Universal modal radiation laws for all thermal emitters

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Contributed by David A. B. Miller, March 12, 2017 (sent for review January 31, 2017; reviewed by John B. Pendry and Eli Yablonovitch)

We derive four laws relating the absorptivity and emissivity of thermal emitters. Unlike the original Kirchhoff radiation law derivations, these derivations include diffraction, and so are valid also for small objects, and can also cover nonreciprocal objects. The proofs exploit two recent approaches. First, we express all fields in terms of the mode-converter basis sets of beams; these sets, which can be uniquely established for any linear optical object, give orthogonal input beams that are coupled one-by-one to orthogonal output beams. Second, we consider thought experiments using universal linear optical machines, which allow us to couple appropriate beams and black bodies. Two of these laws can be regarded as rigorous extensions of previously known laws: One gives a modal version of a radiation law for reciprocal objects—the absorptivity of any input beam equals the emissivity into the “backward” (i.e., phase-conjugated) version of that beam; another gives the overall equality of the sums of the emissivities and the absorptivities for any object, including nonreciprocal ones. The other two laws, valid for reciprocal and nonreciprocal objects, are quite different from previous relations. One shows universal equivalence of the absorptivity of each mode-converter input beam and the emissivity into its corresponding scattered output beam. The other gives unexpected equivalences of absorptivity and emissivity for broad classes of beams. Additionally, we prove these orthogonal mode-converter sets of input and output beams are the ones that maximize absorptivities and emissivities, respectively, giving these beams surprising additional physical meaning.

Kirchhoff radiation laws | thermal radiation | optical modes | solar energy conversion | mode conversion

Radiation laws relating the absorptivity and emissivity of an object are at the core of the thermal physics of radiation and are particularly interesting for understanding limits to efficiency of solar energy conversion (e.g., refs. 1 and 2), for example. The core relation is Kirchhoff’s radiation law (3–6), which equates the absorptivity and emissivity of a surface. This law is often extended to a “directional” version, which equates the absorptivity of a given direction of input beam and the emissivity into the opposite or “reversed” direction. Typical textbook approaches (5, 6) to the directional law trace back to Planck’s rederivation (4) of Kirchhoff’s approach (3). Both Kirchhoff’s and Planck’s treatments explicitly make two assumptions: (i) The optical properties of the object are reciprocal (e.g., excluding Faraday rotation); (ii) diffraction is neglected, presuming objects much larger than a wavelength and using ray rather than wave optics. It is, however, now known that such a directional radiation law is not correct for nonreciprocal systems (1, 7), and nanophotonic structures for the control of thermal radiation (8–22) can have feature sizes much smaller than the wavelength. Given the fundamental thermodynamic importance and the technological relevance of radiation laws, we need to understand just what are the valid laws that cover nonreciprocal behavior and subwavelength structures and whether there are some deeper universal laws for all linear optical systems. Fundamental constraints on nonreciprocal thermal radiation are specifically important in determining the limit of thermal energy conversion. For example, it is known that the Landsberg limit for conversion of thermal radiation (such as solar radiation) to electricity (23), which represents the upper thermodynamic limit in terms of efficiency for such conversion, can be achieved only in systems where the reciprocity is broken (24, 25).

Here, we take an approach using orthogonal beams or “modes” to describe the radiation, thus automatically including diffraction. We derive three related and general laws, valid for reciprocal and nonreciprocal linear optical objects. These give equivalences of absorptivities and emissivities of broad classes of modes, beyond anything anticipated from the previous Kirchhoff laws. Then, adding reciprocity, we prove a fourth law—a rigorous “modal” version of the directional Kirchhoff law for reciprocal systems, showing that the absorptivity from any mode or beam is equal to the emissivity into its “backward” (strictly, phase-conjugated) version.

Approach

Our approach uses two recently proposed concepts in constructing our proofs. First, we use the “mode-converter” basis sets to describe the fields (26). These sets, which can be derived for any linear optical object, give a complete set of orthogonal input fields or modes that couple one by one, through interaction with the object, to a corresponding complete set of orthogonal output fields or modes. These specific modes, which implicitly obey all diffraction phenomena, are essential here: we use them in constructing the proofs, two of the related general laws we derive have to be stated in terms of them, and the discussion here gives an additional physical meaning to them in terms of absorption and emission. Second, we use a universal linear optical component or machine (27) that allows us to construct useful thought experiments.

As in the original Kirchhoff derivation (3), we presume we are working at or near one (angular) frequency ω of radiation so all of the functions we use below are then essentially only spatial functions (with also vector polarization character). For mathematical

Significance

Radiation laws must relate the fraction of incident radiation absorbed by an object and the amount of radiation emitted when it is hot so that objects can come to the same temperature just by exchanging electromagnetic radiation. Such laws are fundamentally important and set limits to practical applications such as in the conversion of light to electricity and in heat and thermal management generally. Kirchhoff’s classic results work well in many situations, but fail in others (specifically for “nonreciprocal” materials), and were derived using simplified models that do not apply to modern nanotechnology and light beams. We derive revised versions of laws that avoid these problems and discover additional and unexpected radiation laws that substantially expand the fundamental relations between optical absorption and emission.

Author contributions: D.A.B.M., L.Z., and S.F. designed research, performed research, contributed new analytic tools, and wrote the paper.

Reviewers: J.B.P., Imperial College London; and E.Y., University of California, Berkeley. The authors declare no conflict of interest.

Ffreely available online through the PNAS open access option.

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This article contains supporting information online at www.pnas.org/lookup/suppl/doi:10.1073/pnas.1701606114/-/DCSupplemental.
convenience, we use complex fields; as usual, at the end we can return to real fields by taking the real part. We can consider all such spatial functions to be multiplied by a monochromatic time-varying function exp(iωt). More physically, we can presume we are considering a very small bandwidth around such a center frequency and a time duration of interest that is long enough to consider equilibrium but short enough that we do not notice any phase differences between frequencies within this bandwidth over the time of interest. In what follows, we presume we have corresponding frequency filters, which pass this narrow bandwidth perfectly and reflect all other wavelengths, in front of all black bodies we use in our arguments.

**Modes, Basis Sets, and Singular-Value Decomposition.** To start, we consider how a linear optical object \( O \) takes an incident input field \( |\psi_{\text{In}}\rangle \) and turns it into an output field of the form \( |\psi_{\text{Out}}\rangle \) (Fig. 1) as a result of a linear interaction that we represent with a linear operator or “object” matrix \( S \) (Terminology for the Matrix \( S \)). (This interaction does not include thermal emission from the object, which we add later.) So, mathematically,

\[
|\psi_{\text{Out}}\rangle = S|\psi_{\text{In}}\rangle
\]

(Note we use Dirac bra-ket linear algebra notation here—see Dirac “Bra-Ket” Notation for its standard definitions and properties. This means we can avoid explicit statements of vector or polarization properties, although such properties are implicitly covered. This generality means our analysis also covers other linear fields, such as acoustic waves.)

For any such matrix \( S \), it is always possible to perform the singular value decomposition (SVD) (26, 28). One statement of such a decomposition is

\[
S = V_S D_{\text{diag}} U_S,
\]

where \( U_S \) and \( V_S \) are unitary matrices and \( D_{\text{diag}} \) is a diagonal matrix. We can also rewrite this as

\[
S = \sum_p s_p |\psi_{\text{Out}p}\rangle \langle \psi_{\text{In}p}|,
\]

Formally, the singular values \( s_p \) are the diagonal elements of \( D_{\text{diag}} \), \( |\psi_{\text{In}p}\rangle \) are the rows of \( U_S \), and \( |\psi_{\text{Out}p}\rangle \) are the columns of \( V_S \). The set of orthogonal input functions or “mode-converter input modes” \( \{|\psi_{\text{In}n}\rangle\} \) constitutes the so-called right singular vectors of \( S \) and can be deduced as the eigenfunctions of \( S^*S \); that is, the solutions of

\[
S^*S|\psi_{\text{In}n}\rangle = |s_n|^2|\psi_{\text{In}n}\rangle
\]

and similarly the set of orthogonal output functions or “mode-converter output modes” \( \{|\psi_{\text{Out}n}\rangle\} \) constitute the left singular vectors of \( S \) and are the eigenfunctions of \( SS^* \), as in

\[
SS^*|\psi_{\text{Out}p}\rangle = |s_p|^2|\psi_{\text{Out}p}\rangle.
\]

The specific elements in the two sets are related by

\[
S|\psi_{\text{In}p}\rangle = s_p |\psi_{\text{Out}p}\rangle
\]

and

\[
S^*|\psi_{\text{Out}p}\rangle = s_p^* |\psi_{\text{In}p}\rangle.
\]

These SVD function sets \( \{|\psi_{\text{In}p}\rangle\} \) and \( \{|\psi_{\text{Out}p}\rangle\} \) have an important physical meaning: When one of the orthogonal input beams or functions \( |\psi_{\text{In}p}\rangle \) interacts with the object, it leads to one and only one of the orthogonal output beams or functions \( |\psi_{\text{Out}p}\rangle \), with a complex amplitude coupling strength \( s_p \) (Fig. 1). Equivalently, each such orthogonal input mode \( |\psi_{\text{In}p}\rangle \) is mapped or “converted” by the object to give an amplitude only in the corresponding orthogonal output mode \( |\psi_{\text{Out}p}\rangle \), thus defining a set of orthogonal “channels” “through” the object. Hence every such linear optical device can be described as a mode converter (26).

Other than the usual arbitrariness of linear combinations of any possible degenerate eigensolutions, these sets of functions are unique and complete; no other complete set of orthogonal input functions generates orthogonal output functions, and these orthogonal “mode-converter pairs” \( |\psi_{\text{In}p}\rangle \text{ and } |\psi_{\text{Out}p}\rangle \) always exist and can be used as basis sets to describe any possible linear relation from input to output. In contrast to plane wave basis descriptions, these beams can be straightforwardly normalized to represent beams of unit power. With unit input power in \( |\psi_{\text{In}p}\rangle \) from Eq. 6 the resulting amplitude in the output beam of form \( |\psi_{\text{Out}p}\rangle \) is \( |s_p|^2 \).

**A Mode-Separating Machine.** For our thought experiments and proofs here, we work with “single-mode” black bodies. Such a black body allows only one optical mode (at a given frequency and polarization state) to propagate in and out of it. (see Existence and Required Properties of Black Bodies for an extended discussion). For example, we could imagine this black body is connected to the external world only through a single-mode optical fiber that also allows only one polarization state to propagate.

We want to take the output from such a single-mode black body and losslessly construct from it any desired input wave \( |\psi_I\rangle \) to shine on the object \( O \). We also want simultaneously to be able losslessly to collect any wave in any desired output beam mode \( |\psi_O\rangle \) and route it as the input back into such a single-mode black body. In this way, we can consider thermal equilibrium between the object \( O \) and such black bodies.

In our thought experiments, we consider multiple single-mode black bodies and multiple orthogonal input fields to object \( O \) and orthogonal output fields from object \( O \) that are connected losslessly in and out of these black bodies. Because all of these various input and output fields may be overlapping in space, the question arises whether we can make an optical machine that can losslessly perform all of the necessary transformations and combinations. Fortunately, recent work (27) shows it is possible, at least in principle, to design and make such a universal lossless linear optical machine.

Here, we presume we can in principle make such an optical machine. The function of this optical machine is, first, to take light in specific single-mode input ports and map each such input into a specific desired input wave incident on the object \( O \) inside a free-space region \( F \). The resulting waves on the object can be chosen to be any orthogonal set, and specifically these waves can be completely overlapping in space. Second, the machine can collect light in any specific orthogonal set of waves from the object, including waves that can be completely overlapping, and map each such orthogonal output wave to a different specific output port on the right of the machine (see Internal Structure of a...
singular values mean strong absorption, as is clarified below.) We presume we have chosen $N$ large enough that these sets describe all of the waves of interest to a sufficient degree of approximation.

Machine $M$ has $N$ input ports and output ports that we can think of as single-mode waveguides. Because of the machine’s design, we have waves only entering the input ports and exiting the output ports; we never have waves in the other directions in these ports. (The machine itself, because it separates forward and backward waves, is nonreciprocal, although that nonreciprocity does not mean we are restricting object $O$ to be either reciprocal or nonreciprocal.) A unit amplitude in input port $p$ is converted by the machine $M$ to unit amplitude in the input mode-converter basis wave $|\psi_{\text{In},p}\rangle$ inside the machine, which is then incident on object $O$. Similarly, a unit amplitude of wave in the mode-converter output function $|\psi_{\text{Out},p}\rangle$ is converted by the machine $M$ into unit amplitude in output port $p$. The circulator $C_p$ routes the output light from port $p$ to be the input light to black body $B_p$, and routes the output light from $B_p$ to input port $p$. In what follows, we can think of black body $B_p$ as being $B_1$ at the top of Fig. 2, but any one of the black bodies can be chosen.

**Derivation of Modal Radiation Laws**

**Absorptivity and Emissivity of a Mode-Converter Pair.** Now, using machine $M$ of Fig. 2, we derive a relation between the absorptivity of a given mode-converter input wave $|\psi_{\text{In},p}\rangle$ and the emissivity into the corresponding scattered mode-converter output wave $|\psi_{\text{Out},p}\rangle$. Because of our use of the mode-converter basis sets in machine $M$, black body $B_p$ receives only power that is scattered (into output mode $|\psi_{\text{Out},p}\rangle$) from the input mode $|\psi_{\text{In},p}\rangle$—power that itself comes only from the black body $B_p$ output power—plus any power that is emitted into this output mode $|\psi_{\text{Out},p}\rangle$ from the object $O$. Here we are using the mode-converter basis sets precisely because we want to avoid considering any scattering from other input modes into the output mode of interest and any emitted power in other output modes; none of the output power from any of the other black bodies is scattered back into the black body $B_p$.

All power in the input wave $|\psi_{\text{In},p}\rangle$ comes from the output of black body $B_p$. For an incident power $P_{\text{in}}$ in a wave of the form $|\psi_{\text{In},p}\rangle$, from Eq. 6 the scattered output power $P_{\text{scat}}$ (which is all in the output wave form of $|\psi_{\text{Out},p}\rangle$) is

$$P_{\text{scat}} = |s_p|^2 P_{\text{in}}.$$  

[8]

All of the power not scattered is absorbed by the lossy object $O$. So the absorbed power is $P_{\text{abs}} = (1 - |s_p|^2) P_{\text{in}}$. So the fraction of the incident power absorbed is, by definition, the absorptivity

$$\alpha_{M,p} = \frac{P_{\text{abs}}}{P_{\text{in}}} = 1 - |s_p|^2.$$  

[9]

Now, because we presume the object $O$ to be at some nonzero temperature $T_{eq}$, it is emitting radiation. By definition of the emissivity $e_{M,p}$ for this mode, the emitted thermal power into mode $|\psi_{\text{Out},p}\rangle$ (at the frequency $\omega$ or in the narrow frequency range of interest around it) is $P_{\text{therm}} = e_{M,p} P_{\text{abs}}$, where $P_{\text{abs}}$ is the power that a black body at this same temperature $T_{eq}$ would emit into a single mode at this frequency or in the narrow frequency range of interest.

The total power going back into black body $B_p$ is the sum of this thermal emitted power in mode $|\psi_{\text{Out},p}\rangle$ and the power scattered into mode $|\psi_{\text{Out},p}\rangle$. Using Eq. 8 with $P_{\text{in}} = P_{\text{abs}}$, the total power going back into black body $B_p$ is therefore

$$P_{\text{Bin}} = e_{M,p} P_{\text{abs}} + |s_p|^2 P_{\text{abs}}.$$  

[10]

But if the black body $B_p$ is to be in thermal equilibrium with the object $O$ at this temperature $T_{eq}$ this input power $P_{\text{Bin}}$ to the
black body must equal its output power, which is \( P_B \); i.e., \( P_{B_{in}} = P_B \) in thermal equilibrium. So, from Eq. 10 and dividing both sides by \( P_B \) and rearranging,

\[
\varepsilon_{M_p} = 1 - |s_p|^2. \tag{11}
\]

But comparing this to Eq. 9, we see that

\[
\varepsilon_{M_p} = \alpha_{M_p}. \tag{12}
\]

This gives an important first result that is different from Kirchhoff’s results.

**Law 1.** For an object \( O \) with mode-converter input and output mode sets \( \{\psi_{In}\} \) and \( \{\psi_{Out}\} \), respectively, the absorptivity \( \alpha_{M_p} \) from any such input mode \( \psi_{In} \) equals the emissivity \( \varepsilon_{M_p} \) into the corresponding output mode \( \psi_{Out} \).

We can also prove the converse of this statement (Proof of the Converse to the First Law and Fig. S2).

If the absorptivity \( \alpha_{M_p} \) from an input mode equals the emissivity \( \varepsilon_{M_p} \) into the corresponding scattered output mode, then these input and output modes are a mode-converter pair.

Note that Law 1 and, mathematically, Eq. 12 are not the conventional statement of Kirchhoff’s directional radiation law; that refers to the absorptivity from one mode and the emissivity into the backward (i.e., the phase-conjugated) version of the same mode. The equality in our Law 1 here is between the absorptivity of the mode-converter input mode and the emissivity into the corresponding mode-converter output mode—that is, the output mode into which the input mode scatters.

This approach makes no assumptions about the scattering object other than that it is linear. Specifically, that object can also be nonreciprocal, so we have derived a condition for absorptivity and emissivity of nonreciprocal as well as reciprocal objects.

**Universal Modal Radiation Law.** Now consider that we want to connect machine \( Q \) that maps the input ports one by one to some possibly different set of orthogonal input functions \( \{\psi_{QIn}\} \) and maps some possibly different set of orthogonal output functions \( \{\psi_{QOut}\} \) one by one to the corresponding output ports. Formally, we can regard these new functions as being related by arbitrary unitary transforms \( U \) and \( V \) to the mode-converter basis functions, i.e., \( \psi_{QIn} = U\psi_{In} \) and \( \psi_{QOut} = V\psi_{Out} \), and we can consider these sets to span the same spaces as the sets \( \{\psi_{In}\} \) and \( \{\psi_{Out}\} \).

In general, if we had a unit input power in a wave \( \psi_{In} \) from a given black body \( B_p \), we would have a scattered output vector \( \psi_{QIn} = S\psi_{In} \) power \( \psi_{QIn} = \psi_{QIn}^*S\psi_{In} \). So the fraction of this unit input power that is absorbed is

\[
\alpha_{Q} = 1 - \langle \psi_{QIn}^*S\psi_{In} \rangle, \tag{13}
\]

which by definition is the absorptivity for this input mode \( \psi_{QIn} \).

At this point, we can usefully define a matrix \( A \),

\[
A = I_{O} - S^{1/2}, \tag{14}
\]

where \( I_{O} \) is the identity matrix in the input space. Then Eq. 13 can be written \( \alpha_{Q} = \langle \psi_{QIn}^*A\psi_{QIn} \rangle \). Because the choice of the actual input mode onto the object here is arbitrary (because \( U \) is arbitrary), we can write for any input mode \( |i\rangle \) on the object that the corresponding absorptivity is

\[
\alpha_{i} = \langle i|A|i\rangle \tag{15}
\]

and for this reason we can call \( A \) the “absorptivity” matrix.

Now, in the situation in Fig. 2 with machine \( Q \), the absorbed power in black body \( B_p \) consists of the emission of the object \( O \) to it, as well as the parts of the emitted power from all of the other bodies that get scattered by object \( O \) into \( B_p \). In thermal equilibrium of all of the black bodies with the object \( O \) at some temperature \( T_{eq} \), that absorbed power into \( B_p \) must balance the emitted power \( P_B \) from \( B_p \). By definition of the object matrix \( S \), for unit amplitude in an input port \( n \), the amplitude that is scattered to an output port \( m \) is the corresponding matrix element \( \langle \psi_{QOm}|S|\psi_{QIn}\rangle \equiv S_{nm} \). Hence, the fraction of the power from black body \( B_m \) that is scattered to black body \( B_m \) is \( |S_{nm}|^2 \) and so a scattered power of \( P_B \) is \( |S_{nm}|^2 \). Because all of these output modes \( \langle \psi_{QOm}\rangle \) are by choice orthogonal, the total scattered power from all of the other black bodies to black body \( B_m \) is the sum of these powers over all of the output modes; that is,

\[
P_B \sum_{n} |S_{nm}|^2 \equiv P_B \sum_{n} \langle \psi_{QOm}|S|\psi_{QIn}\rangle \langle \psi_{QIn}|S^\dagger|\psi_{QOm}\rangle.
\]

Here we have used the equivalence \( \sum_{n} \langle \psi_{QOn}|S|\psi_{QIn}\rangle \equiv I_{O} \) (i.e., the identity operator for the input space). By definition of the emissivity \( \varepsilon_{QOn} \), the emitted power into the output mode \( \langle \psi_{QOm}\rangle \) is \( \varepsilon_{QOm}P_B \), so the equilibrium power balance in black body \( B_m \) requires

\[
P_B = \varepsilon_{QOm}P_B + P_B \sum_{n} |S_{nm}|^2. \tag{17}
\]

Equivalently, if we define a matrix \( E \),

\[
E = I_{O} - S^1, \tag{18}
\]

and for this reason we can call \( E \) the “emissivity” matrix.

Now, consider a specific arbitrary normalized input mode \( |i\rangle \), which can be expanded on the mode-converter input basis functions as

\[
|i\rangle = \sum_{m} c_m |\psi_{In}^m\rangle. \tag{20}
\]

From Eq. 15 and using Eqs. 4 and 9, we have

\[
\alpha_{i} = \sum_{n,m} c_m^*c_n \langle \psi_{In}^m|I_{O} - S^1|\psi_{In}^n\rangle
\]

\[
= \sum_{n,m} c_m^*c_n (1 - |s_m|^2) \langle \psi_{In}^m|\psi_{In}^n\rangle
\]

\[
= \sum_{n,m} c_m^*c_n (1 - |s_m|^2) \delta_{mn} = \sum_{m} |c_m|^2 (1 - |s_m|^2)
\]

\[
= \sum_{m} |c_m|^2 \alpha_{inm}. \tag{21}
\]

Let us also construct a (normalized) output mode \( |j\rangle \) expanded on the mode-converter output basis functions

\[
|j\rangle = \sum_{m} c_m \exp(i\theta_{inm}) |\psi_{Out}^m\rangle. \tag{22}
\]

Explicitly now we allow additional arbitrary phase factors \( \exp(i\theta_{inm}) \); the \( \theta_{inm} \) are arbitrary (real) phases associated with each expansion coefficient \( c_m \). This means that, although we are constructing this output mode of interest with expansion coefficients that have the same magnitudes as the expansion coefficients for the input mode \( |i\rangle \), we are allowing these output mode expansion
coefficients to have arbitrary phases. From Eqs. 19 and 22, using Eqs. 8 and 11,
\[ e_f = \sum_{n,m} c_{nm}^s \exp(-i\theta_n) c_{nm} \exp(i\theta_m)(\psi_{\text{Out},1} - SS^\dagger|\psi_{\text{Out},m}|) \]
\[ = \sum_{n,m} c_{nm}^s \exp(-i\theta_n) c_{nm} \exp(i\theta_m) (1 - |s_n|^2) \delta_{nm} = \sum_{m} c_{m}^s \exp(i\theta_m). \]

[23]
So, finally, using our Law 1 (and as stated in Eq. 12) [or, equivalently, noting the same (1 - |s_n|^2) factors in the sums in both Eqs. 21 and 23],
\[ e_f = a_1, \]
which is the second result of our paper. Note here that the phases \( \theta_m \) do not matter; all that does matter is that for our input mode \( |i\) \) of interest, its power splitting between the input mode-converter modes is the same as the power splitting between the output mode-converter modes in our output mode \( |f\) \) of interest. Hence entire broad sets of input modes have emissivities equal to an entire broad set of output modes. We can state this law as follows.

**Law 2.** For an output beam whose power is split in given fractions among the mode-converter output modes, its emissivity is the same as the absorptivity of an input beam whose power is split in the same fractions among the corresponding mode-converter input modes.

Law 2 is not anticipated in the original Kirchhoff approach, and it is valid for reciprocal and nonreciprocal objects.

We can also draw two corollaries from this argument, which are immediately apparent from Eqs. 21 and 23, respectively, and both follow simply from Law 2.

**Law 2, corollary 1.** All input beams whose power is split in the same fractions among the mode-converter input modes have the same absorptivity.

**Law 2, corollary 2.** All output beams whose power is split in the same fractions among the mode-converter output modes have the same emissivity.

Incidentally, our Law 1 above (also Eq. 12), can now be seen to be a special case of this Law 2 for the case where one of the \( |c_n|^2 = 1 \) and all of the others are therefore zero.

**Alternative Interpretation of Mode-Converter Input and Output Modes.** Note that the output mode-converter basis functions are also eigenfunctions of \( E \) and the input mode-converter basis functions are also eigenfunctions of \( A \). Explicitly, from Eqs. 4 and 5
\[ A|\psi_{\text{In},1}\rangle = (I - S^*S)|\psi_{\text{In},1}\rangle = (1 - |s_1|^2)|\psi_{\text{In},1}\rangle \]  
[25]
\[ E|\psi_{\text{Out},1}\rangle = (I_0 - SS^*)|\psi_{\text{Out},1}\rangle = (1 - |s_1|^2)|\psi_{\text{Out},1}\rangle. \]  
[26]
By the maximization properties of eigenvalues for Hermitian operators, this means that the input mode-converter basis functions \( \{|\psi_{\text{In},1}\rangle\} \) are the orthogonal functions that correspond to the maximum absorptivities, and the corresponding output mode-converter basis functions \( \{|\psi_{\text{Out},1}\rangle\} \) are the orthogonal functions that correspond to the maximum emissivities. In principle, these sets of functions could be found in a set of experiments designed to find the orthogonal beams with the maximum absorptivities and emissivities, which gives an alternative physical meaning to these sets of beams. For example, we could empirically and iteratively first find the beam with the largest absorptivity, then the beam orthogonal to that with the next largest absorptivity, and so on, which would physically construct the set \( \{|\psi_{\text{In},1}\rangle\} \). We could follow a similar procedure with emissivities to construct \( \{|\psi_{\text{Out},1}\rangle\} \). Note this conceptual approach works for both reciprocal and nonreciprocal objects, in contrast to one other physical method for iteratively establishing these mode-converter functions that may work only for reciprocal objects (29).

**An Integrated Radiation Law.** Note from [25] that
\[ \text{Tr}(A) \equiv \sum_p \langle \psi_{\text{In},p}|A|\psi_{\text{In},p}\rangle = \sum_p E_{mp} \]
\[ = \sum_p \langle \psi_{\text{In},p} |(1 - |s_p|^2)|\psi_{\text{In},p}\rangle = \sum_p (1 - |s_p|^2). \]
[27]
and from [26] that
\[ \text{Tr}(E) \equiv \sum_p \langle \psi_{\text{Out},p}|E|\psi_{\text{Out},p}\rangle = \sum_p E_{mp} \]
\[ = \sum_p \langle \psi_{\text{Out},p} |(1 - |s_p|^2)|\psi_{\text{Out},p}\rangle = \sum_p (1 - |s_p|^2). \]
[28]
where by “Tr” we mean the trace of the operator (i.e., the sum of its diagonal elements). Hence, the sum of the absorptivities of the mode-converter input modes and the sum of the emissivities of the mode-converter output modes are equal. More generally, though, because the trace of an operator is independent of the (complete) basis set used to represent it, and because, as in [15] and [19], the absorptivities and emissivities on any basis sets of input and output modes, respectively, are given just by the diagonal elements on those basis sets, then from [27] and [28]
\[ \text{Tr}(A) = \text{Tr}(E). \]
[29]
so we can state an integrated radiation law, valid for both reciprocal and nonreciprocal objects:

**Law 3.** For an arbitrary linear object, the sum of the absorptivities of any complete set of input modes is equal to the sum of the emissivities of any complete set of output modes.

**A Modal Radiation Law for Reciprocal Systems.** For Law 4, we first define reciprocity; we do this directly, rather than by appeal to other laws of physics (such as Maxwell's equations), although such other laws can clarify what systems show reciprocity. If we shine a normalized input field \( |\psi_i\rangle \) on a linear scattering object, then we can write the resulting scattered field as a complex amplitude \( A |\psi_s\rangle \), i.e., a field \( s|\psi_s\rangle \). Then we define reciprocity for this object as follows:

Given that an arbitrary input field \( |i\rangle \) gives rise to a scattered field \( s(f) = 3|s\rangle \) from an object \( O |i\rangle \) and \( |f\rangle \) both being normalized, then object \( O \) is reciprocal if and only if input field \( |f\rangle \), which is the phase conjugate of \( |f\rangle \), gives rise to a scattered field \( s(i) = 3|s\rangle \), where \( |s\rangle \) is the phase conjugate of \( |s\rangle \).

By “phase conjugate” here we mean that, for a complex field, we take the complex conjugate of the spatial part of the field, although not the temporal part. So, for example, the phase conjugate of a “right-going” plane wave \( \exp[i(ot + kx)] \) is the “left-going” plane wave \( \exp[i(ot - kx)] \) and the phase conjugate of a beam expanding to the “right” from a beam waist would converge from the right back onto the original waist (30). [Note, incidentally, that such phase conjugation is not necessarily or generally the same as time reversal (30, 31).]

For a reciprocal electromagnetic system as described by symmetric permittivity and permeability tensors, its scattering matrix, expressed on (real) input and output mode basis sets that satisfy \( |i\rangle = |i\rangle \) and \( |f\rangle = |f\rangle \), is symmetric (31). Our reciprocity definition is equivalent to the standard description of the scattering matrix of a reciprocal system and is more appropriate in the present situation because we do not assume the modal basis to be real.

Note that the constant \( s \) is the same in both the “forward” and backward (phase-conjugated) versions, which means that the backward version is attenuated by the same real factor as the forward version in such scatterings and that the phase delay in the forward scattering is the same as the phase delay in the backward scattering.
For a reciprocal scatterer, because the reciprocity conditions apply for any input field, they must apply to the mode-converter basis functions. In particular, from Eq. 6 and our reciprocity definition,\[ S(\psi_{\text{out}} p) = S(p, \psi_{\text{in}} p). \] (30)

(Note we do not use the complex conjugate of $s_p$ in Eq. 30, as stated above, both the phase delay and the magnitude of the scattering must be the same for the forward and backward versions of the scattering by a reciprocal scatterer.)

These resulting sets of functions $\{\psi_{\text{out}} p\}$ and $\{\psi_{\text{in}} p\}$ are necessarily complete sets for describing input and output functions, respectively; they are orthonormal because the sets $\{\psi_{\text{out}} p\}$ and $\{\psi_{\text{in}} p\}$ are, and they have the same number of elements, with the same singular values. Hence, for this reciprocal case, they form equally good sets of mode-converter pairs and have the same kinds of mathematical properties as those original sets. (The reason we have some choice about how we write these sets is because each eigenvalue $|s_p|^2$ in [4] and [5] is necessarily twofold degenerate in this reciprocal case; both the input function $\psi_{\text{in}} p$ and the input function $\psi_{\text{out}} p$ have the same $|s_p|^2$, as do the output functions $\psi_{\text{out}} p$ and $\psi_{\text{in}} p$.)

Hence, as in Eq. 7, we can write \[ S(\psi_{\text{in}} p) = s_p^* S(\psi_{\text{out}} p). \] [31]

Hence for some input mode $\psi_i$, as in Eq. 20, with absorptivity $\alpha$, as in Eq. 21, the emissivity of $\psi_i$ can be calculated using Eqs. 9, 19, and 31 as \[ e_{\psi_i} = 1 - (1 - \alpha) S(\psi_i) = \sum_m |c_m|^2 \left(1 - |s_m|^2\right) = \alpha. \] [32]

Hence, for a reciprocal object, not only have we now proved the standard statement of the directional version of Kirchhoff’s radiation law, that the absorptivity for any given direction of beam is equal to the emissivity back into that direction, but also we have proved a more general version, valid for any form of input beam.

**Law 4.** For any given input beam $|\psi_i\rangle$ at a given frequency incident on a reciprocal object, the absorptivity is equal to the emissivity into the phase-conjugated version $|\psi_i^*\rangle$ of that beam.

This therefore is a fully modal statement of a radiation law for reciprocal thermal emitters and absorbers; it applies for any input beam and the corresponding reversed beam, not just for plane waves, and it is derived with all diffraction effects fully included. We also note that as indicated by Law 2, $|\psi_i^*\rangle$ is only one of the many possible modes to which the emissivity is equal to the absorptivity of the input mode $|\psi_i\rangle$. Thus, our result, Law 2, generalizes the standard Kirchhoff’s law even in the reciprocal case.

**Conclusions**

Here we have derived four laws for thermal radiation. These give general laws for the absorptivity and emissivity of modes of electromagnetic radiation for linear optical objects, with Laws 1, 2, and 3 applying for both reciprocal and nonreciprocal objects. Law 4 generalizes the directional Kirchhoff radiation law for reciprocal optical objects, equating the absorptivity of any input beam with the emissivity into the backward (phase-conjugated) version of that beam. We have constructed these laws using the mathematical and physical properties of the mode-converter input and output basis functions that exist for any linear optical object and that can be deduced by singular-value decomposition. We have also shown that these types of mode-converter input and output functions are the orthogonal functions with the largest absorptivities and emissivities, respectively, which allows a different way of physically interpreting and establishing what they are. Because of the generality of the approach here, these results could apply to other kinds of waves, such as thermal equilibration by acoustic waves, for example.

We expect these laws will provide a clearer and expanded foundation for discussion of thermal emission and absorption generally and for solar energy conversion in particular and give us additional tools to analyze and describe thermal emitters in a rigorous modal basis.

**Acknowledgments.** This work was supported in part by the Department of Energy “Light-Material Interactions in Energy Conversion” Energy Frontier Research Center under Grant DE-SC0001293 (to S.F. and L.Z.) and by Air Force Office of Scientific Research Grant FA9550-15-1-0335 (to D.A.B.M.).

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