ij}. \qquad \square$$

From the preceding theorem we can see that (FR) for an incomplete graph is a special case of (FR) for a complete graph with some of the lengths of edges between extreme points shrunk to zero.

We have seen that for complete graphs no constraints are redundant, while for incomplete graphs, redundant constraints correspond to s-unconnected subgraphs which

are easily found. When we restrict (FR) to $e^T x = 0$ some non-redundant constraints become redundant. As a result, the shape of (FR) can change as we leave the hyperplane $e^T x = 0$. The following theorem gives a nice characterization of which non-redundant constraints are redundant on (FR) restricted to $e^T x = 0$.

Theorem 3.16. *A non-redundant set θ is redundant on (FR) restricted to $e^T x = 0$ if and only if θ^c is redundant on all of (FR).*

Lemma 3.17 shows the "if" part.

Lemma 3.17. *θ is redundant of (FR) restricted to $e^T x = 0$ if θ^c is redundant on all of (FR).*

Proof: Suppose θ^c is redundant . Then the set of maximally s-unconnected subsets of θ^c has more than one element, say $r \geq 2$. Let ψ_1 be the union of $r-1$ maximally s-unconnected subsets of $\theta^c$and $\psi_2$be equal the the remaining maximally s-unconnected subset of θ^c. For y in the base of (FR) then we have

$$e_{\psi_1}^T y \leq t_{\psi_1}$$
$$e_{\psi_2}^T y \leq t_{\psi_2}$$
$$-e^T y \leq 0$$

Adding the three inequalities yields

$$e_\theta^T y \leq t_{\psi_1} + t_{\psi_2} = t_\theta.$$

Therefore θ is redundant on (FR) restricted to $e^T x = 0$. □

In order to prove the "only if" part of the theorem we need to prove some lemmas for the following system for which the existence of a solution characterizes redundancy for non-redundant θ on the base of (FR).

$$\sum_{\psi \in \mathcal{P}\{1,\ldots,N\}\setminus\{\theta\}} \pi_\psi e_\psi - \pi_e e = e_\theta \tag{7}$$

$$\sum_{\psi \in \mathcal{P}\{1,\ldots,N\}\setminus\{\theta\}} \pi_\psi t_\psi \leq t_\theta \tag{8}$$

$$\pi_\psi \geq 0,\ \forall \psi \in \mathcal{P}\{1,\ldots,N\}\setminus\{\theta\},\ \pi_e > 0 \tag{9}$$

where $\mathcal{P}\{1,...,N\}$ is the power set of $\{1,...,N\}$ not including Ω. We can rewrite (7) as

$$\sum_{\psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta\}:\, n\in\psi} \pi_\psi = 1+\pi_e,\ \forall n\in\theta \tag{10}$$

and

$$\sum_{\psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta\}:\, n\in\psi} \pi_\psi = \pi_e,\ \forall n\notin\theta. \tag{11}$$

These also imply that for all $n\in\Omega$ there is a $\psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta\}$ with $n\in\psi$ and $\pi_\psi > 0$. To see why π_e must be strictly positive and not just non-negative, consider that if it were zero then θ would be redundant.

Lemma 3.18. *Suppose there is a set of multipliers π_ψ, π_e that satisfy* (7) *to* (9). *If for some $\zeta\in\mathcal{P}\{1,...,N\}\setminus\{\theta\}$, $\pi_\zeta > 0$* and *$\pi_{\zeta^c} > 0$, then there is a set of multipliers $\hat{\pi}_\psi, \hat{\pi}_e$ also satisfying* (7) *to* (9) *with at most one of $\hat{\pi}_\zeta$ and $\hat{\pi}_{\zeta^c}$ strictly positive and $\hat{\pi}_\psi = \pi_\psi$ for $\psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta,\zeta,\zeta^c\}$.*

Proof: Without loss of generality let $\pi_\zeta \geq \pi_{\zeta^c}$. Set

$$\begin{aligned}
\hat{\pi}_{\zeta^c} &= 0\\
\hat{\pi}_\zeta &= \pi_\zeta - \pi_{\zeta^c}\\
\hat{\pi}_e &= \pi_e - \pi_{\zeta^c}\\
\hat{\pi}_\psi &= \pi_\psi\,, \psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta,\zeta,\zeta^c\}.
\end{aligned}$$

So, $\hat{\pi}_\psi, \hat{\pi}_e, \hat{\pi}_\zeta\,, \hat{\pi}_{\zeta^c}$ satisfy (9).

$$\sum_{\psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta\}} \hat{\pi}_\psi t_\psi = \hat{\pi}_\zeta t_\zeta + \sum_{\psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta,\zeta,\zeta^c\}} \hat{\pi}_\psi t_\psi$$

$$= \pi_\zeta t_\zeta - \pi_{\zeta^c} t_{\zeta^c} + \sum_{\psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta,\zeta,\zeta^c\}} \pi_\psi t_\psi$$

$$< \sum_{\psi\in\mathcal{P}\{1,...,N\}\setminus\{\theta\}} \pi_\psi t_\psi \leq t_\theta.$$

So (8) is satisfied.

$$\hat{\pi}_\zeta e_\zeta - \hat{\pi}_e e = (\pi_\zeta - \pi_{\zeta^c})e_\zeta - (\pi_e - \pi_{\zeta^c})e$$

$$= \pi_\zeta e_\zeta - \pi_e e + \pi_{\zeta^c}(e - e_\zeta)$$

$$= \pi_\zeta e_\zeta + \pi_{\zeta^c} e_{\zeta^c} - \pi_e e.$$

So (7) is satisfied. □

The next lemma follows from the application of Lemma 3.18 at most 2^{N-2} times.

Lemma 3.19. *There are redundancy multipliers satisfying* (7) *to* (9) *for a set* θ *if and only if there are multipliers satisfying* (7) *to* (9) *with at most one of* π_ψ *and* π_{ψ^c} *strictly positive for every* $\psi \in \mathcal{P}\{1, ..., N\}\backslash\{\theta\}$, *that is,* $\pi_\psi \cdot \pi_{\psi^c} = 0$.

The next two lemmas show that if we have multipliers then we can construct multipliers such that if two sets are disjoint then at most one of the corresponding multipliers is strictly positive.

Lemma 3.20. *Suppose there is a set of multipliers* π_ψ, π_e *satisfying* (7) *to* (9). *If for some* $\eta, \zeta \in \mathcal{P}\{1, ..., N\}\backslash\{\theta\}$, $\zeta \cap \eta = \emptyset$, $\pi_\eta > 0$ *and* $\pi_\zeta > 0$, *then there is a set of multipliers* $\hat{\pi}_\psi, \hat{\pi}_e$ *also satisfying* (7) *to* (9) *with* $\hat{\pi}_\eta \cdot \hat{\pi}_\zeta = 0$. *Moreover, the* $\hat{\pi}_\psi$ *are such that* $\hat{\pi}_\psi \cdot \hat{\pi}_{\psi^c} = 0$ *for every* $\psi \in \mathcal{P}\{1, ..., N\}\backslash\{\theta\}$.

Proof: We can assume by the previous lemma that $\hat{\pi}_\psi \cdot \hat{\pi}_{\psi^c} = 0$ for every $\psi \in \mathcal{P}\{1, ..., N\}\backslash\{\theta\}$. Without loss of generality, let $\pi_\zeta \geq \pi_\eta$. Set

$$\begin{aligned}
\hat{\pi}_{\zeta\cup\eta} &= \pi_{\zeta\cup\eta} + \pi_\eta \\
\hat{\pi}_\zeta &= \pi_\zeta - \pi_\eta \\
\hat{\pi}_\eta &= 0 \\
\hat{\pi}_e &= e \\
\hat{\pi}_\psi &= \pi_\psi, \psi \in \mathcal{P}\{1, ..., N\}\backslash\{\theta, \zeta, \eta, \zeta \cup \eta\}.
\end{aligned}$$

So (9) is satisfied and because

$$\sum_{\psi\in\mathcal{P}\{1,...,N\}\backslash\{\theta\}} \hat{\pi}_\psi e_\psi = \sum_{\psi\in\mathcal{P}\{1,...,N\}\backslash\{\theta\}} \pi_\psi e_\psi$$

(7) is satisfied. Because t is submodular $t_{\zeta\cup\eta} \leq t_\zeta + t_\eta$, so

$$\sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta\}} \hat{\pi}_\psi t_\psi = \pi_\eta t_{\zeta\cup\eta} - \pi_\eta t_\zeta + \sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta,\eta\}} \pi_\psi t_\psi$$
$$\leq \sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta\}} \pi_\psi t_\psi.$$

So (8) is satisfied. □

Again, the next lemma follows from the application of Lemma 3.20 at most 2^{N-2} times.

Lemma 3.21. *There are set of multipliers* π_ψ, π_e *satisfying* (7) *to* (9) *if and only if there are multipliers* $\hat{\pi}_\psi, \hat{\pi}_e$ *also satisfying* (7) *to* (9) *with* $\hat{\pi}_\eta \cdot \hat{\pi}_\zeta = 0$ *for all* $\eta, \zeta \in \mathcal{P}\{1, \ldots, N\}\setminus\{\theta\},\ \zeta \cap \eta = \emptyset$.

The next lemma establishes the construction of multipliers with the preceding properties plus shows that all the non-zero multipliers must have the same value.

Lemma 3.22. *Suppose there is a set of multipliers* π_ψ, π_e *satisfying* (7) *to* (9) *with* $\alpha \cap \beta \neq \theta$, $\pi_\alpha > 0$ *and* $\pi_\beta > 0$, *then there is a set of multipliers* $\hat{\pi}_\psi, \hat{\pi}_e$ *also satisfying* (7) *to* (9) *with either* $\hat{\pi}_\alpha \cdot \hat{\pi}_\beta = 0$. *Moreover, for* $\pi_\alpha > 0$, $\pi_\alpha = \pi_e$.

Proof: By the preceding lemmas we can assume the π_ψ are such that $\hat{\pi}_\eta \cdot \hat{\pi}_\zeta = 0$ *for all* $\eta, \zeta \in \mathcal{P}\{1, \ldots, N\}\setminus\{\theta\},\ \zeta \cap \eta = \emptyset$. Without loss of generality let $\pi_\alpha \leq \pi_\beta$.

Case 1: $\alpha \cup \beta \neq \Omega$

Set

$$\begin{aligned}
\hat{\pi}_{\alpha\cup\beta} &= \pi_{\alpha\cup\beta} + \pi_\alpha \\
\hat{\pi}_{\alpha\cap\beta} &= \pi_{\alpha\cap\beta} + \pi_\alpha \\
\hat{\pi}_\beta &= \pi_\beta - \pi_\alpha \\
\hat{\pi}_\alpha &= 0 \\
\hat{\pi}_e &= \pi_e \\
\hat{\pi}_\psi &= \pi_\psi, \psi \in \mathcal{P}\{1, \ldots, N\}\setminus\{\theta, \alpha, \beta, \alpha\cap\beta, \alpha\cup\beta\}.
\end{aligned}$$

So (9) is satisfied. Since

$$\hat{\pi}_{\alpha\cup\beta} e_{\alpha\cup\beta} + \hat{\pi}_{\alpha\cap\beta} e_{\alpha\cap\beta} + \hat{\pi}_\alpha e_\alpha + \hat{\pi}_\beta e_\beta$$

$$= \pi_{\alpha\cup\beta}e_{\alpha\cup\beta} + \pi_\alpha(e_{\alpha\cup\beta} + e_{\alpha\cap\beta}) + \pi_{\alpha\cap\beta}e_{\alpha\cap\beta} + (\pi_\beta - \pi_\alpha)e_\beta$$

$$= \pi_{\alpha\cup\beta}e_{\alpha\cup\beta} + \pi_{\alpha\cap\beta}e_{\alpha\cap\beta} + \pi_\alpha e_\alpha + \pi_\beta e_\beta,$$

(7) is satisfied. (8) holds since by submodularity since

$$\sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta\}} \hat{\pi}_\psi t_\psi =$$

$$\sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta,\alpha\}} \pi_\psi t_\psi + \pi_\alpha t_{\alpha\cup\beta} + \pi_\alpha t_{\alpha\cap\beta} - \pi_\alpha t_\beta \leq$$

$$\sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta,\alpha\}} \pi_\psi t_\psi + \pi_\alpha t_\alpha = \sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta\}} \pi_\psi t_\psi.$$

Case 2: $\alpha \cup \beta = \Omega$, $\alpha \cap \beta \neq \theta$

Same as case 1 except we do not use $\alpha \cup \beta$ and set $\widehat{\pi}_e = \pi_e - \pi_\alpha$.

Set

$$\begin{aligned}
\widehat{\pi}_{\alpha\cap\beta} &= \pi_{\alpha\cap\beta} + \pi_\alpha \\
\widehat{\pi}_\beta &= \pi_\beta - \pi_\alpha \\
\widehat{\pi}_\alpha &= 0 \\
\widehat{\pi}_e &= \pi_e - \pi_\alpha \\
\hat{\pi}_\psi &= \pi_\psi, \psi \in \mathcal{P}\{1,\ldots,N\}\setminus\{\theta,\alpha,\beta,\alpha\cap\beta,\alpha\cup\beta\}.
\end{aligned}$$

(9) is clearly satisfied. Since

$$\widehat{\pi}_{\alpha\cap\beta}e_{\alpha\cap\beta} + \widehat{\pi}_\alpha e_\alpha + \widehat{\pi}_\beta e_\beta - \widehat{\pi}_e e$$

$$= \pi_{\alpha\cap\beta}e_{\alpha\cap\beta} + \pi_\alpha e_{\alpha\cap\beta} + (\pi_\beta - \pi_\alpha)e_\beta - (\pi_e - \pi_\alpha)e$$

$$= \pi_{\alpha\cap\beta}e_{\alpha\cap\beta} + \pi_\beta e_\beta - \pi_e e + \pi_\alpha e_{\alpha\cap\beta} - \pi_\alpha e_\beta + \pi_\alpha e$$

$$= \pi_{\alpha\cap\beta}e_{\alpha\cap\beta} + \pi_\beta e_\beta - \pi_e e + \pi_\alpha e_{\alpha\cap\beta} + \pi_\alpha e_{\beta^c}$$

$$= \pi_{\alpha\cap\beta}e_{\alpha\cap\beta} + \pi_\beta e_\beta - \pi_e e + \pi_\alpha e_\alpha$$

(7) is satisfied. (8) holds since by submodularity since

$$\sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta\}} \hat{\pi}_\psi t_\psi =$$

$$\sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta,\alpha\}} \pi_\psi t_\psi + \pi_\alpha t_{\alpha\cap\beta} - \pi_\alpha t_\beta \leq$$

$$\sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta,\alpha\}} \pi_\psi t_\psi + \pi_\alpha t_\alpha = \sum_{\psi\in\mathcal{P}\{1,\ldots,N\}\setminus\{\theta\}} \pi_\psi t_\psi.$$

(10) implies that $\pi_\alpha = \pi_e$ for $\pi_\alpha > 0$ since otherwise there would be another subset β with $\pi_\beta > 0$ and $n \in \alpha \cap \beta, n \notin \theta$. □

The next lemma establishes the "only if" part of Theorem 3.16.

Lemma 3.23. *If a non-redundant set θ is redundant on (FR) restricted to $e^T x = 0$, then θ^c is redundant on all of (FR).*

Proof: Since all the positive multipliers are the same and correspond to sets that contain θ and are otherwise disjoint, (7) and (8) imply that there are exactly two such positive multipliers, say ψ_1 and ψ_2. Moreover this means that $\pi_{\psi_1} = \pi_{\psi_2} = 1$. Therefore, from (8) we get $t_{\psi_1} + t_{\psi_2} \leq t_\theta$ and by submodularity $t_\theta + t_{\psi_1 \cup \psi_2} = t_{\psi_1 \cap \psi_2} + t_{\psi_1 \cup \psi_2} \leq t_{\psi_1} + t_{\psi_2}$, so $t_{\psi_1} + t_{\psi_2} = t_\theta$. Therefore by Corollary 3.2, $\psi_1 \backslash \theta$ and $\psi_2 \backslash \theta$ are s-unconnected and so θ^c is redundant by Theorem 3.5.

A consequence of the results so far is the next interesting corollary about the base of the feasible region.

Corollary 3.24. *If y is an extreme point of (FR), then so is $-y$. There are an even number of faces of the base of (FR). Given a face of the base of (FR) then there exists another face of (FR) parallel to the given face with the same size and shape as the given face.*

Wedge Decomposition for Polyhedra

In order to facilitate the integration of our density function over a polyhedron we wish to express it as a sum of iterated integrals. A convenient method for doing so is to break down the polyhedron into shapes we call wedges. A wedge in n dimensions is the intersection of n or fewer half-spaces such that there is a boundary point common to all the half-spaces. Let $Ax \geq b$ be the polyhedron of interest. Recall that if a constraint is

redundant with respect to a system then the reverse of that constraint and the system has no solution. We use a version of Farkas' Lemma from Cottle, Pang, and Stone (1992):

The system $Ax \geq b$ *has a solution if and only if* $y^T A = 0,\ y \geq 0,\ y^T b > 0$ *has no solution.*

Let $\mathfrak{C}(P)$ be the content of polyhedron P and $\alpha = \{2, ..., N\}$. Then we write

$$\mathfrak{C}(Ax \geq b) = \mathfrak{C}(A_{\alpha.}x \geq b_\alpha) - \mathfrak{C}(A_{\alpha.}x \geq b_\alpha,\ A_{1.}x \leq b_1). \tag{12}$$

Our scheme is to break down $Ax \geq b$ into wedges by reversing and dropping constraints as in (12) until we end up with either a wedge or an inconsistent system. The following theorem gives the main step in the procedure:

Theorem 3.25. *Given constraints* $a_i^T x \geq b_i,\ i = 1, ..., p$ *that form a wedge, and* $a_{p+1}^T x \geq b_{p+1}$ *such that the* $a_i,\ i = 1, ..., p+1$ *are linearly dependent, there is a sequence of reversals of constraints such that we can express the content of* $a_i^T x \geq b_i,$ $i = 1, ..., p+1$ *as the sum of wedges.*

Proof: Since $a_i^T x \geq b_i,\ i = 1, ..., p$ forms a wedge the $a_i,\ i = 1, ..., p$ are linearly independent so we can assume without loss of generality that $b_i = 0, i = 1, ..., p$. If $b_{p+1} = 0$, then the version of Farkas Lemma we use has $y \neq 0$ instead of $y^T b > 0$, but the procedure is otherwise entirely the same. Assuming that $a_{p+1}^T x \geq b_{p+1}$ is not redundant in $a_i^T x \geq b_i,\ i = 1, ..., p$. Since a_{p+1} is dependent on $a_i,\ i = 1, ..., p$ there are multipliers $y_i \neq 0$ such that $\sum_{i=1}^{p} y_i a_i = y_{p+1} a_{p+1}$. If $y_{p+1} a_{p+1} < 0$, reverse constraint $a_{p+1}^T x \geq b_{p+1}$ to get the systems:

$\{ - a_{p+1}^T x \geq \ - b_{p+1};\ a_i^T x \geq b_i : i = 1, ..., p\}$ and $\{a_i^T x \geq b_i : i = 1, ..., p\}$ which sum to get the original system. The second system is a wedge by hypothesis. The first now has $- y_{p+1} a_{p+1} > 0$. To get a system as specified in Farkas' Lemma, we now simply reverse and drop every constraint with a multiplier $y_i > 0,\ i = 1, ..., p$. □

The basic procedure for eliminating constraints involves choosing linearly independent rows on the left hand side, solving for the other rows, and then reversing and dropping constraints until we have a wedge or an infeasible system. Theorem 3.25 shows that this is possible. In some cases it is possible to be more efficient by reversing constraints that have few other constraints that touch it. Theorem 3.11 characterizes touching constraints for Gale-Hoffman systems. For instance, if we have a complete graph of N nodes, by reversing an order one constraint we make any constraint that does not contain that node redundant.

We conclude this chapter with a discussion of the application of our procedure to D.C. power flow networks. Our procedure involves the following major steps: characterization of level of unserved energy in terms of piecewise linear functions, that are linear on polyhedral sets, development of the multi-area production cost curve, the wedge decomposition, and the integration of our density representation. Of these steps, only the first, the characterization of level of unserved energy in terms of piecewise linear functions, depends on using the transportation model of network flows. Pereira, Pinto, Oliveira, and Cunha (1987) claim that the characterization we show in Chapter Two for Gale-Hoffman systems that the unserved energy is equal to the most violated constraint in the feasibility system holds as well for D.C. power flow networks. Whether it does or not, it should be noted that we can always characterize the level of unserved energy in terms of a piecewise linear function on polyhedra, since we only need to consider the range over which a particular basis is optimal. While this characterization is not as nice as the other in that it leads to potentially many more regions of integration, it is still viable.

Chapter 4

Integration

The algorithm for multi-area production costing involves construction of an Edgeworth series representation of the probability density. Chapter Two gave a brief development of the series which we show in (1) below.

$$f_X(x) = f_0(x) + \eta^i f_i(x) + \eta^{ij} f_{ij}(x)/2! + \eta^{ijk} f_{ijk}/3! + ..., \quad (1)$$

where $f_i(x) = (-1)^1 \partial f_0(x)/\partial x^i$, $f_{ij}(x) = (-1)^2 \partial^2 f_0(x)/\partial x^i \partial x^j, ...$

Recall that $f_X(x)$ is the function we wish to approximate and that $f_0(x)$ is the approximating function. While one is free to choose $f_0(x)$ to be any reasonable (in the sense that the development of the series still holds) density, in order to make use of the power of the series, one should choose a function that as much as possible is close to the function one wishes to estimate and that eases computation of (in our case) the integral of (1). For the model presented here we use the multivariate normal for four reasons. First, at least for one area distributions the empirical density generated from data is fairly closely fit by the normal density. Partly this is due to the fact that we are adding up a number of independent random variables (the generators' available capacities) to get our density, so that the Law of Large Numbers starts to hold. Second, the multivariate normal function has been intensively studied and in particular, a number of methods for integration of the multivariate normal exist. Third, because we choose the approximating multivariate

normal to have the same mean and covariance as the approximated density the second and third terms in (1) are zero. So, since we are truncating after five terms our approximate expression is

$$f_X(x) \simeq f_0(x) + \eta^{ijk} f_{ijk}/3! + \eta^{ijkl} f_{ijkl}/4! \qquad (2)$$

Fourth, previous work in electric power production costing in the probabilistic simulation framework has, in general, used the normal density as the approximating function, and we wished to extend this model directly to the multi-area context.

In order to integrate (2) , we need some method for integration of the multivariate normal and multivariate Edgeworth series using the multivariate normal as the approximating function. The first part of this chapter discusses integration of the multivariate normal density and presents our new result for one dimensional numerical integration of the trivariate normal density. As far as we can tell, this is a new result that lowers the "dimensionality" of the problem of integration of the multivariate normal by two. Previous results were only able to lower the dimensionality of the problem by one. In the second part of this chapter we document integration of the multivariate Edgeworth series based on integration of the multivariate normal density. Included in this section is the development of expressions for repeated integration of the multivariate normal density.

One Dimensional Numerical Integration of the Trivariate Normal Density

Calculation of the integral of multivariate normal density over a rectangle (in our case, unbounded) is a problem with a long history that has applications in a wide variety of areas and has received extensive study. Applications include those in statistics (Iyengar (1988) cites some), econometrics (Dutt (1976)), and finance (Selby and Hodges (1987) and Ritchken, Sanakarasubramanian, and Vijh (1993)). A number of approaches to it have been taken in attacking this problem including Monte-Carlo simulation, infinite series

approximations, and Gaussian quadrature. Tong (1990) gives an overview of the problem and the various methods used to analyze it. These methods have had various degrees of success, yet have still, essentially, been confronted with at best an $n-1$ dimensional numerical integration problem. Closed forms have not been produced except for a couple of special cases. Integration of the bivariate normal was investigated by Sheppard (1898) and he produced a closed form result in the special case of the distribution having an identically zero mean. David (1953) produced a closed form result for the trivariate case with the zero means. Abrahamson (1964) devised a method for one dimensional numerical integration of the quadrivariate case, again, with zero means. The method presented here is in the spirit of the analysis presented in these three papers. We analyze the surface area of the intersection of the orthant with the ellipses formed by the level sets of the density. With this approach and a new construction we were able to devise a method for one dimensional numerical integration of the trivariate normal with arbitrary means. This new result has implications beyond the trivariate normal since it means that the dimensionality of the multivariate problem can now be reduced by two dimensions rather than just one (see Plackett (1954) for reduction of the order of the integral) by conditioning on three of the variables.

Our approach to one dimensional integration of the trivariate normal has its starting point in the analysis presented in Appendix A of an idea for one dimensional numerical integration of the multivariate normal. We will first, quickly summarize the details we use here.

It should first be clear that the problems of calculating the probability content of a wedge or orthant for a multivariate normal random vector is equivalent to calculating the value of the integral of multivariate normal density over the positive orthant. That is, if we let $\phi_{-b,\Sigma}(x)$ be the multivariate normal density with mean vector $-b$ and covariance matrix Σ, we wish to calculate

$$\int_0^\infty \cdots \int_0^\infty \phi_{-b,\Sigma}(x) dx_1 \cdots dx_N \tag{3}$$

We focus here only on the case where Σ is positive definite. Let $\Sigma = AA^T$, where A is some convenient factorization such as Cholesky or the symmetric square root (see, for example, Golub and Van Loan (1989)). If we transform the variables from x to $A^{-1}x + A^{-1}b$, then (1) becomes

$$\int_{Ax \geq b} \phi_{0,I}(x) dx_1 \cdots dx_N, \tag{4}$$

where $\phi_{0,I}$ is the independent multivariate normal with mean 0 (see Tong (1990)). Define the N-dimensional sphere centered at the origin, $S_{N,r} \equiv \{x \in \mathbb{R}^N : x^T x = r^2\}$. Now the level sets of $\phi_{0,I}$ are $S_{N,r}$. Our approach to calculating (4) is to sum the area given by the intersection of $Ax \geq b$ and $S_{N,r}$ divided by the surface area of $S_{N,r}$ weighted by $\phi_{0,I}$ over r from zero to infinity. To express this more formally write $P(r)$ for the surface area of the intersection of $Ax \geq b$ and $S_{N,r}$ divided by the surface area of $S_{N,r}$. Then we can rewrite (4) as

$$2/(2\pi)^{\frac{1}{2}} \int_0^\infty P(r) e^{-r^2/2} dr. \tag{5}$$

If we have a way to readily calculate $P(r)$, then we can apply our favorite one dimensional numerical integration algorithm to (5) and find the answer to the problem of interest.

To calculate $P(r)$ the basic problem of course is to calculate the surface area of the intersection of $Ax \geq b$ and $S_{N,r}$. In Appendix A this is attempted by constructing an iterated integral by integrating over the right hand side of $Ax \geq b$.

Without loss of generality assume that Σ is the correlation matrix and that A is the Cholesky factorization of Σ, so A is lower triangular. This considerably simplifies the discussion that follows and as a result we have $A_{i.}A_{j.} = \rho_{ij}$ where ρ_{ij} is the correlation between the i^{th} and j^{th} random variables. The vector $-b$ now no longer is the mean, but rather the variance adjusted mean since we have divided each row by the variance of the corresponding random variable. A further simplification helps the discussion as well. Rather than considering the intersection of $Ax \geq b$ and $S_{N,r}$ we perform a change of

variables $x = ry$, and consider the intersection of $Ax \geq b$ and $S_{N,1}$where we have adjusted b by dividing by r. For simplicity, we write S_N for $S_{N,1}$. Also, define

$$\mathcal{A}_n(r) \equiv \frac{2r^{n-1}\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)},$$

the $n-1$ dimensional "surface area" of $S_{n,r}$ (see, for example, Kendall (1961)).

If we take the straightforward approach to calculating $P(r)$ then we, not surprisingly, run into elliptic integrals. In the trivariate case, however, we can decompose $\{x : Ax \geq b,\ x^Tx = 1\}$, the region of intersection of $Ax \geq b$ and $S_{3,1}$, into simpler regions whose areas we can find analytically. For the first part of this analysis assume that the vertex of $Ax \geq b$ is inside the sphere. For a set $\theta \subseteq \{1, ..., N\}$, define the notation $A_{\theta.}$ to be the matrix made up of the rows of A that are given by the set θ. $A_{.\theta}$ is the matrix made up of the columns given by the subset θ. Let ζ_1 be equal to the vector x such that $A_{\{2,3\}.}x = b_{\{2,3\}},\ A_{1.}x \geq b_1,\ x^Tx = 1,$

ζ_2 be equal to the vector x such that $A_{\{1,3\}.}x = b_{\{1,3\}},\ A_{2.}x \geq b_2,\ x^Tx = 1,$ and

ζ_3 be equal to the vector x such that $A_{\{1,2\}.}x = b_{\{1,2\}},\ A_{3.}x \geq b_3,\ x^Tx = 1.$

In words, ζ_i is found by taking the intersection of the other two halfspaces and the sphere. The ζ_i are linearly independent and define the plane $h^Tx = \bar{b}$. It is clear that $\{x : Ax \geq b,\ x^Tx = 1\}$ lies entirely on one side of $h^Tx = \bar{b}$, so without loss of generality define h and $\bar{b}$ such that

$$\{x : Ax \geq b,\ x^Tx = 1\} \subset \{x : h^Tx \geq \bar{b}\}$$

and $h^Th = 1$. So we can find the area of $\{x : Ax \geq b,\ x^Tx = 1\}$ by finding the area of $\{x : h^Tx \geq \bar{b},\ x^Tx = 1\}$ and subtracting the areas of

$$\{x : A_{1.}x \geq b_1,\ h^Tx \leq \bar{b}, x^Tx = 1\},$$

$$\{x : A_{2.}x \geq b_2,\ h^Tx \leq \bar{b},\ x^Tx = 1\},\ \text{and}$$

$$\{x : A_{3.}x \geq b_3,\ h^Tx \leq \bar{b},\ x^Tx = 1\}.$$

So, to implement our approach we need to find the area of the sphere cut off by a single plane once, and find the area of the sphere cut off by two planes three times. In

order to find the area of $\{x : h^T x \geq \overline{b},\ x^T x = r^2\}$ we refer to the formula developed in Appendix A to find that the area is given by

$$2\pi \int_{\overline{b}}^{1} (r^2 - y^2)^{\frac{3-3}{2}} dy = 2\pi(r - \overline{b}). \tag{6}$$

In order to find the area of $\{x : A_{i.}x \geq b,\ h^T x \leq \overline{b}, x^T x = 1\}$, first note that $\rho = A_{i.}h$, the angle between $A_{i.}x = b$ and $h^T x = \overline{b}$, is acute so that following the result in Appendix A the integral does not break into two pieces. Let $\overline{\rho} = (1 - \rho^2)^{\frac{1}{2}}$. Then the formula for the area is given by

$$\begin{aligned} &2\pi^{\frac{1}{2}}/\Gamma(\tfrac{1}{2}) \int_{b_i}^{\overline{\rho}} \int_{\overline{b}}^{\rho y_1 + [\overline{\rho}^2(1-y_1^2)]^{\frac{1}{2}}} (1 - \tfrac{1}{\overline{\rho}^2}\left[y_1^2 - 2\rho y_1 y_2 + y_2^2\right])^{-\frac{1}{2}} dy_2 dy_1 \\ &= 2 \int_{b_i}^{\overline{\rho}} \int_{\overline{b}}^{\rho y_1 + [\overline{\rho}^2(1-y_1^2)]^{\frac{1}{2}}} (1 - \tfrac{1}{\overline{\rho}^2}\left[y_1^2 - 2\rho y_1 y_2 + y_2^2\right])^{-\frac{1}{2}} dy_2 dy_1 \end{aligned}$$

Perform a change of variables

$$y = \begin{bmatrix} 1 & 0 \\ \rho & \overline{\rho} \end{bmatrix} z$$

to get

$$= 2\,\overline{\rho} \int_{b_i}^{\overline{\rho}} \int_{\frac{-\rho}{\overline{\rho}}\overline{b} + \frac{1}{\overline{\rho}} z_1}^{(1-z_1^2)^{\frac{1}{2}}} (1 - z_1^2 - z_2^2)^{-\frac{1}{2}} dz_2 dz_1$$

$$= 2\,\overline{\rho} \int_{b_i}^{\overline{\rho}} \left[\arcsin\left(\frac{z_2}{(1-z_1^2)^{\frac{1}{2}}}\right)\right] \Bigg|_{\frac{-\rho}{\overline{\rho}}\overline{b} + \frac{1}{\overline{\rho}} z_1}^{(1-z_1^2)^{\frac{1}{2}}} dz_1$$

$$= 2\,\overline{\rho} \int_{b_i}^{\overline{\rho}} \left[\frac{\pi}{2} - \arcsin\left(\frac{\frac{-\rho}{\overline{\rho}}\overline{b} + \frac{1}{\overline{\rho}} z_1}{(1-z_1^2)^{\frac{1}{2}}}\right)\right] dz_1,$$

after some tedious integration by parts a lengthy closed form is possible. We achieved this last part of the result via Mathematica.

The case where the vertex of the wedge is not in the sphere, but the intersection is non-empty can be broken down into a number of cases:

1. The facets of dimension one of the wedge all intersect the sphere. In this case, there are two separate areas of intersection of the wedge with the surface of

the sphere. We can apply the same construction as when the vertex is inside the sphere for each of these areas.

2. Two of the three facets of dimension one of the wedge intersect the sphere. Without loss of generality, suppose that the facet given by $A_{2.}x = b_2$ and $A_{3.}x = b_3$ does not intersect the sphere. Let c_1 be the surface area of $\{x : A_{\{1,2\}.}x \geq b_{\{1,2\}}\} \cap S_3$ and c_2 be the surface area of $\{x : A_{1.}x \geq b_1, A_{3.}x \leq b_3\} \cap S_3$. Then the surface area of $\{x : Ax \geq b\} \cap S_3$ is $c_1 - c_2$. These surface areas can be calculated analytically as shown above.
3. One of the three facets of dimension one of the wedge intersect the sphere. In this case the surface area is given by the intersection of the two halfspaces associated with the facet and the sphere.
4. None of the facets of dimension one of the wedge intersect the sphere. In this case the surface area equals the surface area of the sphere minus the sum of the surface area of the intersection of the sphere and the complement of each halfspace.

Integration of the Multivariate Edgeworth Series

In this section we document the method used to integrate the truncated multivariate Edgeworth series. The essence of this exercise is to transform integrals and repeated integrals of the multivariate normal times a polynomial into sums of terms involving just integrals of the multivariate normal for which we have algorithms to compute. We begin this section with a number of transformation laws for these integrals and end it by applying the results to the multi-area production cost curve derived in Chapter Two.

Let $\phi_{\mu,\Sigma}(x)$ be the multivariate normal density with mean μ and variance matrix Σ. We assume for the following that Σ is positive definite.

Then (following Tong (1990))

$$\phi_{\mu,\Sigma}(x) = \frac{1}{(2\pi)^{N/2}|\Sigma|^{\frac{1}{2}}} e^{-Q_N(x;\mu,\Sigma)/2}$$

where

$$Q_N(x;\mu,\Sigma) = (x-\mu)^{\mathrm{T}}\Sigma^{-1}(x-\mu).$$

So the multivariate normal distribution is

$$\Phi_{\mu,\Sigma}(x) = \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_N}\phi_{\mu,\Sigma}(y)dy_N\cdots dy_1.$$

Let $P_n(x)$ be a polynomial in x of order n. First, we develop a representation of $\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_N}P_n(y)\phi_{\mu,\Sigma}(y)dy_N\cdots dy_1$ in terms of $\Phi_{\mu,\Sigma}(x)$ and lower order integrals. We can simplify the discussion that follows by assuming, without loss of generality that, $\mu = 0$.

We start off by considering terms of order one, that is, involving a constant times a single variable, that is,

$$d_i \equiv \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_N} y_i\phi_{0,\Sigma}(y)dy_N\cdots dy_1 \tag{7}$$

Now the derivatives of $\phi_{0,\Sigma}(x)$ divided by $\phi_{0,\Sigma}(x)$ are the hermite polynomials, the first of which is $\Sigma^{-1}x$. So, we can rewrite (7) as

$$\begin{aligned}
&[\Sigma^{-1}]_{ii}^{-1}\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_N}[\Sigma^{-1}]_{i\cdot}y\phi_{0,\Sigma}(y)dy_N\cdots dy_1 - \\
&[\Sigma^{-1}]_{ii}^{-1}\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_N}\sum_{j\neq i}[\Sigma^{-1}]_{ij}y_j\phi_{0,\Sigma}(y)dy_N\cdots dy_1 \\
&= [\Sigma^{-1}]_{ii}^{-1}(\prod_{j\neq i}\int_{-\infty}^{x_j})\phi_{0,\Sigma}(y)(\prod_{j\neq i}dy_j)\Big|_{y_i=x_i} - \sum_{j\neq i}([\Sigma^{-1}]_{ij}/[\Sigma^{-1}]_{ii})d_j
\end{aligned} \tag{8}$$

Define k_i to be the first term in (8). k_i is equivalent to the integral of a multivariate normal of dimension $N-1$. That is, if we condition on $y_i = x_i$, then the conditional distribution has mean

$$x_i\Sigma_{\beta_i,i}$$

and variance

$$\Sigma_{\beta_i,\beta_i} - \Sigma_{\beta_i,i}\Sigma_{i,\beta_i}$$

where

$$\beta_i = \{1, ..., i-1, i+1, ...N\}.$$

So,

$$k_i = \Pr\{y_i \leq x_i\}[\Sigma^{-1}]_{ii}^{-1}(\prod_{j\neq i}\int_{-\infty}^{x_j})\phi_{x_i\Sigma_{\beta_i,i},\Sigma_{\beta_i,\beta_i}-\Sigma_{\beta_i,i}\Sigma_{i,\beta_i}}(y)(\prod_{j\neq i}dy_j).$$

Now, if we define $G \in \mathbb{R}^{N\times N}$ such that

$$G_{ij} = [\Sigma^{-1}]_{ij}/[\Sigma^{-1}]_{ii},$$

then we have

$$G\begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} = \begin{pmatrix} k_1 \\ \vdots \\ k_N \end{pmatrix}.$$

Since

$$G^{-1} = \begin{pmatrix} \Sigma_{.1}[\Sigma^{-1}]_{11} & \cdots & \Sigma_{N.}[\Sigma^{-1}]_{NN} \end{pmatrix},$$

therefore

$$d_i = \sum_{j=1}^{N}\Sigma_{ij}[\Sigma^{-1}]_{jj}k_j. \tag{9}$$

When the polynomial is of order higher than one we can use integration by parts successively to express the integral in terms of sums of integrals of the multivariate normal of lower orders. For example,

$$\prod_{i=1}^{N}\int_{-\infty}^{x_i} y_i^{r_i}\,\phi_{0,\Sigma}(y)dy_N\cdots dy_1$$

$$= x_k^{r_k}\prod_{i=1}^{N}\int_{-\infty}^{x_i} y_1^{r_1}\cdots y_{k-1}^{r_{k-1}}y_{k+1}^{r_{k+1}}\cdots y_N^{r_N}\,\phi_{0,\Sigma}(y)dy_N\cdots dy_1$$

$$-\int_{-\infty}^{x_k}\{\prod_{i\neq j}\int_{-\infty}^{x_i}\int_{-\infty}^{z} y_1^{r_1}\cdots y_{k-1}^{r_{k-1}}y_k^{r_k-1}y_{k+1}^{r_{k+1}}\cdots y_N^{r_N}\,\phi_{0,\Sigma}(y)dy_k dy_N\cdots dy_{k-1}dy_{k+1}\cdots dy_1\}dz.$$

Using (9) and integration of parts also lets us solve repeated integrals of the multivariate normal, such as,

$$\int_{-\infty}^{x_N}\Phi_{0,\Sigma}(x_1, \cdots, x_{N-1}, y_N)dy_N$$

$$= x_N \Phi_{0,\Sigma}(x_1, \cdots, x_{N-1}, x_N) - \int_{-\infty}^{x_N} y_N \frac{d}{dy_N} \Phi_{0,\Sigma}(x_1, \cdots, x_{N-1}, y_N) dy_N$$

$$= x_N \Phi_{0,\Sigma}(x_1, \cdots, x_{N-1}, x_N) - d_N$$

$$= x_N \Phi_{0,\Sigma}(x_1, \cdots, x_{N-1}, x_N) - \sum_{j=1}^{N} \Sigma_{Nj} [\Sigma^{-1}]_{jj} k_j.$$

We now discuss the integration of the multivariate production cost curve using the multivariate normal in a truncated Edgeworth series as our representation of the density function. In Chapter Two we derived the multi-area production cost curve which can be expressed as

$$C = F_{I+1} \sum_{\theta \subset \Omega} \sum_{A \in D(w_\theta)} \text{sgn}(A) \int_{-\infty}^{0} (\prod_{i=1}^{N} \int_{-\infty}^{0}) \; h\,(-[A^{-1}(\mathrm{t} + w + e_N z)]) \prod_{i=1}^{N} dw_i \; dz,$$

where $D(w_\theta)$ is the wedge decomposition of w_θ and $\text{sgn}(A)$ is the polarity of A in $A \in D(w_\theta)$. That is, $\text{sgn}(A) = +1$ if we add the wedge and -1 if we subtract the wedge. Our approximation to the density is

$$h(x) = \phi_{\mu_h, \Sigma_h}(x) + \eta_{ijk} \phi^{ijk}_{\mu_h, \Sigma_h}(x)/3! + \eta_{ijkl} \phi^{ijkl}_{\mu_h, \Sigma_h}(x)/4!$$

where η_{ijk} and η_{ijkl} are the third and fourth order cumulants of h.

Integrating the first term we get

$$\int_{-\infty}^{0} (\prod_{i=1}^{N} \int_{-\infty}^{0}) \; \phi_{\mu_h, \Sigma_h}\,(-[A^{-1}(\mathrm{t} + w + e_N z)]) \prod_{i=1}^{N} dw_i \; dz. \tag{10}$$

Now,

$$(-A^{-1}(\mathrm{t} + w + e_N z) - \mu_h)^T \Sigma_h^{-1} (-A^{-1}(\mathrm{t} + w + e_N z) - \mu_h)$$
$$= (w + e_N z + \mathrm{t} + A\mu_h)^T A^{-T} \Sigma_h^{-1} A^{-1} (w + e_N z + \mathrm{t} + A\mu_h)$$

If we let $\overline{\mu} = -(\mathrm{t} + A\mu_h)$ and $\overline{\Sigma} = A^T \Sigma_h A$, then (10) is equivalent to

$$\int_{-\infty}^{0} (\prod_{i=1}^{N} \int_{-\infty}^{0}) \; \phi_{\overline{\mu}, \overline{\Sigma}}\,(w + e_N z) \prod_{i=1}^{N} dw_i \; dz$$
$$= \int_{-\infty}^{0} \Phi_{\overline{\mu}, \overline{\Sigma}}\,(e_N z) dz,$$

which we can compute using the methods given above.

Chapter 5

Computation

In this chapter we present some results on a three area system using the algorithm developed in this dissertation. Data for the supply and demand characteristics of the system is based on that presented in Bloom (1992) and summarized in tables at the end of this chapter. Table 5.1 gives the data for the generators which was assumed to be identical in each area. The generators consist of 36 units with varying forced outage rate and running cost characteristics. Demand for each of the areas was derived from the load duration curve in Table 5.2. that represents a period of one week (168 hours). The demand for areas two and three was shifted so that the resulting system had correlation matrix:

$$\begin{bmatrix} 1 & .333 & .667 \\ .333 & 1 & .667 \\ .667 & .333 & 1 \end{bmatrix}.$$

Unserved energy was priced at 1.2 times the running cost of the most expensive unit and the resulting first six moments for the production cost curve are listed in Table 5.3. Since the generators are independent the cross moments for the production cost curve are the products of the individual moments. Six moments were used for the computation which was done on a 486-66MHZ computer using Mathematica. The three area system results in twenty wedges for which computation time was within a couple of minutes, but also not optimized for running time.

For the experiment we set the capacity of all of the arcs equal to t and varied t from 0 to 1000. Figure 5.1 shows the change in production cost of about 6% as we change the capacity of the arcs. Production cost did not decline appreciably further (less than a percent) as the arc capacity was increased above 1000 to infinity. The accuracy of the computation was checked by running a Monte-Carlo simulation on the system and was found to be within a 99% confidence interval that was .75% wide in all cases. The sample size to generate this confidence interval was 10^6 and needed several hours to run on a 200MHZ pentium-pro computer. This shows that the ability of our approximate model to reflect the underlying exact model is quite good.

Figure 5.1: Three Area Production Cost vs. Arc Capacity

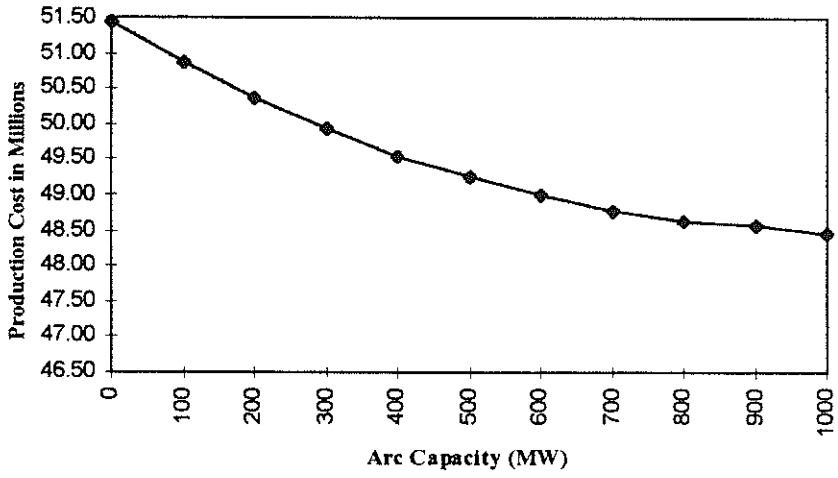

Table 5.1: Test System Unit Data

# Units	Capacity (MW)	Running Cost ($/MWH)	Forced Outage Rate
2	550	8.112	.150
1	800	15.352	.200
2	600	15.504	.180
1	500	15.504	.180
1	400	15.124	.130
1	200	15.960	.090
1	800	24.287	.160
2	600	24.838	.150
1	400	25.100	.124
1	200	26.462	.081
2	400	38.224	.124
11	200	40.299	.081
10	50	55.860	.105

Table 5.2: Test System Load Duration Curve

Load (MW)	Duration
3149.271	1.000000
4529.213	0.714290
5498.220	0.535710
6277.735	0.357140
6569.775	0.238100
6796.421	0.148810
7007.462	0.089286
7088.461	0.059524
7255.659	0.029762
7311.391	0.017857
7431.031	0.000000

Table 5.3: Moments of Test System Production Cost Curve

Order	Moment
1	$5.3781\ 10^{3}$
2	$3.9785\ 10^{7}$
3	$3.1756\ 10^{11}$
4	$2.6453\ 10^{15}$
5	$2.2662\ 10^{19}$
6	$1.9804\ 10^{23}$

Chapter 6

Extensions

In this chapter we discuss directions for future research based on the findings presented in this dissertation. While there is much of a direct and practical nature that can still be developed such as computation for higher order networks, there are four more theoretical ideas that are based on our research that are explored here. First, we examine the idea of finding upper and lower bounds and other types of approximations to the multi-area production cost for large networks by using smaller or more computationally tractable networks. Second, a sampling scheme is proposed in order to overcome the problem that the number of wedges explodes with the size of the network. Third, the theory developed in Appendix A and Chapter Four is applied to the problem of the multivariate normal integral. Fourth, a method for stochastic linear programming is conjectured.

Multi-Area Production Costing via Approximations with Smaller Networks

The approach to multi-area production costing developed in this dissertation is limited in its direct application to large networks for two reasons. First, as the size of the network increases, the number of Gale-Hoffman constraints also increase. For "dense" networks, this will be an exponential increase, although in Chapter Three it was shown that there are many network topologies for which the number of non-redundant

constraints only grows as the square or cube of the number of nodes. Second, the wedge decomposition algorithm maybe theoretically exponential. While computational experiments need to be performed to see to what extent our algorithm can be directly applied, it seems clear that the network size can quickly exceed capacity for direct computation.

Lee (1990) developed an approximation algorithm based on using the Bloom (1990) two area algorithm for calculating unserved energy. In this approach all two area partitions of the network are examined and ranked via their "importance". In phase two, the unserved energy is calculated using all the partitions of the network that are "important" enough. This approach worked well on a four area system.

We begin with the observation that production cost is decreasing in arc capacity.[2] If we increase the arc capacity to infinity then we have, in effect, combined two areas into one. On the other hand, if we decrease an arc's capacity to zero, we will have disconnected two nodes. Increasing the arc capacity or combining areas then, gives us a lower bound to production cost and decreasing the arc capacity or disconnecting areas gives us an upper bound to production cost. One approach to approximating the cost of a larger network would be to change the arc capacities to get network topologies that are more computationally tractable and yield bounds to the target network cost. For instance, in the network illustrated below in figure 6.1, we can increase arc capacities to get the smaller network shown in figure 6.2.

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[2]The author wishes to acknowledge the helpful discussions on this idea with Jeremy A. Bloom.

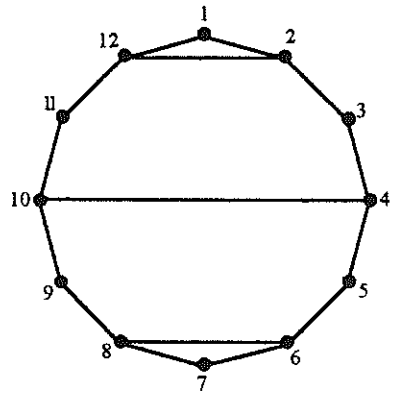

Figure 6.1 : A Twelve Area Network

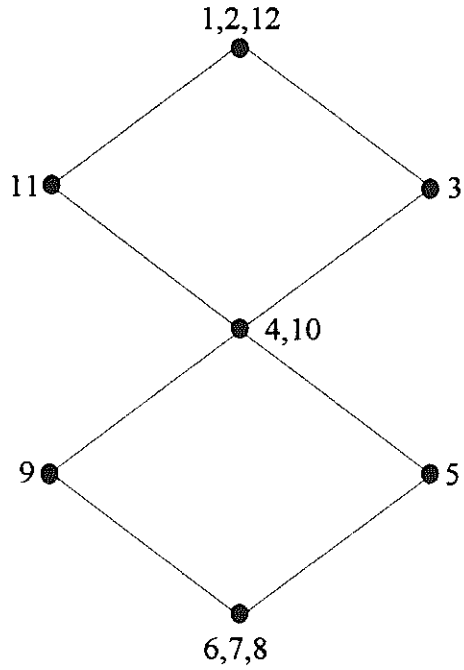

Figure 6.2 : Combine Nodes to Get Lower Bound

Similar methods suggest themselves to obtain an upper bound, but we will not belabor them here. What we would like to point out is that analysis of demand and supply might suggest ways of changing arc capacities that would lead to closer approximations. Division of the state space is one possibility that suggests itself. Or, two connected nodes with highly correlated demand might be candidates for combining.

Another avenue of research that seems promising is based on Corollary 3.24. Networks that are completely connected have Gale-Hoffman regions that are very highly structured. The base has an almost fractal-like property (see, for instance, Falconer (1990), for rigorous discussion of self-replicating structures) in that the base in a higher dimension has sides made of replicas of the base in the next lower dimension. Moreover, when the arc capacities are all equal then the base is a polyhedron with all the edges of the polyhedron having equal length (Theorem 3.15), so that the polyhedron is regular. The resulting "super-symmetry" suggests that the wedge decomposition would produce regions that have the same shape and thus could be combined to greatly simplify the calculation of production cost.

A Wedge Sampling Method

Since the increase in the number of wedges with the size and interconnectivity of the network quickly makes direct computation difficult, we also think it is promising avenue to sample over the wedges to estimate the production cost. Direct enumeration of the wedges or, better, calculation of the number of wedges without actually generating them seems unpromising since we do not have a theory of how many will be produced in general. Rather, we can sample the wedges via the following process. First, we sample from the constraints. Second, we can view the process of generating the wedges associated with that constraint as the process of traversing a binary tree from the root node to a leaf. We say that the *polarity* of the root node is $+1$. At each node other than a leaf we branch to two nodes. On the node on the left we drop one of the constraints and on the right we reverse it. The polarity of the node on the left is the same as its parent node, while the polarity of the node on the right is the opposite of its parent. The value that we use for our estimate of the overall value is the polarity times the value we derive from calculation of the wedge. It is unclear whether this sampling procedure has a low

enough variance to be valuable, but clearly there are various variance reduction schemes that could be applied to it. One remark is that rather than each sample calculating just one point, we calculate many points with each sample. The meaning of this tradeoff in work will be more clear in our discussion of stochastic linear programming below.

The Multivariate Normal Integral

In Chapter Four and Appendix A, we developed a number of results concerning the multivariate normal integral. The main result of immediate interest is the univariate numerical integration of the trivariate normal. Unfortunately, direct extension of this result to higher dimensions is not possible. A pair of ideas along with results of Appendix A[3] do suggest an approach that could reduce the multivariate normal integral to a univarite numerical integration. The first idea is that although the integrals generated in Appendix A result in elliptic integrals, these elliptic functions in turn when integrated generate linear combinations of other elliptic functions. As a result each integral in turn generates a linear combination of elliptic functions and the process of integration is an affine transformation on a space determined by elliptic functions. In this way the integral is at worst solvable via one dimensional numerical integration. The second idea addresses the problem that for each correlation that is positive the integral splits in two. It seems reasonable, however, that there should be a method for "triangulating" a wedge with not too many other wedges such that all the triangulating wedges have acute angles (those of less than $\pi/2$) between the constituent hyperplanes. So, what remains to be solved is, first, finding the function that transforms the integrand in the space spanned by the elliptic functions and, second, the developing an acute wedge triangulation of another wedge.

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[3]The author wishes to acknowledge collaboration with Jeremy A. Bloom in developing these ideas.

Stochastic Linear Programming

Stochastic linear programming addresses the problem of solving linear programs when some of the parameters are random variables. One can summarize the essential part of the problem as

$$\begin{aligned} & E \min \ \overline{c}x \\ \text{s.t.} \quad & \overline{A}x \leq \overline{b} \\ & x \ \geq 0 \end{aligned} \tag{P}$$

where $\overline{c}$, $\overline{b}$, $\overline{A}$, are now random variables and E is the expectation of the solution to the minimization problem.

The method developed in this dissertation addresses the problem when only $\overline{c}$ and $\overline{b}$ are random variables as follows. Let θ be a subset of $\{1, 2, ..., n\}$. This subset of the x variables is a feasible, optimal solution over a convex polyhedron given by costing out the solution and checking the solution to see if it is feasible. That is,

$$x_\theta = [(A_{.\theta} I_{.\theta^c})^{-1} \overline{b}]_\theta ,$$

$$x_{\theta^c} = 0,$$

so the feasibility condition is

$$[(A_{.\theta} I_{.\theta^c})^{-1} \overline{b}]_\theta \geq 0.$$

The optimality condition is

$$\overline{c}_{\theta^c} - [(A_{.\theta} I_{.\theta^c})^{-1} A_{.\theta^c}]^T \overline{c}_\theta \geq 0,$$

and the value of the program when the basis is feasible and optimal is

$$[(A_{.\theta} I_{.\theta^c})^{-1} \overline{b}]_\theta^{\mathrm{T}} \overline{c}_\theta .$$

Write $\Re(\theta) = \{(\overline{b}, \overline{c}) : [(A_{.\theta} I_{.\theta^c})^{-1} \overline{b}]_\theta \geq 0, \ \overline{c}_{\theta^c} - [(A_{.\theta} I_{.\theta^c})^{-1} A_{.\theta^c}]^T \overline{c}_\theta \geq 0\}$ and let $f(\overline{b}, \overline{c})$ be the joint density function for $(\overline{b}, \overline{c})$.

Then, the solution to (P) is given by

$$\sum_{\theta \subset \{1,..,N\}} \int_{\Re(\theta)} [(A_{.\theta} I_{.\theta^c})^{-1} \overline{b}]_\theta^{\mathrm{T}} \overline{c}_\theta \, f(\overline{b}, \overline{c}) \, d\overline{b} \, d\overline{c}.$$

When $f(\bar{b}, \bar{c})$ can be expressed directly or as an Edgeworth series then we can directly apply the methods developed in this thesis. If N is large we can apply the sampling scheme outlined above.

This method contrasts with the approach of Infanger (1992) and Pereira, Cunha, Oliveira, Mazumdar, and Yengar (1990) that solves (P) by picking out single points in the state space and then solving the linear program. They use various variance reduction techniques to improve the characteristics of their samples. The method we propose, in contrast, samples over regions of the state space and then integrates to find the expected value over the regions. Certainly, variance reduction methods could be investigated with respect to our sample as well. Whether the technique we propose is computationally attractive depends on how hard it is to perform the integration and the variance characteristics of our sample. Since we are averaging over many points in the state space for each sample wedge, it would seem reasonable to think that the variance of our sample would be relatively low.

Appendix A

An Approach to One Dimensional Numerical Integration of the Multivariate Normal

This appendix presents an approach suggested by Jeremy A. Bloom[4] to one dimensional numerical integration of the multivariate normal density over an orthant. Some of the analysis presented here is used in developing the algorithm for one dimensional numerical integration of the trivariate normal presented in Chapter Three. It also forms the basis for parts of Chapter Six.

We are concerned with the problem of calculating the value of the integral of the multivariate normal density over the positive orthant. That is, if we let $\phi_{-b,\Sigma}(x)$ be the multivariate normal density with mean vector $-b$ and covariance matrix Σ, we wish to calculate

$$\int_0^\infty \cdots \int_0^\infty \phi_{-b,\Sigma}(x) dx_1 \cdots dx_N \tag{1}$$

We focus here only on the case where Σ is positive definite. Let $\Sigma = AA^T$, where A is some convenient factorization such as Cholesky or the symmetric square root (see, for example, Golub and Van Loan (1989)). If we transform the variables from x to $A^{-1}x + A^{-1}b$, then (1) becomes

$$\int_{Ax \geq b} \phi_{0,I}(x) dx_1 \cdots dx_N, \tag{2}$$

where $\phi_{0,I}$ is the independent multivariate normal with mean 0 (see Tong (1990)). Define the N-dimensional sphere centered at the origin, $S_{N,r} \equiv \{x \in \mathbb{R}^N : x^T x = r^2\}$. So, the

[4]Private communication.

level sets of $\phi_{0,I}$ are $S_{N,r}$. Our approach to calculating (2) is to sum the area given by the intersection of $Ax \geq b$ and $S_{N,r}$ divided by the surface area of $S_{N,r}$ weighted by $\phi_{0,I}$ over r from zero to infinity. To express this more formally, write $P(r)$ for the surface area of the intersection of $Ax \geq b$ and $S_{N,r}$ divided by the surface area of $S_{N,r}$. Then we can rewrite (2) as

$$2/(2\pi)^{\frac{1}{2}} \int_0^\infty P(r) e^{-r^2/2} dr. \tag{3}$$

If we have a way to readily calculate $P(r)$, then we can apply our favorite one dimensional numerical integration algorithm to (3) and find the answer to the problem of interest.

To calculate $P(r)$ the basic problem is to calculate the surface area of the intersection of $Ax \geq b$ and $S_{N,r}$. Note that we get the same answer to (3) if for $x \in \mathbb{R}^N$, we consider $P(r)$ for the surface area of the intersection of $Ax \geq b$ and $S_{N+m,r}$, $m \in \mathbb{N} \cup \{0\}$. This fact becomes useful in actually performing the integration. We attempt this by constructing an iterated integral by integrating over the right hand side of $Ax \geq b$.

Without loss of generality, assume that Σ is the correlation matrix and that A is the Cholesky factorization of Σ, so A is lower triangular. This considerably simplifies the discussion that follows and as a result we have $A_{i.}A_{j.} = \rho_{ij}$ where ρ_{ij} is the correlation between the i^{th} and j^{th} random variables. The vector $-b$ now no longer is the mean, but rather the variance adjusted mean since we have divided each row by the variance of the corresponding random variable. A further simplification helps the discussion as well. Rather than considering the intersection of $Ax \geq b$ and $S_{N,r}$ we perform a change of variables $x = ry$, and consider the intersection of $Ax \geq b$ and $S_{N,1}$ where we have adjusted b by dividing by r. For simplicity, we write S_N for $S_{N,1}$. Also, define $\mathcal{A}_n(r)$ $\equiv \frac{2r^{n-1}\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}$, the surface area of $S_{n,r}$ (see, for example, Kendall (1961)).

Structure of the Integrand

We begin by noting the hyperplane $A_{i.}x_i = b_i$ and the sphere S_N intersect when $b_i \leq r = 1$ and their intersection is a sphere of dimension $N-1$ with center at the point

$$\xi = A_{.i}b_i$$

at a distance

$$q = |b_i|.$$

from the center of S_N, that is, the origin. So, the radius of the intersection is, by the Pythagorean theorem, $(r-q^2)^{\frac{1}{2}} = (1-q^2)^{\frac{1}{2}}$. Note that ξ is orthogonal to the hyperplane $A_{i.}x_i = b_i$ and points to the center of S_N. To compute the surface area of S_N in the region $A_{i.}x \geq b_i$ we can sum over the surface areas of the spheres of dimension $N-1$ given by the intersection of $A_{i.}x_i = y$ and S_N varying y from b_i to $r = 1$. That is, we integrate radially outward from the center of the sphere along the vector ξ, starting at $y = b_i$ until the radius of the intersection becomes zero at $y = 1$. At each value of y the area element is a ring with "circumference" (in $N-2$ dimensions) equal to the surface area of an $(N-1)$-sphere with radius $(r^2-y^2)^{\frac{1}{2}}$, $\mathcal{A}_{N-1}([r-y^2]^{\frac{1}{2}})$. The width of this ring depends on the slope of the surface of S_N as

$$r(r^2-y^2)^{-\frac{1}{2}}dy = (1-y^2)^{-\frac{1}{2}}dy.$$

Hence the area is

$$\begin{aligned} & \int_{b_i}^{r} r(r^2-y^2)^{-\frac{1}{2}}\mathcal{A}_{N-1}([r^2-y^2]^{\frac{1}{2}})dy \\ &= \int_{b_i}^{1}(1-y^2)^{-\frac{1}{2}}\mathcal{A}_{N-1}([1-y^2]^{\frac{1}{2}})dy \\ &= \frac{2\pi^{\frac{1}{2}(N-1)}}{\Gamma(\frac{1}{2}(N-1))}\int_{b_i}^{1}(1-y^2)^{\frac{N-3}{2}}dy. \end{aligned}$$

Now consider the intersection of S_N with $M \leq N$ of the hyperplanes. Let α be some subset of $\{1,...,N\}$ of cardinality M. So, we consider the intersection of $A_{\alpha.}x = b_\alpha$ and S_N . By using the Karush-Kuhn-Tucker conditions we can find that the result is an $(N-M)$-sphere with center at point

$$\xi = (A_{\alpha.})^T(A_{\alpha.}(A_{\alpha.})^T)^{-1}b_\alpha = (A_{\alpha.})^T\Sigma_{\alpha\alpha}^{-1}b_\alpha$$

and a distance

$$q = [b_\alpha^T (A_{\alpha.}(A_{\alpha.})^T)^{-1} b_\alpha]^{\frac{1}{2}} = [b_\alpha^T \Sigma_{\alpha\alpha}^{-1} b_\alpha]^{\frac{1}{2}}$$

from the center of S_N (the origin). The radius of this sphere is $(r^2 - q^2)^{\frac{1}{2}}$.

We propose to compute the surface area of S_N in the wedge

$$W = \{x \in \mathbb{R}^N : A_{\alpha.} x \geq b_\alpha\}$$

as a nested set of integrals of the form discussed above, each of which varies on a component of the right-hand side vector b_α of the wedge. Define $\alpha_i = \{1, ..., i\}$. The intersection of S_N with the $(N-m)$-plane P_{N-m} defined by $A_{\alpha_m.}$ and y_{α_m} is an $(N-m)$-sphere with center at the point ξ_m and distance q_m from the center of S_N, where ξ_m and q_m can be calculated using the above formulas. We make use of the fact that A is lower triangular to get:

$$\begin{aligned}
q_m^2 &= y_{\alpha_m}^T A_{\alpha_m\alpha_m}^{-T} A_{\alpha_m\alpha_m}^{-1} y_{\alpha_m} \\
\xi_m &= (A_{\alpha_m.})^T (A_{\alpha_m.}(A_{\alpha_m.})^T)^{-1} b_{\alpha_m} \\
&= (A_{\alpha_m\alpha_m} 0)^T (A_{\alpha_m\alpha_m}(A_{\alpha_m\alpha_m})^T)^{-1} b_{\alpha_m} \\
&= \begin{bmatrix} A_{\alpha_m\alpha_m}^T \\ 0 \end{bmatrix} A_{\alpha_m\alpha_m}^{-T} A_{\alpha_m\alpha_m}^{-1} b_{\alpha_m} \\
&= \begin{bmatrix} A_{\alpha_m\alpha_m}^{-1} b_{\alpha_m} & 0 \end{bmatrix}
\end{aligned}$$

Define the vectors

$$\begin{aligned}
z_{m+1} &= \xi_{m+1} - \xi_m \\
z_1 &= \xi_1.
\end{aligned}$$

z_m is parallel to P_{N-m} and orthogonal to $P_{N-(m+1)}$. So, $z_{m+1}^T \xi_m = 0$. By the Pythagorean theorem,

$$\begin{aligned}
z_{m+1}^T z_{m+1} &= q_{m+1}^2 - q_m^2 \text{ and} \\
z_1^T z_1 &= q_1^2.
\end{aligned}$$

The vector z_m is the radial direction of the sphere S_{N-m} in the plane P_{N-m}. In general, the normal vector of the $(N-1)$-plane $A_{(m+1).}x = y_{m+1}$ does not lie in P_{N-m}, but the vector $(A_{(m+1).}z_m)z_m$ is normal to the plane $P_{N-(m+1)}$ and is parallel to P_{N-m}.

Using the argument given above, to compute the surface area of S_{N-m} in the region $A_{m+1.}x \geq b_{m+1}$ we can integrate radially outward from the center of the sphere along the vector z_m. The radial distance along this vector is measured by

$$s = (q_{m+1}^2 - q_m^2)^{\frac{1}{2}}.$$

Now,

$$q_{m+1}^2 = y_{\alpha_{m+1}}^T (A_{\alpha_{m+1}.}(A_{\alpha_{m+1}.})^T)^{-1} y_{\alpha_{m+1}}$$

$$= \begin{bmatrix} y_{\alpha_m}^T & y_{m+1} \end{bmatrix} \begin{bmatrix} A_{\alpha_m.}(A_{\alpha_m.})^T & A_{\alpha_m.}(A_{m+1.})^T \\ A_{m+1.}(A_{\alpha_m.})^T & A_{m+1.}(A_{m+1.})^T \end{bmatrix}^{-1} \begin{bmatrix} y_{\alpha_m} \\ y_{m+1} \end{bmatrix}$$

since A is lower triangular and $AA^T = \Sigma$

$$= \begin{bmatrix} y_{\alpha_m}^T & y_{m+1} \end{bmatrix} \begin{bmatrix} A_{\alpha_m\alpha_m} A^T{}_{\alpha_m\alpha_m} & A_{\alpha_m\alpha_m}(A_{m+1,\alpha_m})^T \\ A_{m+1,\alpha_m}(A_{\alpha_m\alpha_m})^T & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_{\alpha_m} \\ y_{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} y_{\alpha_m}^T & y_{m+1} \end{bmatrix} \left(\begin{bmatrix} A_{\alpha_m\alpha_m} & 0 \\ A_{m+1,\alpha_m} & 1 \end{bmatrix} \begin{bmatrix} A^T{}_{\alpha_m\alpha_m} & (A_{m+1,\alpha_m})^T \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} y_{\alpha_m} \\ y_{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} y_{\alpha_m}^T & y_{m+1} \end{bmatrix} \begin{bmatrix} A^T{}_{\alpha_m\alpha_m} & (A_{m+1,\alpha_m})^T \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_{\alpha_m\alpha_m} & 0 \\ A_{m+1,\alpha_m} & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_{\alpha_m} \\ y_{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} y_{\alpha_m}^T & y_{m+1} \end{bmatrix} \begin{bmatrix} A_{\alpha_m\alpha_m}^{-T} & -A_{\alpha_m\alpha_m}^{-T}(A_{m+1,\alpha_m})^T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{\alpha_m\alpha_m}^{-1} & 0 \\ -A_{m+1,\alpha_m}A_{\alpha_m\alpha_m}^{-1} & 1 \end{bmatrix} \begin{bmatrix} y_{\alpha_m} \\ y_{m+1} \end{bmatrix}$$

$$= y_{\alpha_m}^T A_{\alpha_m\alpha_m}^{-T} A_{\alpha_m\alpha_m}^{-1} y_{\alpha_m} + y_{\alpha_m}^T A_{\alpha_m\alpha_m}^{-T} (A_{m+1,\alpha_m})^T A_{m+1,\alpha_m} A_{\alpha_m\alpha_m}^{-1} y_{\alpha_m}$$
$$- 2y_{m+1} A_{m+1,\alpha_m} A_{\alpha_m\alpha_m}^{-1} y_{\alpha_m} + y_{m+1}^2$$

since the first term is q_m^2 and $\xi_m = \left[A^{-1}_{\alpha_m \alpha_m} y_{\alpha_m} \quad 0 \right]$

$$= q_m^2 + \xi_m^T (A_{m+1.})^T A_{m+1.} \xi_m - 2 y_{m+1} A_{m+1.} \xi_m + y_{m+1}^2$$

$$= q_m^2 + (y_{m+1} - A_{m+1.} \xi_m)^2.$$

Therefore,

$$q_{m+1}^2 = q_m^2 + (y_{m+1} - A_{m+1.} \xi_m)^2$$

and

$$\begin{aligned} s &= (q_{m+1}^2 - q_m^2)^{\frac{1}{2}} \\ &= \left| y_{m+1} - A_{m+1.} \xi_m \right|. \end{aligned}$$

At each value of y_{m+1} the area element is a ring with "circumference" (in $N - (m+2)$ dimensions) equal to the surface area of an $(N - (m+1))$-sphere $S_{N-(m+1)}$ with radius $(r^2 - q_m^2 - s^2)^{\frac{1}{2}} = (r^2 - q_{m+1}^2)^{\frac{1}{2}}$, that is, $\mathcal{A}_{N-(m+1)}([r^2 - q_{m+1}^2]^{\frac{1}{2}})$. The width of this ring depends on the slope of the surface of S_{N-m} as

$$\begin{aligned} &(r^2 - q_m^2)^{\frac{1}{2}} (r^2 - q_m^2 - s^2)^{-\frac{1}{2}} ds \\ &= (r^2 - q_m^2)^{\frac{1}{2}} (r^2 - q_{m+1}^2(y_{m+1}))^{-\frac{1}{2}} dy_{m+1}. \end{aligned}$$

The integration starts at $y_{m+1} = b_{m+1}$ and continues outward until the radius of the intersection $S_{N-(m+1)}$ is zero,

$$\begin{aligned} 0 &= r^2 - q_{m+1}^2(y_{m+1}) \\ &= r^2 - q_m^2 - (y_{m+1} - A_{m+1.} \xi_m)^2 \quad \text{or} \\ y_{m+1} &= A_{m+1.} \xi_m + (r^2 - q_m^2)^{\frac{1}{2}}. \end{aligned}$$

Hence the area is

$$\int_{b_{m+1}}^{r^2 - q_{m+1}^2(y_{m+1})=0} (r^2 - q_m^2)^{\frac{1}{2}} (r^2 - q_{m+1}^2(y_{m+1}))^{-\frac{1}{2}} \mathcal{A}_{N-(m+1)}([r^2 - q_{m+1}^2(y_{m+1})]^{\frac{1}{2}}) dy_{m+1}$$

It would be tempting to think that the limits of integration for y_{m+1} stay the same if we added another hyperplane, but as figure A.1 shows, this is not the case. The dotted line shows the point at which $r^2 - q_{m+2}^2(y_{m+2}) = 0$, yet there is still some of the sphere to be integrated over. As a result we have to spend a number of pages that follow on determining the limits of integration.

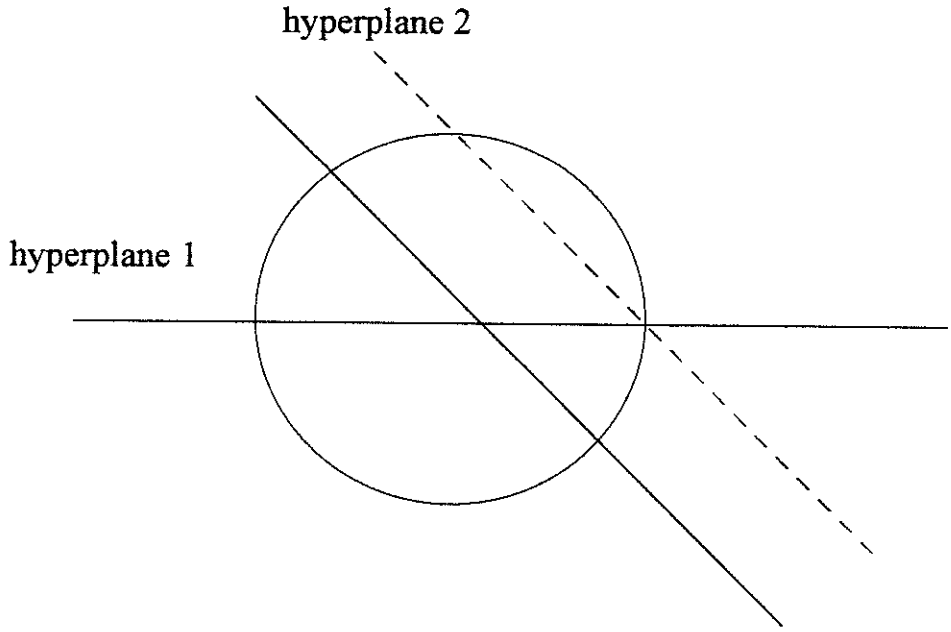

Figure A.1: Limits of Integration of Intersection of Wedge and Sphere

Before we do that however let us first, having come so far, determine what the integrand looks like. For the surface area of S_N in $A_{\alpha_M} x \geq b_{\alpha_M}$, we iterate the argument given above to obtain

$$\int_{l(y_1)}^{u(y_1)} \cdots \int_{l(y_M)}^{u(y_M)} (r^2 - q_m^2)^{\frac{1}{2}} \prod_{m=0}^{M-1} \left(\frac{(r^2 - q_m^2(y_m))}{(r^2 - q_{m+1}^2(y_{m+1}))} \right)^{\frac{1}{2}} \mathcal{A}_{N-M}([r^2 - q_M^2(y_M)]^{\frac{1}{2}}) dy_M \cdots dy_1$$

substituting the definition of $\mathcal{A}$ and canceling the numerator and denominator of successive factors:

$$= \frac{2\pi^{\frac{1}{2}(N-M)}}{\Gamma(\frac{1}{2}(N-M))} \int_{l(y_1)}^{u(y_1)} \cdots \int_{l(y_M)}^{u(y_M)} \left(\frac{r^2}{(r^2 - q_M^2(y_M))} \right)^{\frac{1}{2}} ([r^2 - q_M^2(y_M)]^{\frac{1}{2}})^{N-M-1} dy_M \cdots dy_1$$

$$= \frac{2\pi^{\frac{1}{2}(N-M)}}{\Gamma(\frac{1}{2}(N-M))} \int_{l(y_1)}^{u(y_1)} \cdots \int_{l(y_M)}^{u(y_M)} r\,(r^2 - q_M^2(y_M))^{\frac{N-M-2}{2}} dy_M \cdots dy_1$$

for $r = 1$

$$= \frac{2\pi^{\frac{1}{2}(N-M)}}{\Gamma(\frac{1}{2}(N-M))} \int_{l(y_1)}^{u(y_1)} \cdots \int_{l(y_M)}^{u(y_M)} (1 - q_M^2(y_M))^{\frac{N-M-2}{2}} dy_M \cdots dy_1 \qquad (4)$$

Limits of Integration

The foregoing discussion has implicitly assumed that the sphere S_N and $A_{\alpha_M} x = b_{\alpha_M}$ have a non-empty intersection. This section derives limits of integration, that is, the $l(y_i)$ and $u(y_i)$ in (1) for the general case. It turns out that the integral can break into a number of different pieces depending on the correlations of the random variables. Clearly, the limits of integration are bounded for each y_i by b_i below and $y_i : r^2 - q_i^2(y_i) = 0$ above. First define some notation.

Let α be a subset of the index set $\{1, ..., N\}$, H_α the hyperplane defined by the set of constraints $A_\alpha x = b_\alpha$, q_α the distance from the origin to H_α, W_α the half space defined by taking the constraints in W that are indexed by α. For two disjoint sets α and β and index $i \notin \alpha, \beta$ we write $w^-_{(\alpha),i,(\beta)} < w^+_{(\alpha),i,(\beta)}$ for the y_i that satisfy

$$(y_\alpha \quad y_i \quad b_\beta)(\Sigma_{\alpha\cup\{i,\beta\},\alpha\cup\{i,\beta\}})^{-1} \begin{pmatrix} y_\alpha \\ y_i \\ b_\beta \end{pmatrix} = r^2.$$

When we write w without the β subscript, $w^{\pm}_{(\alpha),i}$ are the values of y_i that satisfy

$$(y_\alpha \; y_i)\,(\Sigma_{\alpha\cup\{i\},\alpha\cup\{i\}})^{-1}\begin{pmatrix} y_\alpha \\ y_i \end{pmatrix} = r^2.$$

And finally when we write w without the α, $w^{\pm}_{i,(\beta)}$ are the values of y_i that satisfy

$$(y_i \; b_\beta)\,(\Sigma_{\{i,j\},\{i,j\}})^{-1}\begin{pmatrix} y_i \\ b_\beta \end{pmatrix} = r^2.$$

Since $w^{\pm}_{(\alpha),i}$ is the value of y_i that makes $q_{\alpha\cup\{i\}} = r$. We have

$$\begin{aligned} 0 &= r^2 - q^2_{\alpha\cup\{i\}} \\ &= r^2 - q^2_\alpha - (\, w_{(\alpha),i} - a_i^T\xi_\alpha)^2 \text{ or} \\ w^{\pm}_{(\alpha),i} &= a_i^T\xi_\alpha \pm (r^2 - q^2_\alpha)^{\frac{1}{2}} \end{aligned}$$

The following lemma establishes a property of the w's that are used in deriving the limits of integration.

Lemma A.1. Let γ_1, γ_2, and γ_3 be disjoint subsets of $\{1, ..., N\}$, $i \notin \gamma_1 \cup \gamma_2 \cup \gamma_3$, and $i \in \{1, ..., N\}$. Also, let $H_{\gamma_1\cup\gamma_2\cup\gamma_3} \cap S \neq \emptyset$. Then

$$w^{+}_{(\gamma_1)i\,(\gamma_2)} \geq \quad w^{+}_{(\gamma_1\cup\gamma_3)i\,(\gamma_2)} \tag{5}$$

$$w^{+}_{(\gamma_1)i\,(\gamma_2)} \geq \quad w^{+}_{(\gamma_1)i\,(\gamma_2\cup\gamma_3)} \tag{6}$$

$$w^{-}_{(\gamma_1)i\,(\gamma_2)} \leq \quad w^{-}_{(\gamma_1\cup\gamma_3)i\,(\gamma_2)} \tag{7}$$

$$w^{-}_{(\gamma_1)i\,(\gamma_2)} \leq \quad w^{-}_{(\gamma_1)i\,(\gamma_2\cup\gamma_3)} \tag{8}$$

Proof: Since b is arbitrary we only need to prove (5) and (7). Evidently, $w_i^- \leq -r \leq w_{(\gamma_1)i}^- \leq w_{(\gamma_1)i}^+ \leq r \leq w_i^+$ since if $y_i \geq r$, $W_i \cap S = \emptyset$.

Because $H_{\gamma_1 \cup \gamma_3} \subset H_{\gamma_1}$, $H_{\gamma_1 \cup \gamma_3} \cap H_i(w_{(\gamma_1)i}^{\pm}) \cap S \in H_{\gamma_1} \cap H_i(w_{(\gamma_1)i}^{\pm}) \cap S$.

So, $w_{(\gamma_1)i}^- \leq w_{(\gamma_1 \cup \gamma_2)i}^- \leq w_{(\gamma_1 \cup \gamma_2)i}^+ \leq w_{(\gamma_1)i}^+$.

□

As a consequence of the lemma if $\alpha \subset \beta$, then $H_\beta \subset H_\alpha$ and so $q_\beta \geq q_\alpha$ since $\xi_\beta \in H_\alpha$. Let γ_1, γ_2 be a disjoint partition of α and $i \notin \beta$. Then $w_{(\gamma_1)i(\gamma_2)}^+ \geq w_{(\gamma_1 \cup \beta \setminus \alpha)i(\gamma_2)}^+$ and $w_{(\gamma_1)i(\gamma_2)}^+ \geq w_{(\gamma_1)i(\gamma_2 \cup \beta \setminus \alpha)}^+$. This is true because at $y_i = w_{(\gamma_1 \cup \beta \setminus \alpha)i(\gamma_2)}^+$, q_β is non-decreasing in y_i. If it were not non-decreasing, then we would be at $y_i = w_{(\gamma_1 \cup \beta \setminus \alpha)i(\gamma_2)}^-$.

We separate the question into two cases: that where the vertex of W lies on or inside the sphere and that where it is outside the sphere. In the following we denote the integrand, which changes from one iterated integral to another, by $\mathcal{I}$ and the whole integral by $\mathfrak{I}$.

Case One: The vertex of W is on or inside the sphere.

In this case the integration is performed by sweeping the hyperplanes out in the feasible directions. Since the vertex of W is inside the sphere we start out with

$$w_{(\alpha),i}^- < b_i < w_{(\alpha),i}^+ \text{ for } y_\alpha = b_\alpha.$$

If we increase the y_α so that $b_i > w^+_{(\alpha),i}$ then $S \cap W_\alpha$ is infeasible and that part of the integral is zero. If for some y_α, $w^-_{(\alpha),i} = b_i$, then the i^{th} constraint is redundant in the sense that it does not cut off any part of the sphere $S \cap W_\alpha$ so we can drop the innermost integral by increasing the order of the integrand (or project to a lower dimension). Furthermore, for $\hat{y}_\alpha \geq y_\alpha$ we then have $w^-_{(\alpha),i} \geq b_i$ since otherwise the region of integration would not be convex. We now demonstrate this with the following two lemmas.

Lemma A.2. Let the vertex of W be in the interior of S. If for some $y'_\alpha \geq b_\alpha$, $y'_\alpha \neq b_\alpha$, $b_i = w^+_{(\alpha),i}$, then $b_i \geq w^+_{(\alpha),i}$ for $\hat{y}_\alpha \geq y'_\alpha$, $\hat{y}_\alpha \neq y'_\alpha$.

Proof: Without loss of generality let $\alpha = \{1, ..., N-1\}$ and $i = N$. Let $W' = \{x : Ax \geq \begin{pmatrix} y'_\alpha \\ b_N \end{pmatrix}\}$ and $\hat{W} = \{x : Ax \geq \begin{pmatrix} \hat{y}_\alpha \\ b_N \end{pmatrix}\}$. Since $\hat{y}_\alpha \geq y'_\alpha$, $\hat{W} \subset W'$. Because $b_N = w^+_{(\alpha),N}$ then $W' \cap S = \xi_N$. Therefore, $\hat{W} \cap S \subset \{\xi_N\}$, so $b_N \geq w^+_{(\alpha),N}$ for $\hat{y}_\alpha \geq y'_\alpha$, $\hat{y}_\alpha \neq y'_\alpha$.

□

Lemma A.3. Let the vertex of W be in the interior of S. If for some $y'_\alpha \geq b_\alpha$, $y'_\alpha \neq b_\alpha$, $b_N = w^-_{(\alpha),N}$, then $b_N \leq w^-_{(\alpha),N}$ for $\hat{y}_\alpha \geq y'_\alpha$, $\hat{y}_\alpha \neq y'_\alpha$.

Proof: Without loss of generality let $\alpha = \{1, ..., N-1\}$ and $i = N$. As before, let $W' = \{x : Ax \geq \begin{pmatrix} y'_\alpha \\ b_N \end{pmatrix}\}$ and $\hat{W} = \{x : Ax \geq \begin{pmatrix} \hat{y}_\alpha \\ b_N \end{pmatrix}\}$. Further define $W'_\alpha = \{x : A_{N-1}x \geq y'_\alpha\}$ and $\hat{W}_\alpha = \{x : A_{N-1}x \geq \hat{y}_\alpha\}$. So we have $\hat{W} \subset W'$ and $\hat{W}_\alpha \subset W'_\alpha$. Because $b_i = w^-_{(\alpha),i}$, $\{x : a_N^T x \geq b_N\} \cap W'_\alpha \cap S = W'_\alpha \cap S$. Therefore, $\{x : a_N^T x \geq b_N\} \cap \hat{W}_\alpha \cap S = \hat{W}_\alpha \cap S$, so $b_N \leq w^-_{(\alpha),N}$ for $\hat{y}_\alpha \geq y'_\alpha$, $\hat{y}_\alpha \neq y'_\alpha$.

□

Now we develop the cases for $N = 2$ and 3, followed by the general case.

Case $N = 2$:

Proposition A.4. For $N = 2$, the integral is given by

$$\mathfrak{I} = \int_{b_1}^{w_{1(2)}^+} dy_1 \int_{b_2}^{w_{(1)2}^+} \mathcal{I}\, dy_2 + \int_{w_{1(2)}^+}^{r} H(r\,\rho_{12} - b_2)\mathcal{I}\, dy_1 .$$

Proof: The integral can be written in general form as

$$\mathfrak{I} = \int_{b_1}^{r} dy_1 \int_{b_2 \vee w_{(1)2}^- \le y_2 \le w_{(1)2}^+} \mathcal{I}\, dy_2 .$$

If $N = 2$, the integral breaks into at most two pieces. When $y_1 = w_1^+ = r$, if $\xi_1(y_1)$ is feasible, then the second or inner hyperplane no longer cuts off part of the sphere formed by the intersection of the outer hyperplane with S by lemma A.3. So the lower bound for y_2 changes at that point to move up to where the inner hyperplane can now intersect with the outer hyperplane. In this case, however we are (of course) integrating over the whole sphere and so the inner integral can be dropped with a suitable change in the integrand. If $\xi_1(r)$ is not feasible then the integral is empty for $y_1 > w_{1(2)}^+$ by lemma A.2. $\xi_1(r)$ not feasible means $a_2^T \xi_1(r) < b_2$. Or since $\xi_1 = y_1 a_1$, we get $y_1 a_2^T a_1 < b_2$. So this is the case when $r\,\rho_{12} < b_2$.
So we can rewrite the integral more precisely as

$$\mathfrak{I} = \int_{b_1}^{w^+_{1(2)}} dy_1 \int_{b_2}^{w^+_{(1)2}} \mathcal{I}\, dy_2 + \int_{w^+_{1(2)}}^{r} H(r\,\rho_{12} - b_2)\, \mathcal{I}\, dy_1$$

where H is the unit step function.

□

Note that because of lemmas A.2 and A.3 we only need to compare two numbers to see the form of the second part of the integral, rather than comparing two functions.

Case $N = 3$:

Proposition A.5. For $N = 3$, the integral is given by

$$\mathfrak{I} = \int_{b_1}^{w^+_{1(2,3)}} dy_1 \int_{b_2}^{w^+_{(1)2(3)}} dy_2 \int_{b_3}^{w^+_{(1,2)3}} \mathcal{I}\, dy_3 +$$

$$\int_{b_1}^{w^+_{1(2,3)}} dy_1 \int_{w^+_{(1)2(3)}}^{w^+_{(1)2}} \mathcal{I} H(a_3^T \xi_2(b_1, w^+_{2(1,3)}) - b_3)\, dy_2 +$$

$$\int_{w^+_{1(2,3)}}^{w^+_{1(2)}} dy_1 \int_{b_2}^{w^+_{(1)2}} \mathcal{I} H(a_3^T \xi_2(w^+_{1(2,3)}, b_2) - b_3)\, dy_2 \quad +$$

$$\int_{w^+_{1(2)}}^{r} H(r\,\rho_{12} - b_2)\, H(r\,\rho_{13} - b_3)\, \mathcal{I}\, dy_1$$

Proof: We can write the integral as

$$\mathfrak{I} = \int_{b_1}^{r} dy_1 \int_{b_2 \vee w^-_{(1)2} \le y_2 \le w^+_{(1)2}} dy_2 \int_{b_3 \vee w^-_{(1,2)3} \le y_3 \le w^+_{(1,2)3}} \mathcal{I}\, dy_3.$$

By lemma 1, $w^+_{1(2,3)} \le w^+_{1(2)} \le r$ and $w^+_{(1)2(3)} \le w^+_{(1)2}$ so we know the order of possible breakpoints in the integral is correct. At $y_1 = w^+_{1(2,3)}$, we have by definition $b_2 = w^{\pm}_{(1)2(3)}$ and $b_3 = w^{\pm}_{(1,2)3}$. By lemmas A.2 and A.3 we know that for

$b_1 < y_1' < w^+_{1(2,3)}$, $w^-_{(1),2} < b_2 < w^+_{(1),2}$ and for $b_1 < y_1' < w^+_{1(2,3)}$ and $b_2 < y_2' < w^+_{(1)2(3)}$, $w^-_{(1,2),3} < b_3 < w^+_{(1,2),3}$.

Therefore, the integral can be rewritten as

$$\mathfrak{I} = \int_{b_1}^{w^+_{1(2,3)}} dy_1 \int_{b_2}^{w^+_{(1)2(3)}} dy_2 \int_{b_3}^{w^+_{(1,2)3}} \mathcal{I}\, dy_3 +$$

$$\int_{b_1}^{w^+_{1(2,3)}} dy_1 \int_{w^+_{(1)2(3)}}^{w^+_{(1)2}} dy_2 \int_{w^-_{(1,2)3}}^{w^+_{(1,2)3}} \mathcal{I}\; H(a_3^T \xi_2(b_1, w^+_{2(1,3)}) - b_3)\, dy_3 +$$

$$\int_{w^+_{1(2,3)}}^{r} dy_1 \int_{b_2 \vee w^-_{(1)2} \le y_2 \le w^+_{(1)2}} dy_2 \int_{b_3 \vee w^-_{(1,2)3} \le y_3 \le w^+_{(1,2)3}} \mathcal{I}\, dy_3.$$

When $w^+_{1(2,3)} < y_1' < w^+_{1(2)}$, $b_2 \le w^+_{(1)2}$ then by lemmas A.2 and A.3, either the third constraint cuts all of $W_{\{1,2\}} \cap S$ or it is redundant. That is, either $b_3 \ge w^+_{(1,2)3}$ or $b_3 \le w^-_{(1,2)3}$.

So,

$$\int_{w^+_{1(2,3)}}^{r} dy_1 \int_{b_2 \vee w^-_{(1)2} \le y_2 \le w^+_{(1)2}} dy_2 \int_{b_3 \vee w^-_{(1,2)3} \le y_3 \le w^+_{(1,2)3}} \mathcal{I}\, dy_3 =$$

$$\int_{w^+_{1(2,3)}}^{w^+_{1(2)}} dy_1 \int_{b_2}^{w^+_{(1)2}} dy_2 \int_{w^-_{(1,2)3}}^{w^+_{(1,2)3}} \mathcal{I}\; H(a_3^T \xi_2(w^+_{1(2,3)}, b_2) - b_3)\; dy_3 +$$

$$\int_{w^+_{1(2)}}^{r} dy_1 \int_{b_2 \vee w^-_{(1)2} \le y_2 \le w^+_{(1)2}} dy_2 \int_{b_3 \vee w^-_{(1,2)3} \le y_3 \le w^+_{(1,2)3}} \mathcal{I}\, dy_3.$$

For $y_1 \ge w^+_{1(2)}$, the integral depends on the second and third constraints separately. Note that lemmas two and three imply that if constraint three is redundant (makes the integral empty) for $w^+_{1(2,3)} \le y_1 \le w^+_{1(2)}$ and $b_2 \le y_2 \le w^+_{(1)2}$ then constraint three is redundant (makes the integral empty) for $y_1 \ge w^+_{1(2)}$. Therefore,

$$\int_{w^+_{1(2)}}^{r} dy_1 \int_{b_2 \vee w^-_{(1)2} \le y_2 \le w^+_{(1)2}} dy_2 \int_{b_3 \vee w^-_{(1,2)3} \le y_3 \le w^+_{(1,2)3}} \mathcal{I}\, dy_3 \quad =$$

$$\int_{w^+_{1(2)}}^{r} dy_1 \int_{w^-_{(1)2}}^{w^+_{(1)2}} dy_2 \int_{w^-_{(1,2)3}}^{w^+_{(1,2)3}} H(r\,\rho_{12} - b_2)\, H(r\,\rho_{13} - b_3)\, \mathcal{I}\, dy_3 =$$

$$\int_{w^+_{1(2)}}^{r} H(r\,\rho_{12} - b_2)\, H(r\,\rho_{13} - b_3)\, \mathcal{I}\, dy_1$$

Therefore,

$$\mathfrak{I} = \int_{b_1}^{w^+_{1(2,3)}} dy_1 \int_{b_2}^{w^+_{(1)2(3)}} dy_2 \int_{b_3}^{w^+_{(1,2)3}} \mathcal{I}\, dy_3 +$$

$$\int_{b_1}^{w^+_{1(2,3)}} dy_1 \int_{w^+_{(1)2(3)}}^{w^+_{(1)2}} dy_2 \int_{w^-_{(1,2)3}}^{w^+_{(1,2)3}} \mathcal{I}\, H(a_3^T \xi_2(b_1, w^+_{2(1,3)}) - b_3)\, dy_3 +$$

$$\int_{w^+_{1(2,3)}}^{w^+_{1(2)}} dy_1 \int_{b_2}^{w^+_{(1)2}} dy_2 \int_{w^-_{(1,2)3}}^{w^+_{(1,2)3}} \mathcal{I}\, H(a_3^T \xi_2(w^+_{1(2,3)}, b_2) - b_3)\, dy_3 \quad +$$

$$\int_{w^+_{1(2)}}^{r} dy_1 \int_{w^-_{(1)2}}^{w^+_{(1)2}} dy_2 \int_{w^-_{(1,2)3}}^{w^+_{(1,2)3}} H(r\,\rho_{12} - b_2)\, H(r\,\rho_{13} - b_3)\, \mathcal{I}\, dy_3 .$$

The above expression then simplifies into the desired result.

□

We show without proof the integral for $N = 4$.

$N = 4$

$$\mathfrak{I} = \int_{b_1}^{w^+_{1(2,3,4)}} dy_1 \int_{b_2}^{w^+_{(1)2(3,4)}} dy_2 \int_{b_3}^{w^+_{(1,2)3(4)}} dy_3 \int_{b_4}^{w^+_{(1,2,3)4}} \mathcal{I}\, dy_4 +$$

$$\int_{b_1}^{w^+_{1(2,3,4)}} dy_1 \int_{b_2}^{w^+_{(1)2(3,4)}} dy_2 \int_{w^+_{(1,2)3(4)}}^{w^+_{(1,2)3}} dy_3 \int_{w^+_{(1,2,3)4}}^{w^+_{(1,2,3)4}} \mathcal{I}\, H(a_4^T \xi_3(y_1,y_2,y_3) - b_4)\, dy_4 +$$

$$\int_{b_1}^{w^+_{1(2,3,4)}} dy_1 \int_{w^+_{(1)2(3,4)}}^{w^+_{(1)2(3)}} dy_2 \int_{b_3}^{w^+_{(1,2)3}} dy_3 \int_{w^+_{(1,2,3)4}}^{w^+_{(1,2,3)4}} \mathcal{I}\, H(a_4^T \xi_3(y_1,y_2,y_3) - b_4)\, dy_4 +$$

$$\int_{b_1}^{w^+_{1(2,3,4)}} dy_1 \int_{w^+_{(1)2(3)}}^{w^+_{(1)2}} dy_2 \int_{w^+_{(1,2)3}}^{w^+_{(1,2)3}} dy_3 \int_{w^+_{(1,2,3)4}}^{w^+_{(1,2,3)4}} \mathcal{I}$$
$$H(a_3^T \xi_2(y_1,y_2) - b_3)\, H(a_4^T \xi_2(y_1,y_2) - b_4)\, dy_4 +$$

$$\int_{w^+_{1(2,3,4)}}^{w^+_{1(2,3)}} dy_1 \int_{b_2}^{w^+_{(1)2(3)}} dy_2 \int_{b_3}^{w^+_{(1,2)3}} dy_3 \int_{w^+_{(1,2,3)4}}^{w^+_{(1,2,3)4}} \mathcal{I}\, H(a_4^T \xi_3(y_1,y_2,y_3) - b_4)\, dy_4 +$$

$$\int_{w^+_{1(2,3,4)}}^{w^+_{1(2,3)}} dy_1 \int_{w^+_{(1)2(3)}}^{w^+_{(1)2}} dy_2 \int_{w^+_{(1,2)3}}^{w^+_{(1,2)3}} dy_3 \int_{w^+_{(1,2,3)4}}^{w^+_{(1,2,3)4}} \mathcal{I}$$
$$H(a_3^T \xi_2(y_1,y_2) - b_3)\, H(a_4^T \xi_2(y_1,y_2) - b_4)\, dy_4 +$$

$$\int_{w^+_{1(2,3)}}^{w^+_{1(2)}} dy_1 \int_{b_2}^{w^+_{(1)2}} dy_2 \int_{w^+_{(1,2)3}}^{w^+_{(1,2)3}} dy_3 \int_{w^+_{(1,2,3)4}}^{w^+_{(1,2,3)4}} \mathcal{I}$$
$$H(a_3^T \xi_2(y_1,y_2) - b_3)\, H(a_4^T \xi_2(y_1,y_2) - b_4)\, dy_4 +$$
$$\int_{w^+_{1(2)}}^{r} dy_1 \int_{w^+_{(1)2}}^{w^+_{(1)2}} dy_2 \int_{w^+_{(1,2)3}}^{w^+_{(1,2)3}} dy_3 \int_{w^+_{(1,2,3)4}}^{w^+_{(1,2,3)4}}$$
$$H(r\,\rho_{14} - b_4)\, H(r\,\rho_{24} - b_4)\, H(r\,\rho_{34} - b_4)\;\; \mathcal{I}\, dy_4 \quad =$$

$$\int_{b_1}^{w^+_{1(2,3,4)}} dy_1 \int_{b_2}^{w^+_{(1)2(3,4)}} dy_2 \int_{b_3}^{w^+_{(1,2)3(4)}} dy_3 \int_{b_4}^{w^+_{(1,2,3)4}} \mathcal{I}\, dy_4 +$$

$$\int_{b_1}^{w^+_{1(2,3,4)}} dy_1 \int_{b_2}^{w^+_{(1)2(3,4)}} dy_2 \int_{w^+_{(1,2)3(4)}}^{w^+_{(1,2)3}} \mathcal{I}\, H(a_4^T \xi_3(y_1,y_2,y_3) - b_4)\, dy_3 +$$

$$\int_{b_1}^{w^+_{1(2,3,4)}} dy_1 \int_{w^+_{(1)2(3,4)}}^{w^+_{(1)2(3)}} dy_2 \int_{b_3}^{w^+_{(1,2)3}} \mathcal{I}\, H(a_4^T \xi_3(y_1,y_2,y_3) - b_4)\, dy_3 +$$

$$\int_{b_1}^{w^+_{1(2,3,4)}} dy_1 \int_{w^+_{(1)2(3)}}^{w^+_{(1)2}} \mathcal{I}\, H(a_3^T \xi_2(y_1,y_2) - b_3)\, H(a_4^T \xi_2(y_1,y_2) - b_4)\, dy_2 +$$

$$\int_{w^+_{1(2,3,4)}}^{w^+_{1(2,3)}} dy_1 \int_{b_2}^{w^+_{(1)2(3)}} dy_2 \int_{b_3}^{w^+_{(1,2)3}} \mathcal{I}\, H(a_4^T \xi_3(y_1, y_2, y_3) - b_4)\, dy_3 +$$

$$\int_{w^+_{1(2,3,4)}}^{w^+_{1(2,3)}} dy_1 \int_{w^+_{(1)2(3)}}^{w^+_{(1)2}} \mathcal{I}\, H(a_3^T \xi_2(y_1, y_2) - b_3)\, H(a_4^T \xi_2(y_1, y_2) - b_4)\, dy_2 +$$

$$\int_{w^+_{1(2,3)}}^{w^+_{1(2)}} dy_1 \int_{b_2}^{w^+_{(1)2}} \mathcal{I}\, H(a_3^T \xi_2(y_1, y_2) - b_3)\, H(a_4^T \xi_2(y_1, y_2) - b_4)\, dy_2 +$$

$$\int_{w^+_{1(2)}}^{r} H(r\,\rho_{14} - b_4)\, H(r\,\rho_{24} - b_4)\, H(r\,\rho_{34} - b_4)\;\; \mathcal{I}\, dy_1$$

$N = k.$

The general case follows by induction on the previous cases.

Theorem A.6.

Let $\alpha_i = \{1, ..., i-1\}$ and $\beta_{i,j} = \{i+1, i+2, ..., j-2, j-1, j\}$ for $i < j$.

Define $\mathfrak{m}_{i,j} = \begin{cases} w^+_{(\alpha_i)i(\beta_{i,j})} & \text{for } i \leq j \text{ and } j \leq N. \\ b_i & \text{for } j \geq N. \\ w^-_{(\alpha_i)i} & \text{for } j < i. \end{cases}$

and $k_0 = N$. Then

$$\mathfrak{I} = \prod_{i=1}^{N} \int_{b_i \vee w^-_{(\alpha_j)i} \leq y_2 \leq w^+_{(\alpha_j)i}} \mathcal{I}\, dy_i =$$

$$\prod_{n=1}^{N} \sum_{k_n = k_{n-1}}^{n} \int_{\mathfrak{m}_{n,k_n+1}}^{\mathfrak{m}_{n,k_n}} H(a_n^T \xi_{n-1} - b_n)\, \mathcal{I} dy_n$$

Proof: The proof for $N = 2$ is given in proposition 1 above. By induction

$$\prod_{n=1}^{N-1} \int_{b_i \vee w^-_{(\alpha_j)i} \leq y_2 \leq w^+_{(\alpha_j)i}} \mathcal{I}\, dy_i =$$

$$\prod_{n=1}^{N-1} \sum_{k_n = k_{n-1}}^{n} \int_{\mathfrak{m}_{n,k_n+1}}^{\mathfrak{m}_{n,k_n}} H(a_n^T \xi_{n-1} - b_n)\, \mathcal{I} dy_n \,,$$

so,

$$\prod_{n=1}^{N} \int_{b_i \vee w^-_{(\alpha_i)i} \le y_2 \le w^+_{(\alpha_i)i}} \mathcal{I}\, dy_i \;=$$

$$\prod_{n=1}^{N-1} \sum_{k_n=k_{n-1}}^{n} \int_{\mathfrak{m}_{n,k_{n+1}}}^{\mathfrak{m}_{n,k_n}} H(a_n^T \xi_{n-1} - b_n)\, dy_n \int_{b_N \vee w^-_{(\alpha_N)N} \le y_2 \le w^+_{(\alpha_N)N}} \mathcal{I}\, dy_N \,.$$

By lemma A.1, $w^+_{(\alpha_N)N} \le w^+_{(\alpha_N)N-1}$ so the breakpoints are established. By lemma A.2, if for some y_{α_N}, $b_N \ge w^+_{(\alpha_N)N}$ then $b_N \ge w^+_{(\alpha_N)N}$ for any $y'_{\alpha_N} > y_{\alpha_N}$. And by lemma A.3 if for some y_{α_N}, $b_N \le w^-_{(\alpha_N)N}$ then $b_N \le w^-_{(\alpha_N)N}$ for any $y'_{\alpha_N} > y_{\alpha_N}$. Then

$$\mathfrak{I} = \prod_{i=1}^{N} \int_{b_i \vee w^-_{(\alpha_i)i} \le y_2 \le w^+_{(\alpha_i)i}} \mathcal{I}\, dy_i \;=$$

$$\prod_{n=1}^{N} \sum_{k_n=k_{n-1}}^{n} \int_{\mathfrak{m}_{n,k_{n+1}}}^{\mathfrak{m}_{n,k_n}} H(a_n^T \xi_{n-1} - b_n)\, \mathcal{I} dy_n$$

□

Case Two: The vertex of W is outside the sphere.

Let us assume for the following discussion that $H_i \cap S$ is not empty for every i. Otherwise $W \cap S$ is empty and the integral is zero, or we can drop the non-intersecting constraints to obtain a lower dimensional problem. As a result, $b_1 < w^+_{1(\alpha_1)}$. When the vertex of W is outside the sphere the situation is more complicated because then we do not necessarily have $w^-_{(\alpha),i} < b_i < w^+_{(\alpha),i}$ for $y_\alpha = b_\alpha$, but we have:

$b_1 < w^-_{1(\alpha_1)}$. The general form of the integral is still

$$\mathfrak{I} = \prod_{i=1}^{k} \int_{b_i \vee w^-_{(\alpha_i)i} \leq y_2 \leq w^+_{(\alpha_i)i}} \mathcal{I}\, dy_i \,,$$

but we do not know in general whether we start off for a particular variable (other than the first) at b_i or $w^-_{(\alpha_i)i}$. It is easier to take the constraints in turn and reverse them one by one to calculate the surface area of the complement of $W \cap S$ in S. To facilitate the discussion we define yet more notation:

$W^r_\alpha = \{x : A_\alpha . x \leq b_\alpha\}$, that is W^r_α is W_α with the inequalities reversed

and

$$W^{r_j}_\alpha = W_{\alpha \setminus \{j\}} \cap W^r_{\{j\}}$$

So we can write for $N = k$,

$$\mathfrak{I} = A_k(r) - \sum_{i=1}^{k} \int_{W_{\alpha_i} \cap W^r_i \cap S} \mathcal{I} \prod_{j=1}^{i} dy_j$$

When $N = 2$ the hyperplanes must intersect outside the sphere, otherwise we would have the first case where the vertex of W is inside the sphere. Therefore,

$$\mathfrak{I} = A_2(r) - \int_{W^r_1 \cap S} \mathcal{I} dy_1 - \int_{W^r_2 \cap S} \mathcal{I} dy_2 =$$
$$A_2(r) - \int_{-r}^{b_1} \mathcal{I} dy_1 - \int_{-r}^{b_2} \mathcal{I} dy_2$$

In words, we can break down the $N = 2$ case into two one-dimensional problems.

For $N = 3$,

$$\mathfrak{I} = A_2(r) - \int_{W^r_1 \cap S} \mathcal{I} dy_1 - \int_{W_1 \cap W^r_2 \cap S} \mathcal{I} dy_1\, dy_2 - \int_{W_{\alpha_3} \cap W^r_3 \cap S} \mathcal{I} dy_1\, dy_2 dy_3.$$

We already know how to get the limits for the first two integrals in $\mathfrak{I}$, so we are left with the third integral:

$$\mathfrak{I}_3 = \int_{W_{\alpha_3} \cap W_3^r \cap S} \mathcal{I} dy_1\, dy_2 dy_3.$$

We may assume that $H_3 \cap H_1 \neq \emptyset$ and $H_3 \cap H_2 \neq \emptyset$ since otherwise we could drop the non-intersecting case and then the integral would be of dimension two and we would be done. Since the vertex is outside the sphere we have $w_{3(1,2)}^{-} > b_3$ and so by lemma A.3

$w_{(1,2)3}^{-} > b_3$ for $y_1 > b_1$ and $y_2 > b_2$. Therefore,

$$\begin{aligned}\mathfrak{I}_3 &= \int_{W_{\alpha_3} \cap W_3^r \cap S} \mathcal{I} dy_1 dy_2 dy_3 \\ &= \int_{W_1 \cap W_3^r \cap S} \mathcal{I} dy_1 dy_3 - \int_{W_2^r \cap W_3^r \cap S} \mathcal{I} dy_2 dy_3.\end{aligned}$$

Thus $\mathfrak{I}_3$ is the difference of two two-dimensional integrals for which we already can derive the limits of integration.

When $N = k$ we can inductively calculate all the limits for each part of the sum up to the last summand:

$$\mathfrak{I}_k = \int_{W_{\alpha_k} \cap W_k^r \cap S} \mathcal{I} \prod_{j=1}^{k} dy_j.$$

Here we apply the trick developed for $\mathfrak{I}_3$ to get

$$\mathfrak{I}_k = \int_{W_{\alpha_k} \cap W_k^r \cap S} \mathcal{I} \prod_{j=1}^{k} dy_j = \int_{W_{\alpha_k} \cap W_{\{k-1,k\}}^r \cap S} \mathcal{I} \prod_{j=1}^{k} dy_j + \int_{W_{\alpha_{k-1}} \cap W_k^r \cap S} \mathcal{I} \prod_{j=1}^{k} dy_j.$$

The second integral in the sum has dimension $k-1$, so we may consider it done and concentrate on the first integral. We handle the first integral by reversing another constraint. We continue to reverse constraints until either one of the non-reversed constraints is redundant (so we get a $k-1$) dimensional case, or we have reversed all the constraints and then the intersection with the sphere is empty. So,

$$\mathfrak{I}_k = \sum_{i=k}^{2} \int_{W_{\alpha_{i-1}} \cap W^r_{\{i,\ldots,k\}} \cap S} \mathcal{I} \prod_{j=1}^{k} dy_j$$

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