We generalize the concept of photonic gauge potential in real space by introducing an additional “synthetic” frequency dimension in addition to the real space dimensions. As an illustration, we consider a one-dimensional array of ring resonators, each supporting a set of resonant modes having a frequency comb with spacing $\Omega$, and undergoing a refractive index modulation at the modulation frequency equal to $\Omega$. We show that the modulation phase provides a gauge potential in the synthetic two-dimensional space with the dimensions being the frequency and the spatial axes. Such a gauge potential can create a topologically protected one-way edge state in the synthetic space that is useful for high-efficiency generation of higher-order side bands.

The creation of photonic gauge potential in real space opens a new dimension in the control of light propagation. Such real-space photonic gauge potential can be created either in time-reversal invariant systems, such as a static resonator lattice or metamaterials [1–3], or in systems where time-reversal symmetry is broken with either magneto-optical [4] or dynamic modulation effects [5]. Similar concepts of a gauge potential can be achieved in the acoustic systems [6]. A proper choice of photonic gauge potential can lead to an effective magnetic field for photons [1,5,7], which directly results in the creation of topologically protected one-way edge states. The ability to specify arbitrary gauge potential in real space, moreover, can lead to novel effects, including negative refraction [8], directional-dependent total internal reflection [8], and gauge-field waveguides [9]. The concept of photonic gauge potential in real space is also intimately connected to the concept of photonic gauge potential in momentum space [10–15], both of which are of significant importance in the emerging area of topological photonics [16–20].

In this Letter we generalize the concept of the photonic gauge potential in real space by adding a “synthetic” frequency dimension to the real space dimensions. As an illustration, we start by considering a simple one-dimensional coupled resonator model shown in Fig. 1(a). Each resonator supports a set of modes with their frequencies equally spaced at a frequency $\Omega$, forming a frequency comb. We assume coupling only between modes having the same frequency at the nearest neighbor resonators. In addition, we assume that each resonator is modulated at the frequency $\Omega$, which induces coupling between modes in the same resonator with frequencies separated by $\Omega$. The Hamiltonian of the system is then

$$H = \sum_{l,m} \omega_m a_{l,m}^\dagger a_{l,m} + \sum_{l,m} \{\kappa(a_{l,m}^\dagger a_{l+1,m} + a_{l+1,m}^\dagger a_{l,m})$$

$$+ 2g \cos(\Omega t + \varphi_l)(a_{l,m}^\dagger a_{l,m+1} + a_{l,m+1}^\dagger a_{l,m})\} \tag{1}$$

where $a_{l,m}^\dagger (a_{l,m})$ is the creation (annihilation) operator for the $m$-th mode at the $l$-th resonator; $\omega_m = \omega_0 + m\Omega$ gives the frequency for the $m$-th resonant mode, $\kappa$ is the coupling constant between two nearest-neighbor resonators, $g$ is the strength of modulation, and $\varphi_l$ is the associated modulation phase at the $l$-th resonator. Below we will show that this Hamiltonian is equivalent to a tight-binding Hamiltonian in two dimensions subject to an out-of-plane effective magnetic field, with each lattice site corresponding to one mode in a specific resonator.

![Fig. 1.](image_url) (a) A one-dimensional array of ring resonators. Each ring resonator supports a set of resonant modes with the frequencies of the modes forming a frequency comb with equally spaced $\Omega$. The $l$-th ring undergoes a modulation at the modulation frequency $\Omega$ with a modulation phase $\varphi_l$. (b) The system in the panel (a) can be mapped into a tight-binding model in two dimensions, with the extra synthetic dimension being the frequency dimension.
and with the two dimensions corresponding to the one-dimensional space and the frequency axes, respectively. In this construction the frequency axis therefore becomes an extra synthetic dimension.

The concept of synthetic dimension in the study of artificial gauge field has been recently discussed for cold atoms [21], where the synthetic dimension corresponds to the atomic states, and in a photonic system where the synthetic dimension corresponds to the orbital angular momentum of light [22]. In contrast to these works, here we show that the use of the frequency space as the synthetic dimension offers new possibilities for controlling the frequencies of light.

To see the emergence of an effective magnetic field in Eq. (1), we define \( \epsilon_{j,m} \equiv d_{j,m} e^{-i\omega t} \), and apply the rotating wave approximation to obtain from Eq. (1)

\[
H = \sum_{j,m} \left( \frac{\hbar}{2} \left( \epsilon_{j+1,m}^* \epsilon_{j+1,m} + \epsilon_{j+1,m} \epsilon_{j,m}^* \right) \right) + g \left( \epsilon_{j,m}^* \epsilon_{j,m+1} + \epsilon_{j,m+1}^* \epsilon_{j,m} \right).
\]

Equation (2) is identical in form to the Hamiltonian of a quantum particle on a two-dimensional lattice subject to a magnetic field [5], except that here, one of the axes (as labeled by the modal index \( m \)) is the frequency axis. Because there is one modulator per resonator, the hopping phases are uniform along the frequency axis. Nevertheless, Eq. (2) is sufficient to implement the Landau gauge for a uniform magnetic field by choosing the gauge field has been recently discussed for cold atoms [21], and in a photonic system where the synthetic dimension corresponds to the frequency axis therefore becomes an extra synthetic dimension as described in Eq. (1).

To see the emergence of an effective magnetic field in Eq. (1), we define \( \epsilon_{j,m} \equiv d_{j,m} e^{-i\omega t} \), and apply the rotating wave approximation to obtain from Eq. (1)

\[
H = \sum_{j,m} \left( \frac{\hbar}{2} \left( \epsilon_{j+1,m}^* \epsilon_{j+1,m} + \epsilon_{j+1,m} \epsilon_{j,m}^* \right) \right) + g \left( \epsilon_{j,m}^* \epsilon_{j,m+1} + \epsilon_{j,m+1}^* \epsilon_{j,m} \right).
\]

Equation (2) is identical in form to the Hamiltonian of a quantum particle on a two-dimensional lattice subject to a magnetic field [5], except that here, one of the axes (as labeled by the modal index \( m \)) is the frequency axis. Because there is one modulator per resonator, the hopping phases are uniform along the frequency axis. Nevertheless, Eq. (2) is sufficient to implement the Landau gauge for a uniform magnetic field by choosing the gauge field has been recently discussed for cold atoms [21], and in a photonic system where the synthetic dimension corresponds to the frequency axis therefore becomes an extra synthetic dimension as described in Eq. (1).

To see the emergence of an effective magnetic field in Eq. (1), we define \( \epsilon_{j,m} \equiv d_{j,m} e^{-i\omega t} \), and apply the rotating wave approximation to obtain from Eq. (1)

\[
H = \sum_{j,m} \left( \frac{\hbar}{2} \left( \epsilon_{j+1,m}^* \epsilon_{j+1,m} + \epsilon_{j+1,m} \epsilon_{j,m}^* \right) \right) + g \left( \epsilon_{j,m}^* \epsilon_{j,m+1} + \epsilon_{j,m+1}^* \epsilon_{j,m} \right).
\]

Equation (2) is identical in form to the Hamiltonian of a quantum particle on a two-dimensional lattice subject to a magnetic field [5], except that here, one of the axes (as labeled by the modal index \( m \)) is the frequency axis. Because there is one modulator per resonator, the hopping phases are uniform along the frequency axis. Nevertheless, Eq. (2) is sufficient to implement the Landau gauge for a uniform magnetic field by choosing the gauge field has been recently discussed for cold atoms [21], and in a photonic system where the synthetic dimension corresponds to the frequency axis therefore becomes an extra synthetic dimension as described in Eq. (1).

To see the emergence of an effective magnetic field in Eq. (1), we define \( \epsilon_{j,m} \equiv d_{j,m} e^{-i\omega t} \), and apply the rotating wave approximation to obtain from Eq. (1)

\[
H = \sum_{j,m} \left( \frac{\hbar}{2} \left( \epsilon_{j+1,m}^* \epsilon_{j+1,m} + \epsilon_{j+1,m} \epsilon_{j,m}^* \right) \right) + g \left( \epsilon_{j,m}^* \epsilon_{j,m+1} + \epsilon_{j,m+1}^* \epsilon_{j,m} \right).
\]

Equation (2) is identical in form to the Hamiltonian of a quantum particle on a two-dimensional lattice subject to a magnetic field [5], except that here, one of the axes (as labeled by the modal index \( m \)) is the frequency axis. Because there is one modulator per resonator, the hopping phases are uniform along the frequency axis. Nevertheless, Eq. (2) is sufficient to implement the Landau gauge for a uniform magnetic field by choosing the gauge field has been recently discussed for cold atoms [21], and in a photonic system where the synthetic dimension corresponds to the frequency axis therefore becomes an extra synthetic dimension as described in Eq. (1).
where \( \eta \) is the strength of the coupling between the resonator and the waveguide and \( \mathcal{E} \) is the source field in the waveguide.

Equations (9)–(13) provide a description of a physical system consisting of a set of ring resonators coupled together, with each ring modulated by an electro-optic phase modulator. We solve Eq. (9) with a finite difference approach in both space and time. At the places where the coupling occurs or at the locations of the modulators, we calculate the fields at time \( t^+ \) first with Eq. (9), and then apply either Eqs. (10) and (11) or Eq. (12) to compute the fields at \( t^+ \).

In the modulated ring, with the absence of group velocity dispersion, the modulation sidebands at \( \omega_m \) coincide with the modes of the ring at \( \omega_m \). Therefore we have on-resonance coupling between multiple modes. On the other hand, with group velocity dispersion, \( \omega_m \neq \omega_\text{on} \), and the coupling between the modes become off-resonance. As a result, the group velocity dispersion of a waveguide provides a natural “boundary” in the frequency space. As an illustration, we first perform a simulation of a system with six ring resonators as shown in Fig. 3(a). We assume that at a center frequency \( \omega_0 \) the waveguide for the ring has an effective index of \( n_0 = n(\omega_0) = 1.5 \). In the simulation, we include 11 side bands \( (\omega_m = \omega_0 + m\Omega \) and \( m = -5, -4, ..., 5) \), where \( \Omega = 2\pi c/n_0 L \). We choose a waveguide dispersion relation \( \beta(\omega) \), as shown in Fig. 3(b), such that 7 of the side bands with \( m = -3, -2, ..., 3 \) are on-resonance having \( n(\omega_m) = n_0 \). The other 4 side bands have an effective index that differs from \( n_0 \). The dispersion relation here thus is chosen to illustrate a waveguide with a zero group velocity dispersion near frequency \( \omega_0 \). Each of the rings is modulated as described above with an electro-optic modulator with a modulation frequency \( \Omega \), and with a modulation phase \( \phi_l \) in Eq. (3). To excite the system, a continuous-wave signal, having a single frequency at \( \omega_m \), is sent into the left ring resonator \( (l = 1) \). The distribution of the intensity \( |\mathcal{E}_{l,m}|^2 \) as a function of \( l \) and \( m \) at \( t = 400 \) is plotted in Fig. 3(c). We note that there is almost zero intensity for the sidebands with \( m = \pm 5 \) and \( \pm 4 \). Thus the group velocity dispersion indeed provides a boundary in the frequency space. The intensity is concentrated at the edge of the synthetic space forming a topologically protected one-way mode. We plot in Fig. 3(d) the intensity spectra corresponding to the \( \mathcal{E}(t, r) \) field inside each resonator. For the resonators at the spatial edge, \( (l = 1, \) and \( l = 6) \), the intensity spectra have significant components in all on-resonance side bands, while for the resonators at the center of the structure \( (l = 3, \) and \( l = 4) \), the intensity spectra are almost completely concentrated in the on-resonance side bands that have the highest and lowest frequencies.

In a typical ring resonator system under modulation, multiple side bands are generated. On the other hand, in the system considered here the resonators in the center of the structure have amplitudes only in the highest and lowest on-resonance frequency side bands. Therefore, the use of a gauge field in the synthetic space provides the opportunity to create high-efficiency generation of high-order side-bands. As an illustration, we consider the system shown in Fig. 4(a). The system consists of five resonators modulated at a frequency \( \Omega \) with the modulation phase chosen according to Eq. (3(a)). The system consists of five resonators modulated at a frequency \( \Omega \) with the modulation phase chosen according to Eq. (3). At the places where the coupling occurs or at the locations of the modulators, we calculate the fields at time \( t^+ \) first with Eq. (9), and then apply either Eqs. (10) and (11) or Eq. (12) to compute the fields at \( t^+ \).

![Fig. 3.](image-url)

The excitation is injected into the system through an input/output waveguide

\[
\mathcal{E}^l_m(t^+, x^l_r) = \sqrt{1 - \eta^2} \mathcal{E}^l_m(t^+, x^l_r) - i\eta \mathcal{E}_s(t^+, x^l_r),
\]

where \( \eta \) is the strength of the coupling between the resonator and the waveguide and \( \mathcal{E} \) is the source field in the waveguide.
m = ±5 are non-resonant and the rest of the side bands resonant with the ring modes. We couple to the ring resonator at l = 1 (l = 3) with an extra ring resonator having a resonance at ωm=m−4, (ωm=m+4), and are off resonance at all other side-band frequencies. To excite the system, we inject into the additional resonator associated with the l = 1 resonator with a pulse that has its carrier frequency ωm=m and the temporal full width at half-maximum (FWHM) Δt = 50 n0L/c, corresponding to the FWHM of the spectral intensity Δω = 0.055 c/n0L ≪ Ω. Therefore the input pulse only excites a single side band. The topologically protected one-way edge state converts the input at ωm=m−4 to the frequency component at ωm=m4 when the signal propagates along the edge of the lattice. We plot the input and output intensity spectra and the convert spectra in Fig. 4(b). The input field has a single frequency near ωm=m−4 to excite the edge mode in the band gap. The output spectra are measured at detector 2 and we see that the field has been converted to the field with the frequency near ωm=m4 with an efficiency of 81%. Our result shows that this system serves the purpose of a higher-order frequency converter.

Our proposal here can be implemented in various systems in fiber optics or integrated photonics. For the standard fiber system, the electro-optic phase modulation frequency can be up to ~1 GHz. This requires that the resonator be composed of the loop of a fiber with the length of ~0.1 m. We note that the standard single-mode fiber has a zero group velocity dispersion at λ = 1.3 μm, which is useful for our design here. For a silicon resonator modulated at ~100 GHz, the radius of the resonator is ~100 μm [25]. Flat and low intracavity dispersion over a wide wavelength range (~500 nm) can be achieved in a curving waveguide [26]. Further miniaturization of the structure in either platform can be accomplished with the use of slow-light waveguides.

In summary, we generalize the concept of photonic gauge potential to a synthetic space with both the spatial and frequency dimensions, and demonstrate that such a gauge potential leads to an edge state in the synthetic space that is useful for high-efficiency conversion to higher-order side bands. Related to and independent of our work, a recent preprint [27] has proposed the use of similar synthetic gauge potential for simulation of quantum hall effect in four dimensions. We believe that the concept of gauge potential in such synthetic space provides a new dimension for the control of light in both the real and the frequency spaces.

Acknowledgment. The authors acknowledge discussions with I. Carusotto, who alerted us to Ref. [27] when we were in the final stage of preparing this Letter.

REFERENCES