Anisotropic dielectric waveguides

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The guided modes of anisotropic waveguides are known to be leaky when degenerate radiation modes of the orthogonal polarization exist. These losses can be quite large when the principal axes of the dielectric tensor are not aligned with the waveguide axis. We calculate the leakage rate in waveguides that have arbitrarily oriented dielectric tensors, using a perturbation expansion that is valid for weak guidance and weak anisotropy. A relatively simple expression for the loss is obtained and applied to model cases drawn from planar and fiber waveguides with step- and graded-index profiles. The scaling of the loss with profile height and anisotropy is discussed, including the practical case of strongly asymmetric planar waveguides.

1. INTRODUCTION

It is well known that modes that are polarized along the fast axis of an anisotropic waveguide are in many cases leaky. The leakage is caused by the coupling of the guided mode to nearly degenerate radiation modes of the orthogonal polarization. Such degenerate modes exist whenever the birefringence of the waveguide exceeds the index difference between the core and the cladding, and coupling between modes occurs if they are not linearly polarized along principal axes of the dielectric tensor. In planar waveguides the latter condition holds only when certain off-diagonal elements of the dielectric tensor are nonzero. However, coupling is always present in two-dimensional waveguides, because the true modes are not linearly polarized. If, in addition, the off-diagonal elements of the dielectric tensor are nonzero, stronger coupling can occur.

The leakage that is due to stress-induced birefringence has been used in the design of single-polarization glass fibers. In anisotropic crystalline waveguides, such leakage can be orders of magnitude larger and often precludes propagation in directions that are of interest for applications, particularly in nonlinear devices. A number of analyses of the leakage loss in planar structures have been published, including an exact solution of the symmetric planar waveguide, a WKB method, a coupled-mode method, and a multilayer numerical approximation.

Snyder and Ruhl presented a useful weakly guiding perturbation method for a step-profile anisotropic fiber in which one of the principal axes of the dielectric tensor aligned with the waveguide axis. Later Ruhl and Snyder demonstrated a Green function method for the same problem. The leaky modes in fibers in which the principal axes in both the core and the cladding were aligned with the waveguide axis but were misaligned with each other in the transverse plane are discussed in Ref. 10.

In the present study we extend the method of Ref. 1 to waveguides with arbitrarily oriented dielectric tensors. We begin by recasting Maxwell's equations as scalar-wave equations for the Cartesian components of the modal fields, with coupling terms based on waveguide geometry and material anisotropy. A perturbative solution that is valid in the limit of small anisotropy and weak guidance is presented, and an expression for the leakage loss is given. A simple general form for the coupling term is derived and is applied to the calculation of the leakage rate for various planar and circular waveguides.

2. DERIVATION OF THE MODAL FIELDS AND THE LOSS

In this section we recast Maxwell's equations for a waveguide of arbitrary anisotropy as a set of coupled scalar equations for the Cartesian components of the modal fields. We then obtain the modal fields to first order and propagation constants to second order in the coupling. The loss is proportional to the imaginary component of the propagation constant. In the following derivation we follow the perturbation method and the notation of Ref. 1 wherever possible.

A. Modal-Field Equations

A general waveguide is drawn schematically in Fig. 1, where the waveguide axis is along z. From Maxwell's equations and the constitutive relations for the fields we write the equations obeyed by the exact vector modal electric fields \( \vec{E} \) as

\[
\nabla \times \vec{E} + \kappa^2 \epsilon \cdot \vec{E} = \nabla (\nabla \cdot \vec{E}),
\]

\[
\nabla \cdot (\epsilon \cdot \vec{E}) = 0,
\]

where \( \kappa = \omega/c \) is the free-space propagation constant and the magnetic permeability \( \mu \) has been assumed to take the vacuum value \( \mu_0 \). For an ideal waveguide oriented along the z-axis, the dielectric tensor \( \epsilon \) is a function only of the transverse coordinates, i.e., \( \epsilon = \epsilon(x, y) \).

The most general possible dielectric tensor \( \epsilon' \) for a lossless nongyrotropic medium is symmetric and has six independent elements \( \epsilon'_{ij} \), where \( i, j = 1, 2, 3 \) and refer to the crystallographic axes. For simplicity we derive the results for the case in which the medium is uniaxial and the optical axis \( \hat{3} \) lies in the \( y-z \) plane at an angle \( \alpha \) to \( \hat{z} \) (we assume a uniform \( \alpha \) through the waveguide, contrary to what is done in Ref. 9). Referred to these axes, the dielectric
tensor \( \mathbf{\epsilon} \) takes the form
\[
\mathbf{\epsilon} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & \epsilon_{xz} \\ 0 & \epsilon_{xz} & \epsilon_{zz} \end{bmatrix}, \tag{3}
\]
where the \( \epsilon_{ij} \) are given in terms of the \( \epsilon_{ij} \) as
\[
\epsilon_{xx} = \epsilon_{11},
\]
\[
\epsilon_{yy} = \epsilon_{11} \cos^2 \alpha + \epsilon_{33} \sin^2 \alpha,
\]
\[
\epsilon_{zz} = \epsilon_{11} \sin^2 \alpha + \epsilon_{33} \cos^2 \alpha,
\]
\[
\epsilon_{xz} = (\epsilon_{33} - \epsilon_{11}) \sin \alpha \cos \alpha. \tag{4}
\]

Assume a modal field of the form \( \mathbf{E} = \mathbf{\tilde{E}}(x, y) \exp[i(\beta z - \omega t)] \); Eq. (1) becomes
\[
(\nabla^2 + k^2 \epsilon_{xx} - \beta^2) \mathbf{\tilde{E}}_x = \frac{\partial}{\partial x} \left( \frac{\partial \mathbf{\tilde{E}}_x}{\partial x} + \frac{\partial \mathbf{\tilde{E}}_y}{\partial y} + i \beta \mathbf{\tilde{E}}_z \right), \tag{5a}
\]
\[
(\nabla^2 + k^2 \epsilon_{yy} - \beta^2) \mathbf{\tilde{E}}_y = \frac{\partial}{\partial y} \left( \frac{\partial \mathbf{\tilde{E}}_y}{\partial x} + \frac{\partial \mathbf{\tilde{E}}_x}{\partial y} + i \beta \mathbf{\tilde{E}}_z \right), \tag{5b}
\]
\[
(\nabla^2 + k^2 \epsilon_{zz} - \beta^2) \mathbf{\tilde{E}}_z = i \beta \left( \frac{\partial \mathbf{\tilde{E}}_x}{\partial x} + \frac{\partial \mathbf{\tilde{E}}_y}{\partial y} + i \beta \mathbf{\tilde{E}}_z \right), \tag{5c}
\]

and Eq. (2) becomes
\[
\frac{\partial}{\partial x} (\epsilon_{xx} \mathbf{\tilde{E}}_x) + \frac{\partial}{\partial y} (\epsilon_{yy} \mathbf{\tilde{E}}_y + \epsilon_{xy} \mathbf{\tilde{E}}_y) + i \beta (\epsilon_{xz} \mathbf{\tilde{E}}_z + \epsilon_{yx} \mathbf{\tilde{E}}_y) = 0, \tag{6}
\]
where \( \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \).

We can obtain an expression for \( \mathbf{\tilde{E}}_2 \) from Eq. (6). To separate to first order the contribution of \( \mathbf{\tilde{E}}_2 \) of the off-diagonal elements \( \epsilon_{ij} \) from that of the waveguide structure, we make the substitution
\[
\mathbf{\tilde{E}}_2 = -\frac{\epsilon_{yy}}{\epsilon_{zz}} \mathbf{\tilde{E}}_y + \mathbf{\tilde{E}}_2', \tag{7}
\]

With Eqs. (6) and (7) we obtain
\[
\mathbf{\tilde{E}}_2' = -\frac{1}{i \beta} \left( \frac{\epsilon_{xx}}{\epsilon_{zz}} \frac{\partial \mathbf{\tilde{E}}_2}{\partial x} + \frac{\epsilon_{yy}}{\epsilon_{zz}} \frac{\partial \mathbf{\tilde{E}}_2}{\partial y} \right) + \frac{\epsilon_{xy}}{\epsilon_{zz}} \frac{\partial \ln \epsilon_{zz}}{\partial x} + \frac{\epsilon_{yz}}{\epsilon_{zz}} \frac{\partial \ln \epsilon_{zz}}{\partial y}, \tag{8}
\]
where we have assumed that \( \rho \beta \epsilon_{zz} \gg \epsilon_{yy} \), which should hold for all cases of practical interest, and have defined
\[
\mathbf{\tilde{E}}_{yy} = \frac{\epsilon_{yy}}{\epsilon_{zz}} \mathbf{\tilde{E}}_2 - \frac{\epsilon_{yy}^2}{\epsilon_{zz}^2} \mathbf{\tilde{E}}_y. \tag{9}
\]

Eliminating \( \mathbf{\tilde{E}}_2 \), from Eqs. (5a) and (5b) using Eqs. (7) and (8) leads to the coupled equations for the transverse components of the modal electric fields
\[
(\nabla^2 + k^2 \epsilon_{xx} - \beta^2) \mathbf{\tilde{E}}_x = P_{xx} \mathbf{\tilde{E}}_x - P_{xy} \mathbf{\tilde{E}}_y, \tag{10a}
\]
\[
(\nabla^2 + k^2 \epsilon_{yy} - \beta^2) \mathbf{\tilde{E}}_y = P_{xy} \mathbf{\tilde{E}}_x + P_{yy} \mathbf{\tilde{E}}_y, \tag{10b}
\]
where
\[
\begin{align}
P_{xx} \mathbf{\tilde{E}}_x &= \frac{\partial}{\partial x} \left( 1 - \frac{\epsilon_{xx}}{\epsilon_{zz}} \right) \frac{\partial \mathbf{\tilde{E}}_x}{\partial x} - \frac{\partial}{\partial x} \left( \frac{\epsilon_{xx}}{\epsilon_{zz}} \frac{\partial \ln \epsilon_{zz}}{\partial x} \right), \tag{11a} \\
P_{xy} \mathbf{\tilde{E}}_x &= \frac{\partial}{\partial y} \left( 1 - \frac{\epsilon_{xx}}{\epsilon_{zz}} \right) \frac{\partial \mathbf{\tilde{E}}_x}{\partial y} - \frac{\partial}{\partial y} \left( \frac{\epsilon_{xx}}{\epsilon_{zz}} \frac{\partial \ln \epsilon_{zz}}{\partial y} \right) - i \beta \frac{\partial}{\partial x} \left( \frac{\epsilon_{22}}{\epsilon_{zz}} \right), \tag{11b} \\
P_{yy} \mathbf{\tilde{E}}_y &= \frac{\partial}{\partial y} \left( 1 - \frac{\epsilon_{yy}}{\epsilon_{zz}} \right) \frac{\partial \mathbf{\tilde{E}}_y}{\partial y} - \frac{\partial}{\partial y} \left( \frac{\epsilon_{yy}}{\epsilon_{zz}} \frac{\partial \ln \epsilon_{zz}}{\partial y} \right) + k^2 \frac{\epsilon_{yy}}{\epsilon_{zz}} \frac{\partial}{\partial x} (\epsilon_{xx} \mathbf{\tilde{E}}_y), \tag{11c} \\
P_{xy} \mathbf{\tilde{E}}_y &= \frac{\partial}{\partial y} \left( 1 - \frac{\epsilon_{yy}}{\epsilon_{zz}} \right) \frac{\partial \mathbf{\tilde{E}}_y}{\partial y} - \frac{\partial}{\partial y} \left( \frac{\epsilon_{yy}}{\epsilon_{zz}} \frac{\partial \ln \epsilon_{zz}}{\partial y} \right) - i \beta \frac{\partial}{\partial y} \left( \frac{\epsilon_{22}}{\epsilon_{zz}} \right) + k^2 \frac{\epsilon_{yy}}{\epsilon_{zz}} \frac{\partial}{\partial y} (\epsilon_{yy} \mathbf{\tilde{E}}_y). \tag{11d}
\end{align}
\]
If \( \epsilon_{yy} = 0 \), Eqs. (11) reduce to those given in Ref. 1 for waveguides with diagonal dielectric tensors.

If the terms on the right-hand sides of Eqs. (11) vanish, Eqs. (10) become uncoupled scalar-wave equations for the Cartesian components of the modal fields. The solutions of these equations are the well-known linearly polarized (LP) modes that are good approximations for the modes of weakly guiding isotropic waveguides. The second term on the right-hand side (RHS) of each part of Eqs. (11) appears even in isotropic waveguides. In Eqs. (11a) and (11d) these terms represent polarization corrections that are due to the waveguide structure,11 while in Eqs. (11b) and (11c) they cause polarization coupling in two-dimensional waveguides. The first term on the RHS of each part of Eqs. (11) vanishes in isotropic waveguides and represents polarization coupling that is due to the difference between \( \epsilon_{xx} \) or \( \epsilon_{yy} \)—the dielectric constant that the small \( z \) component of the modal field would be expected to see—and \( \epsilon_{zz} \), the dielectric constant that is actually present in an anisotropic structure. The third term on the RHS of each part of Eqs. (11b)–(11d) and the fourth term of Eq. (11d) are present only when a principal axis of the dielectric tensor...
is not aligned with the waveguide axis, and they represent coupling induced by the material anisotropy between the small structural z-polarized component of the predominantly x-polarized mode and the z-polarized component of the predominantly y-polarized mode.

The modal field equations (10) for anisotropic weakly guiding structures can be solved with a perturbation expansion. The uncoupled LP modes are taken as the basis modes, with coupling that is due to the perturbation terms on the RHSs of Eqs. (10). Coupling between modes of the same polarization exists when \( P_i \bar{c}_j \) terms do not vanish, while the \( P_i \bar{c}_j \) terms give rise to coupling of orthogonal polarizations. We are interested primarily in calculating the loss that occurs when a guided mode is coupled to degenerate radiation modes. Generally speaking, guided and radiation modes of the same polarization are not degenerate, but if the material is anisotropic, degeneracy between modes of orthogonal polarizations can exist.

In Subsection 2.3 we carry out the perturbation expansion to second order, the lowest order in which an imaginary component of the propagation constant can appear. If the coupling is between guided modes, a hybrid guided mode results, as is discussed in Ref. 12. If the coupling is between a guided mode and a degenerate mode, the guided mode becomes leaky. Equations (10) can be used to describe both of these phenomena, but in this paper we apply them only to the calculation of leaky modes.

Further discussion of the solutions of Eqs. (10) is facilitated if we assume a specific form for the spatial variation of the elements of the principal dielectric tensor,

\[
\epsilon_{ij}(x, y) = \epsilon_{i\Delta \epsilon i}(1 - 2\Delta f_i(x, y)), \quad J = 1, 3, \tag{12a}
\]

where \( \epsilon_{i\Delta \epsilon i} \) is the maximum value of the \( \epsilon_{i\Delta \epsilon i} \) component, \( 2\Delta \epsilon_{i\Delta \epsilon i} \) is the difference between the dielectric tensor component in the core and the cladding, and \( \Delta f_i \) is known as the profile height. Functions \( f_i(x, y) \) describe the spatial variation of the dielectric tensor components, normalized to vary between 0 and 1. By assuming this form, we have restricted the analysis to systems in which the orientation of the principal axes does not depend on the transverse coordinates. The dielectric tensor components \( \epsilon_{xx} \) and \( \epsilon_{yy} \) are related to \( \epsilon_{ij} \) through the expressions in Eqs. (4) and hence can be written in the same form as Eq. (12a). In this way the profile heights for the transverse polarizations, \( \Delta x \) and \( \Delta y \), are introduced. It is convenient to characterize the anisotropy by the quantities \( \delta_{ij} \), defined by

\[
\delta_{ij}(x, y) = \frac{1}{2} \left[ 1 - \frac{\epsilon_{ij}(x, y)}{\epsilon_{i\Delta \epsilon i}(x, y)} \right], \tag{12b}
\]

and to use the usual definition of the normalized frequencies \( \nu_i = \rho \kappa n_{i,\infty}(2\Delta \iota)^{1/2} \), where \( i = x, y \) and \( n_i = \epsilon_i^{1/2} \) is the refractive index.

Note from the definition of normalized frequency in Table 1 that \( \rho \Delta \) is fixed for the first order and \( \rho \Delta \iota \) is of order unity; it is clear that, for weakly guiding (\( \Delta \ll 1 \)) and weakly anisotropic (\( \delta \ll 1 \)) waveguides, the first two terms on the RHSs of Eqs. (11b) and (11c) are small. Thus when \( \epsilon_{ii} = 0 \), a solution correct to first order in \( \Delta \) and \( \delta \) is generally valid. When \( \epsilon_{ii} \neq 0 \) the third term on the RHSs of Eqs. (11b) and (11c) are of order \( \Delta \iota \); the terms are small when \( \delta \ll \Delta \) but can be large when \( \delta \gg \Delta \). However, we will show in Subsection 2.3 that these coupling terms are actually of order \( \Delta \iota \) and thus can be regarded as perturbations as long as \( \Delta, \delta \ll 1 \).

The zeroth-order solutions, \( \epsilon_{i\Delta \epsilon i} \) and \( \epsilon_{i\Delta \epsilon i} \), obtained from the homogeneous forms of Eqs. (10), are the familiar linearly polarized modes. We can therefore construct modal-field solutions to the coupled-field equations (10) in such a way that they are either purely x polarized or y polarized to lowest order. We assume that a predominantly i-polarized mode, written as \( \epsilon_{i0} \), and its propagation constant \( \beta_{i0} \) can be expanded as power series in the small quantities \( \Delta \) and \( \delta \) (which are treated as being of the same order):

\[
\epsilon_{i0} = (\epsilon_{i0}^{(1)} + \epsilon_{i0}^{(1)} + \epsilon_{i0}^{(1)} + \ldots)i + (\epsilon_{i0}^{(1)} + \epsilon_{i0}^{(1)} + \ldots)j, \tag{13a}
\]

\[
\beta_{i0} = \beta_{i0} + \beta_{i0} + \beta_{i0} + \ldots. \tag{13b}
\]

We can evaluate the series given in Eqs. (13) by substituting them into Eqs. (10) and equating like powers of the perturbation, which gives the approximate equations for a predominantly i-polarized mode:

\[
(V_i^2 + k^2 \epsilon_{ii} - \beta_{i0}^2)\epsilon_{i0}^{(1)} = 0, \tag{14a}
\]

\[
(V_i^2 + k^2 \epsilon_{ii} - \beta_{i0}^2)\epsilon_{i0}^{(1)} = P_i \epsilon_{i0}^{(0)}, \tag{14b}
\]

\[
(V_i^2 + k^2 \epsilon_{ii} - \beta_{i0}^2)\epsilon_{i0}^{(1)} = 2\beta_{i0} \epsilon_{i0}^{(0)} + P_i \epsilon_{i0}^{(0)}, \tag{14c}
\]

\[
(V_i^2 + k^2 \epsilon_{ii} - \beta_{i0}^2)\epsilon_{i0}^{(1)} = (2\beta_{i0} \beta_{i0} + \beta_{i0}^2)\epsilon_{i0}^{(1)}
+ 2\beta_{i0} \epsilon_{i0}^{(1)} + P_i \epsilon_{i0}^{(1)} + P_i \epsilon_{i0}^{(1)}, \tag{14d}
\]

### Table 1. Waveguide Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dielectric Constant</td>
<td>( \epsilon_{xx} = \epsilon_{i}^{1/2} )</td>
<td>( \epsilon_{yy} = \epsilon_{i}^{1/2} )</td>
</tr>
<tr>
<td>Profile</td>
<td>( n_i(x, y) = (\epsilon_{i}^{1/2}) )</td>
<td>( n_i(x, y) = (\epsilon_{i}^{1/2}) )</td>
</tr>
<tr>
<td>Profile Height</td>
<td>( \Delta x = \Delta )</td>
<td>( \Delta y = \Delta \sin^2 \alpha + \Delta y \sin^2 \alpha )</td>
</tr>
<tr>
<td>Normalized Frequency*</td>
<td>( V_i = \rho \kappa n_{i,\infty}(2\Delta \iota)^{1/2} )</td>
<td>( V_i = \rho \kappa n_{i,\infty}(2\Delta \iota)^{1/2} )</td>
</tr>
<tr>
<td>Principal Dielectric Constant</td>
<td>( \epsilon_{ii}(x, y) = \epsilon_{i\Delta \epsilon i}(1 - 2\Delta f_i(x, y)) )</td>
<td>( \epsilon_{ii}(x, y) = \epsilon_{i\Delta \epsilon i}(1 - 2\Delta f_i(x, y)) )</td>
</tr>
<tr>
<td>Principal Profile Height</td>
<td>( \Delta f_i = (1 - \epsilon_{i\Delta \epsilon i}(1 - 2\Delta f_i(x, y)))^{1/2} )</td>
<td>( \Delta f_i = (1 - \epsilon_{i\Delta \epsilon i}(1 - 2\Delta f_i(x, y)))^{1/2} )</td>
</tr>
<tr>
<td>Anisotropy</td>
<td>( \delta_{ij} = (1 - \epsilon_{ij}(x, y))/(2\Delta \iota) )</td>
<td>( \delta_{ij} = (1 - \epsilon_{ij}(x, y))/(2\Delta \iota) )</td>
</tr>
</tbody>
</table>

*\( k = 2\pi/\lambda; \rho \) is the dimension of the core region.
Table 2. Modal Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>y (Guided)</th>
<th>x (Radiation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Propagation constant</td>
<td>( \beta_{(y)} )</td>
<td>( \beta_{(x)} )</td>
</tr>
<tr>
<td>Core parameter</td>
<td>( U_{c} = \frac{\rho(\Delta k_{\text{e},\text{y}}}{} - \beta_{(y)}^{2}\Delta_{y}^{2} )</td>
<td>( U_{c} = \frac{\rho(\Delta k_{\text{e},\text{x}}}{} - \beta_{(x)}^{2}\Delta_{x}^{2} )</td>
</tr>
<tr>
<td>Cladding parameter</td>
<td>( W_{c} = \rho(\beta_{(y)}^{2} - k_{\text{e},\text{y}}}{}^{2}\Delta_{y}^{2} )</td>
<td>( Q_{c} = \rho(\Delta k_{\text{e},\text{c}}}{} - \beta_{(x)}^{2}\Delta_{x}^{2} )</td>
</tr>
<tr>
<td>Normalization constant</td>
<td>( \left( \frac{\rho^{2}}{\Delta_{y}^{2}} \right) )</td>
<td>( \left( \frac{\rho^{2}}{\Delta_{x}^{2}} \right) )</td>
</tr>
</tbody>
</table>

\( \Delta_{y}^{2} = U_{c}^{2} + W_{c}^{2}; \Delta_{x}^{2} = Q_{c}^{2} + U_{c}^{2}; \Delta_{y}^{2} = V_{y}^{2}\Delta_{y}^{2}; \Delta_{x}^{2} = V_{x}^{2}\Delta_{x}^{2}; \) when \( \beta(y) = \beta(x) \).

B. Solution of the Approximate Field Equations

We are interested primarily in calculating the leakage loss resulting from the anisotropy, and so we must obtain the imaginary part of the propagation constant. If we assume an \( i \)-polarized zeroth-order mode, the first-order correction to the propagation constant \( \beta_{(y)} \) is the expectation value of the coupling for the zeroth-order mode and can be obtained from Eqs. (14a) and (14c) with the use of standard perturbation theory. It can be shown that \( \beta_{(y)} \) is real. We then must obtain the second-order correction to find the loss. Multiplying Eq. (14d) by \( e_{(y)}^{(0)} \) and using the orthogonality of \( e_{(y)}^{(0)} \) and \( e_{(z)}^{(0)} \) to \( e_{(y)}^{(0)} \), we arrive at

\[
\beta_{(y)} = -\frac{1}{2\beta_{(y)}} \left[ \beta_{(y)}^{2} + \frac{\left( e_{(z)}^{(0)} P_{e_{(y)}^{(0)}} \right)^{2}}{\left( e_{(z)}^{(0)} \right)^{2}} + \frac{\left( e_{(z)}^{(0)} P_{e_{(y)}^{(0)}} \right)^{2}}{\left( e_{(z)}^{(0)} \right)^{2}} \right],
\]

where we use the symbol \( (g) \) to denote an integral of \( g(x, y) \) over the \( x-y \) plane. As is discussed in Section 1 and in Appendix A, only coupling to degenerate radiation modes of the orthogonal polarization leads to loss, so that only the third term in Eq. (15) contributes to the imaginary part of \( \beta_{(y)} \). An integral expression for \( e_{(y)}^{(1)} \), first given in Ref. 1, is discussed in the context of the present analysis in Appendix A. It is shown there that using the integral expression for \( e_{(y)}^{(1)} \), given by Eq. (A1.4), in Eq. (15) leads to

\[
\beta_{(y)}^{\text{im}} = \begin{cases} \pi r^{2} \frac{\left( e_{(z)}^{(0)} P_{e_{(y)}^{(0)}} \right)^{2}}{2n_{z} n_{y} \left( e_{(z)}^{(0)} \right)^{2}}, & \beta_{(y)} < \beta_{\text{th}} \\ 0, & \beta_{(y)} > \beta_{\text{th}} \end{cases}
\]

where \( \beta_{\text{th}} = k n_{z} n_{y} e_{(z)}^{(0)}(Q_{c}) \) is the \( j \)-polarized LP radiation mode satisfying \( \beta_{j} \Delta_{y} = \beta_{j} \Delta_{x} \), and \( n_{y} \) and \( Q_{c} \), defined in Table 2, are the normalization constant and the cladding parameter of the radiation mode, respectively. The limits to the validity of this expansion are discussed in Subsection 2.C after a simplified form of the coupling coefficient is derived.

C. Simplification of the Coupling Terms \( \langle e_{(y)}^{(0)} P_{e_{(z)}^{(0)}} \rangle \)

The expressions for the coupling terms that are necessary for the evaluation of Eq. (16), given by Eqs. (11), are obviously complicated. In this section we develop relations that will both simplify the calculation and yield physical insight into the leaky modes. The simplified coupling coefficients illustrate more clearly the dependence of the leakage rate on the material anisotropy \( \delta_{ji} \), the profile height \( \Delta_{y} \), and the angle \( \alpha \) between the optical and the waveguide axes, without evaluation of the coupling integral or detailed information about LP modes. The transformed coupling coefficient, which involves the modal fields only in the region of the spatially varying refractive index rather than in all space and does not require the spatial derivatives of the modal fields, often yields more accurate results than does direct application of Eq. (16) when approximations to the modal fields are used.

We assume that the dielectric tensors of the waveguide media are of the form given in Eqs. (3) and (4), that the orientation of the principal axes is the same throughout the structure; and, in addition, that \( e_{x} \perp \epsilon_{y} \perp \epsilon_{z} \), i.e., that the media are negative uniaxial. Since the \( x \)-polarized modes are the slow modes, only the \( y \)-polarized modes exhibit leakage. Thus in evaluating Eq. (16) we take \( i = y \) and \( j = x \). Any of the expressions derived under these assumptions can be applied to positive uniaxial media by interchanging the coordinates \( x \) and \( y \).

Under these assumptions, leaky modes do not exist when \( \alpha = 0 \). Anisotropy in the \( x-y \) plane emerges when \( \alpha \neq 0 \), and leakage begins when \( \alpha \) exceeds a threshold value \( \alpha_{\text{th}} \), as we discuss in Section 3. In planar waveguides the coupling that leads to the leakage is attributable exclusively to the third terms on the RHS’s of Eqs. (11b) and (11c). In two-dimensional guides all three terms on the RHS’s of Eqs. (11b) and (11c) are nonvanishing, but the third terms generally dominate, as is discussed in Section 4. Therefore we focus our attention on these dominant terms and present some results for the other terms in later sections. Retaining only the most important terms in Eq. (11c), we have

\[
\langle e_{(y)}^{(0)} P_{e_{(z)}^{(0)}} \rangle = \frac{\pi r^{2}}{4\beta_{(y)}} \left( e_{x}^{(s)} \frac{\partial e_{x}^{(s)}}{\partial x} \right)
\]

As we know, \( e_{x}^{(s)} \) and \( e_{y}^{(s)} \), satisfy the homogeneous forms of Eqs. (10). For convenience we list them again as follows:

\[
\begin{align*}
(\nabla_{x}^{2} + k_{x}^{2} e_{x}^{(s)} - \beta_{x}^{(s)} e_{x}^{(s)}) e_{x}^{(s)} &= 0, \\
(\nabla_{y}^{2} + k_{y}^{2} e_{y}^{(s)} - \beta_{y}^{(s)} e_{y}^{(s)}) e_{y}^{(s)} &= 0.
\end{align*}
\]

For the radiation mode that is degenerate with the guided mode, i.e., \( \beta_{(y)} = \beta_{x,y}^{(s)} \), it can be readily derived from Eq. (18b) that

\[
(\nabla_{x}^{2} + k_{x}^{2} e_{x}^{(s)} - \beta_{y}^{(s)} e_{y}^{(s)}) \frac{\partial e_{x}^{(s)}}{\partial x} = -k_{y}^{2} e_{y}^{(s)} \frac{\partial e_{y}^{(s)}}{\partial x}.
\]

Multiplying Eq. (19) by \( e_{x}^{(s)} \), and Eq. (18a) by \( e_{x}^{(s)} \), subtracting, and integrating over the \( x-y \) plane, we obtain
Making use of Eq. (20) and a relation
\[(\varepsilon_{xx} - \varepsilon_{yy}) \frac{\partial e_{(0)}^{(0)}}{\partial x} = -\frac{\partial e_{xx}}{\partial x} e_{(0)}^{(0)} e_{(0)}^{(0)}, \]  
(20)
derived from Eqs. (4), we find that
\[\left(\varepsilon_{(0)}^{(1)} e_{(0)}^{(0)} \frac{\partial e_{(0)}^{(0)}}{\partial x}\right) = \cot \alpha \left(\frac{\partial e_{xx}}{\partial x} e_{(0)}^{(0)} e_{(0)}^{(0)}\right). \]  
(22)
Combining Eqs. (12a), (16), (17), and (22), we obtain the relatively simple form for the imaginary part of the propagation constant:
\[\beta_{(y)}^{L_2} = \begin{cases} \frac{\pi \rho^2 \Delta_2 \cot^2 \alpha}{(\varepsilon_{(0)}^{(2)})^2} L_2 Q_v, & \beta_{(y)} < \beta_{th}, \\ 0, & \beta_{(y)} > \beta_{th}. \end{cases} \]  
(23)
where \(\varepsilon_{(0)}^{(0)}\) is the radiation mode that is degenerate with \(e_{(0)}^{(0)}\) and \(\beta_{th}\) is given by Eq. (B7), with \(j = x\). Equation (23) is derived under the assumption that the medium is uniaxial.

With Eq. (23) it can be shown that the perturbation expansion in Eqs. (13) is valid, in the sense that \(|\rho^2 \beta_{(y)}^{L_2} |e_{(y)}^{(1)}| \ll 1\) and \(|\beta_{(y)}^{L_2} |e_{(y)}^{(1)}| \ll |\beta_{(y)}|\), where \(\beta_{(y)}^{L_2}\) is the first-order correction to \(\beta_{(y)}\) and the first-order corrections to the modal fields are much smaller than the LP basis modes, as long as the waveguide is weakly guiding and the media are weakly anisotropic, i.e., as long as \(\Delta_1, \Delta_2, \Delta_3 \ll 1\) and \(\delta_1, \delta_2 \ll 1\). Many of the waveguides used in integrated optics are strongly asymmetric, violating the weak-guidance condition at one interface. It is shown in Section 5 that the perturbative approach is still adequate for such waveguides.

3. SYMMETRIC STEP-PROFILE PLANAR WAVEGUIDE

Despite the existence of exact solutions for the step-profile planar waveguide, it is useful to apply the present approximation technique to this case, both as a test of the method and to develop simple closed-form expressions for the dependence of the loss on various waveguide parameters. The results that are obtained for the step-profile planar guide are prototypical of those that are obtained for more complicated structures in Sections 4 and 5.

We assume a waveguide of the form shown in Fig. 2, with a step profile \(\delta f/\delta x = \delta(x + \rho) - \delta(x - \rho)\), where \(\rho\) is the half-thickness of the guiding layer. Since \(\delta f/\delta x\) is odd, the imaginary part of the propagation constant given by Eq. (23) is nonzero only when the product of the \(y\)-polarized zeroth-order mode \(e_{(y)}^{(0)}\) and the degenerate \(x\)-polarized radiation mode \(e_{(x)}^{(0)}\) is odd. If \(e_{(y)}^{(0)}\) is even, the expressions for the LP modes, \(e_{(y)}^{(0)}\) and \(e_{(x)}^{(0)}\), given in Eqs. (A2.1) and (A2.2), can be used in Eq. (23) to find
\[\beta_{(xy)}^{L_2} = \left\{4\beta_{(xy)} \Delta^2 \cot^2 \alpha \right\} \times \left\{\frac{U_x^2}{V_x^2} \left\{\sin^2 \frac{U_x}{Q_v} \left[1 + (V_x/Q_v) \cos^2 \frac{U_x}{U_x}\right]\right\}\right. \]  
where \(Q_v, U_x, U_y, \) and \(W_y\) are modal parameters that are defined in Table 2, and \(V_x\) is the normalized frequency for the \(y\) polarization that is defined in Table 1. Note that this expression contains three factors, the first of which is independent of the modal parameters, the second of which depends only on the parameters of the guided mode, and the third of which depends only on the parameters of the radiation mode. It will be seen that similar forms hold for the more complex waveguides that are discussed in Section 4. For odd \(e_{(y)}^{(0)}\), the imaginary part of the propagation constant can be obtained from Eq. (24) by interchanging \(\sin U_x\) and \(\cos U_x\).

Several basic features of the loss can be seen in Fig. 3, where the power-loss coefficient, \(2\beta_{th}\), in decibels per centimeter, is plotted against \(\alpha\) for a waveguide with the same parameters as those given in Fig. 6 of Ref. 2. The loss vanishes for \(\alpha\) that is smaller than a threshold angle \(\alpha_{th}\), rises steeply to a maximum, and then rolls off slowly with a series of similarly spaced minima. Quantitative description and qualitative understanding of these phenomena are facilitated by rewriting the expression for the loss as a product of a smooth envelope function \(g\) (shown by the dashed curve in Fig. 3) and a positive definite oscillatory function \(h\), which is of order unity except in special cases to be discussed later in this section. Thus \(\beta_{(xy)}^{L_2} = gh\), where \(g\) and \(h\) are given by
\[g = \begin{cases} 4\Delta^2 \cos^2 \frac{\rho}{Q_v} \frac{W_y U_x^2}{V_x^2(1 + W_y) U_x^2}, & \alpha > \alpha_{th}, \\ 0, & \alpha < \alpha_{th}, \end{cases} \]  
(25)
\[h = \frac{\sin^2 \frac{U_x}{Q_v} \sin^2 \frac{U_y}{Q_v} \cos^2 \frac{U_x}{U_x}}{[(Q_v^2/U_x^2) \sin^2 \frac{U_y}{Q_v} + \cos^2 \frac{U_x}{Q_v}]. \]  
(26)
The basic features of the envelope function are the threshold angle below which the loss vanishes, \(\alpha_{th}\); the maximum value of the loss \(g_{th}\), which occurs at an angle \(\alpha_{th}\); and the slow decrease with increasing \(\alpha\) up to \(\alpha = \pi/2\), where the envelope vanishes. For angles smaller than \(\alpha_{th}\), \(\beta_{(xy)}^{L_2} > \beta_{th}\), degenerate radiation modes exist, and the \(y\)-polarized mode is leaky. Within the range of angles
where degenerate radiation modes exist, the magnitude of the leakage depends, according to Eqs. (16) and (17), on the square of the product of \( \varepsilon_x \) and an overlap integral between the degenerate modes. At large angles the leakage is small because of the sin 2\( \alpha \) dependence of \( \varepsilon_x \), while for small angles the loss peak results from a trade-off between the increase in \( \varepsilon_x \) and the rapid decrease in overlap integral with \( \alpha \). The overlap is large for small \( \alpha \) because the waveguide is nearly isotropic; and, in an isotropic waveguide, degeneracy between a guided and a radiation mode can occur only as the guided mode approaches cutoff. For fixed \( \Delta_x \), \( g \) decreases as \( V_x \) increases. When the profile height is much smaller than the anisotropy, \( g \) is proportional to \( \Delta_x^2 \Delta_{\text{th}}^{-1/2} \delta_{31, \text{th}}^{1/2} \) for fixed \( V_x \).

Using the definitions of the modal parameters given in Table 2, we can obtain the threshold angle \( \alpha_{\text{th}} \) from the expression for \( \beta_{\text{th}} \) in Eq. (A7) [taking \( j = x \) in Eq. (A7)]. We find that

\[
\sin^2 \alpha_{\text{th}} = \frac{\Delta_x}{\delta_{31, \text{th}}} \left( \frac{W_z}{V_x} \right)^2 \quad \text{e}^{-a_{\text{th}}},
\]

(27)

where \( (W_z/V_x)^2 \) is only a function of \( V_x \) and is denoted conventionally as \( b(V_x) \). Both \( \delta_{31, \text{th}} \) and \( \Delta_x/(V_x)^2 \) are independent of \( \alpha \), but \( V_x \) and thus \( W_z \) are functions of \( \alpha \). If \( \Delta_x \) and \( \Delta_{\text{th}} \) are equal, \( V_x \) is independent of \( \alpha \), as is the RHS of Eq. (27); therefore \( \alpha_{\text{th}} \) can be expressed in closed form. Otherwise, both sides of Eq. (27) are functions of \( \alpha_{\text{th}} \) and \( \alpha_{\text{th}} \) can be obtained only numerically. As \( b(V_x) \equiv 1 \), \( \alpha_{\text{th}} \) is small when the profile height is small compared with the anisotropy but can approach \( \pi/2 \) when \( \Delta_x \) approaches \( \delta_{31, \text{th}} \).

\[\alpha_{\text{th}} = \left( \frac{2\pi}{\Delta_x} \right) \left( 1 + \frac{[-(3 + 2\kappa) + [(3 + 2\kappa)^2 + 40\kappa]^{1/2}]}{20\kappa} \right) e^{-a_{\text{th}}}, \]

(28a)

where

\[\kappa = \frac{\Delta_x}{\Delta_{\text{th}}}. \]

(28b)

The second term in Eq. (28a) can be approximated reasonably well by \( 1/3(1 + \kappa) \). Using this approximation, we obtain the maximum value of the envelope function by substituting \( \alpha_{\text{th}} \) into the expression for \( g \) given in Eq. (25), which results in

\[g_{\text{th}} = \gamma \beta_{\text{th}} \Delta_x \delta_{31, \text{th}}, \]

(29a)

where

\[\gamma = \frac{9\sqrt{3}(1 - b)(1 + \kappa)^{3/2}}{(V_x \sqrt{6} + 1)(4 + 3\kappa)^3(3 + 5\kappa)} \quad e^{-a_{\text{th}}}, \]

(29b)

\( \gamma \) is 0.46 at \( V_x = 0 \) and decays as \( V_x \) increases, and can be approximated by \( 0.785/V_x^3 \) when \( V_x \gg 1 \) and \( \Delta_x = \Delta_{\text{th}} \). The last two factors in Eq. (29a) show a linear increase with the anisotropy and the profile height when \( V_x \) is fixed. Comparing the expression for \( g_{\text{th}} \) with that for \( g \) obtained from Eq. (25), and using the expressions for \( U_x \) and \( Q_x \) given in Table 2, we find that as the anisotropy increases, the peak of the envelope increases, but the envelope decreases at large \( \alpha \), i.e., the loss becomes peaked in the small \( \alpha \) region.

The oscillatory function \( h \) [Eq. (26)] is related to the properties of the radiation mode that is degenerate with the guided mode. As \( \alpha \) increases, the denominator in the expression for \( h \) is generally of order unity, so that the nearly periodic minima in the leakage are due primarily to the \( \sin^2 \) \( U_x \) factor in the numerator. These minima can be understood either in a ray picture, where the loss vanishes when the radiation transmitted on successive reflections interferes destructively, or in a modal picture, where the loss vanishes according to the transformed overlap integral [Eq. (23)] when the magnitude of the \( x \)-polarized radiation mode at the core–cladding interface vanishes. The condition for the occurrence of a minimum is \( U_x = m\pi \), where \( m \) is an integer. While these minima are equally spaced in \( U_x \), they are not in general equally spaced in \( \alpha \), because of the nonlinear dependence of \( U_x \) on \( \alpha \) (see Table 2). If \( \Delta_x \ll \delta_{31, \text{th}} \), we can use the approximate expression for \( U_x \) in Table 2 to obtain the angular separations between the minima as

\[\alpha_m - \alpha_{m-1} = c[m(1 - c^2(m - 1)^2)^{1/2} - (m - 1)(1 - c^2 m^2)^{1/2}], \]

(30a)

where \( c \) is independent of \( \alpha \) and is given by

\[c = \frac{\pi}{V_x} \left( \frac{\Delta_x}{\delta_{31, \text{th}}} \right)^{1/2}. \]

(30b)

Noting that \( V_x/\Delta_x^{1/2} \sim \rho \beta_{\text{th}} \), we see that the spacing is independent of \( \Delta_x \), if the thickness of the waveguide is fixed.

Exact solutions for the modal fields and a transcendental equation for \( \beta \) that is solved numerically are presented in Ref. 2. It can be shown that, to first order in \( \Delta \) and \( \delta \), the exact solution for \( \beta \) deduced from Eq. (26) of Ref. 2 and the perturbative result obtained here are identical [except when the degenerate radiation mode is just above the cutoff of an odd guided mode, i.e., when \( V_x \sim (1/2 + m\pi) \)]. The approximate values for the loss that are presented in Fig. 3 are indistinguishable on the scale of the graph from the results obtained from the exact analysis of Ref. 2.

Figure 4 shows the losses of two waveguides with the same \( V_x \) and \( \delta_{31} \) but with different \( \Delta \) (for simplicity, we
\[ \gamma V^2 \tan^2 V = 7.04 \text{ for } \Delta = 0.002 \text{ and } \gamma V^2 \tan^2 V = 0.012 \text{ for } \Delta = 0.0005. \] For large \( V \) the limiting form \( \gamma V^2 \to 0.785/V \) is convenient.

4. OTHER EXAMPLES OF WEAKLY GUIDING WAVEGUIDES

A. Asymmetric Step-Profile Planar Waveguides

Most waveguides that are used in practical applications are strongly asymmetric. In this section we consider the behavior of asymmetric waveguides but keep the assumption of weak guidance at both interfaces. In Section 5 we consider the extension to the case of one weakly guiding and one strongly guiding interface.

The form of the step-profile asymmetric planar waveguide considered in this section is illustrated in Fig. 6, where it is assumed that \( n_{x,cl} < n_{y,cl} \) and \( \delta f/\delta x = (\Delta x_2/\Delta x_1)\delta(x + \rho) - \delta(x) \), with \( \rho \) the thickness of the waveguide. If \( k n_{x,cl} < \beta(y) < k n_{x,cl} \) (which is usually the case for small \( \alpha \), especially when \( \alpha \) is near the threshold angle),

\[ \begin{align*}
\Delta_1 &= \Delta_3 = 0.0002 \\
\Delta_1 &= \Delta_3 = 0.002
\end{align*} \]

Fig. 4. Loss of the lowest-order leaky modes for two symmetric step-profile planar waveguides with the same normalized frequencies \( V = 1.4 \). The refractive indices in the film for both waveguides are chosen as \( n_{x,co} = 2.281 \) and \( n_{x,co} = 2.174 \). The profile heights, assumed to be the same for ordinary and extraordinary polarizations but different for the two waveguides, are 0.0005 for the solid curve and 0.002 for the dashed curve. The wavelength is \( \lambda = 0.6328 \mu m \).

choose \( \Delta = \Delta_3 \); therefore \( \Delta_1 = \Delta = \Delta_3 \) and \( V_r = V_e = V \), and \( \Delta \ll \delta \). For waveguides with fixed \( V \) and \( \Delta_3 \), Eq. (27) indicates that the threshold angle \( \alpha_{th} \) is proportional to the square root of \( \Delta \). Eq. (29a) shows that the peak of the envelope increases linearly with \( \Delta \), and it can be deduced from Eq. (25) that the envelope function \( g \) is proportional to \( \Delta^{1/2} \), a result that is consistent with the results shown in Fig. 4.

While estimates of \( \beta_{max} \) based on the envelope function \( g \) are generally valid for \( \alpha > \alpha_{th} \), in the vicinity of the first loss peak the oscillatory function \( h \) can have a significant effect on the magnitude of the loss in certain cases. If a zero of \( h \) occurs near \( \alpha_{th} \), the peak of the loss is reduced and the location of the peak shifts slightly. If \( Q_x \to 0 \) and \( V_r \to (m + 1/2)\pi \), it can be seen from Eq. (26) that \( h \to \infty \). The calculated loss then diverges, and the perturbation method clearly is no longer valid. This result is consistent with the exact results of Ref. 2, which show that in this limit \( \beta_{max} \) is of order unity, so that the loss is extremely large and the perturbation method can be expected to break down. In the context of the present analysis these conditions correspond to the degeneracy of the guided mode with a radiation mode that is just at cutoff.

Figure 5 illustrates the losses of two waveguides with the same \( \rho \) and \( \delta_3 \) but different \( \Delta \) (again, \( \Delta_1 = \Delta_3 = \Delta \) and \( V_r = V_e = V \)) and therefore different \( V \). From Fig. 5 we can see that the losses for \( \Delta = 0.002 \) are generally larger than those for \( \Delta = 0.0005 \) except in the vicinity of the first loss peak, and the minima of the two losses occur at the same angle. The magnitude of the loss for \( \alpha = \alpha_{th} \) is dominated by the \( \Delta^{1/2} \) scaling of \( g \) that can be seen from Eq. (25), and the positions of the minima follow from Eq. (30). The threshold angle \( \alpha_{th} \), given by Eq. (27), varies only with \( W_F \) for waveguides with fixed \( \rho \), so that the smaller \( \alpha_{th} \) for the waveguide with smaller \( \Delta \) is expected. For \( \alpha = \alpha_{th} \), the magnitude of the loss can be calculated by using the limiting form of \( h \approx \tan^3 V \) obtained from Eq. (26) with \( Q_x \to 1 \). In this limit, \( gh \approx \gamma V^2 \tan^3 V \) for fixed \( \rho \), so that the smaller peak loss for the waveguide with larger \( \Delta \) can be seen from

\[ \begin{align*}
\Delta_1 &= \Delta_3 = 0.0005 \\
\Delta_1 &= \Delta_3 = 0.002
\end{align*} \]
and if we use the expressions for LP modes \(e_{ij}^{(0)}\) and \(e_{ij}^{(1)}\), given in Eqs. (B6) and (B9) and their normalization constants \(\langle e_{ij}^{(0)}\rangle^2\) and \(N_{ij}\), given by Eqs. (B7) and (B12), respectively, Eq. (23) becomes

\[
\beta_{ij}^{(1)} = \chi \left( \frac{\Delta_{ij} \Delta_{kl}}{Q_{kl}} V_{kl}^2 \sin(\pi U_{m} - \phi_{lm}) \right) \left( 1 + \frac{1}{\psi_{kl}} \right),
\]

(31a)

where

\[
\chi = \left( 4 \beta_{ij} \Delta_{kl} \Delta_{m} \cot^2 \alpha \right) \left( \frac{U_{k}^2}{V_{kl}^2} \frac{1}{1 + \frac{1}{\psi_{kl}}} \right).
\]

(31b)

Notice that, following the notation of Refs. 15 and 16, the characteristic dimension \(\rho\), used in the normalized frequency \(V\), is the full thickness for asymmetric waveguides and the half-thickness for symmetric waveguides.

If \(\beta_{ij}^{(0)} < k n_{x,el}\), the radiation modes propagate in both cladding regions and for each propagation constant two kinds of radiation modes exist, so that Eq. (23) becomes a sum of two terms. Using Eq. (B13) for the two types of radiation mode, we obtain

\[
\beta_{ij}^{(1)} = \chi \sum_{m=1,2} \left[ \frac{((\Delta_{kl} \Delta_{m})/(\psi_{kl} V_{m}^2))^{1/2} - (\Delta_{kl} \Delta_{m})^{1/2} \cos(U_{m} - \phi_{lm}))}{(Q_{kl}^2)/(Q_{m}^2)} \right]^{1/2} \left[ 1 + \frac{1}{\psi_{kl}} \right] \left[ (\Delta_{kl} \Delta_{m})^{1/2} \sin(U_{m} - \phi_{lm}) + (Q_{kl}/Q_{m}) \right]^{1/2} \left[ 1 + \frac{1}{\psi_{kl}} \right]^{1/2}.
\]

(32)

where the \(F_m\), parameters in the expressions for the radiation field, are given by Eq. (B15) and \(\phi_{lm} = \cos^{-1}(1/F_m^{1/2})\).

While the form of the expressions for the loss for an asymmetric waveguide, Eqs. (31a) and (32), is substantially more complicated than that for a symmetric waveguide, Eq. (24), the essential behaviors of these waveguides are similar. Figure 7 shows the loss for an asymmetric waveguide whose parameters are the same as those of the symmetric waveguides shown in Fig. 5, except that \(\Delta_{x} = 0.002\) and \(\Delta_{y} = 0.0005\) (again, \(\Delta_{x} = \Delta_{y}\), and thus \(\Delta_{x} = \Delta_{y}\)). We can see that, except in the vicinity of \(\alpha_{th}\), the envelope of the loss for the asymmetric guide falls between the envelopes of the symmetric waveguides, and the oscillations of the loss are weaker for the asymmetric waveguide. The behavior of the envelopes could be anticipated from the first factors in Eqs. (31a) and (32), \(\chi\), which contain the product \(\Delta_{x} \Delta_{y}\). It can be shown that \(\beta_{ij}^{(1)} \frac{(\Delta_{ij} \Delta_{kl})}{(Q_{kl}) V_{kl}^2} > (\Delta_{ij} \Delta_{kl})^{1/2}\) is generally true when \(k n_{x,el} < |\beta_{ij}^{(1)}| < k n_{x,el}\). When \(\beta_{ij}^{(1)} < k n_{x,el}\), the magnitude of one of the \(F_m\) [which could be either \(F_1\) or \(F_2\) depending on the sign of \(\sin(2U_m)\)] is smaller than 1/2, the loss from the coupling to the mode with the smaller \(F_m\) dominates, and \((\Delta_{ij} \Delta_{kl}) (\Delta_{ij} \Delta_{kl})^{1/2} > 1/F_m^{1/2}\) for this smaller \(F_m\). Therefore the oscillatory function represented by the third factor in Eqs. (31a) and (32) has no zeros. The divergence of the calculated loss noted for the symmetric waveguide when the guided mode is degenerate with a near-cutoff radiation mode is also observed in the asymmetric waveguide, and it occurs when \(V_{kl} = \cos^{-1}(\Delta_{ij} \Delta_{kl})^{1/2} (1/2 + m)\pi\), which is again the cutoff condition for the \(m\)th guided mode.

### B. Step-Profile Circular Fiber

Analytical solutions for the modes of anisotropic fibers exist only when the optical and the fiber axes are parallel.\(^{17}\)

However, with weak-guidance and weak-anisotropy assumptions, approximate solutions are possible. Here we consider step-index fibers, as they yield simple results and are prototypical of other index profiles. Such step-profile anisotropic fibers can be obtained experimentally, for example, by growth of organic-crystal cores in glass capillaries\(^{18}\) or by diffusion of claddings in lithium niobate fibers.\(^{19}\) A typical step-profile circular waveguide is sketched in Fig. 8.

Snyder and Ruhl\(^1\) have studied the case of a step-profile fiber whose axis lies along one of the principal axes of the dielectric tensor.\(^1\) In such a waveguide \(\epsilon_{ij} = 0\), and only the structural coupling terms contribute to the loss. In this section we consider the loss in fibers when \(\epsilon_{ij} \neq 0\) and compare it with the loss in the cases in which \(\epsilon_{ij} = 0\).

Even when \(\epsilon_{ij} \neq 0\), the approximate Eqs. (14) have closed-form solutions. Coupling terms that are due both to \(\epsilon_{ij}\) and to structural coupling contribute to the total loss. As a consequence of the orthogonality of the radiation modes involved in these two cases, the two contributions add without interference. The contribution to \(\beta^{(1)}\) from the \(\epsilon_{ij} \neq 0\) coupling term, which couples the lowest-order y-polarized guided mode \(e_{ij}^{(0)}\), with the x-polarized radiation mode having \(\cos\phi\) symmetry, is

\[
\beta_{ij}^{(1)} = (\beta_{ij}^{(1)} \Delta_{x} \Delta_{y} \cot^2 \alpha \left[ \frac{\pi W_{x}^2 J_{x}^2(U_{x})}{V_{x} J_{x}^2(U_{x})} \right] [\rho_{x}^2 \mu_{x}^2(U_{x})].
\]

(33)

Equation (33) is obtained from Eq. (23) by the use of the expressions for the LP modes \(e_{ij}^{(0)}\) and \(e_{ij}^{(1)}\) given in Eqs. (B24) and (B25) and the associated normalization integral and constant, \(\langle e_{ij}^{(0)}\rangle^2\) and \(N_{ij}\), given in Eqs. (B28) and (B29). The definition of the coefficient \(\rho_{ij}\) is given by Eq. (B26). Note that, as in Eq. (24) for the planar waveguide, the expression for the loss contains a factor that is independent of modal parameters, one that is dependent only on the guided mode, and one that is dependent only on the radiation mode.

![Fig. 7. Loss coefficient of the lowest-order leaky mode for the asymmetric waveguide in Fig. 6, where \(\rho = 4.078\,\mu m, \lambda = 0.6328\,\mu m, n_{1,oo} = 2.281, n_{1,em} = 2.174, \Delta_{x1} = \Delta_{y1} = 0.002, and \Delta_{x2} = \Delta_{y2} = 0.0005.\)](image)
Comparing Fig. 9 with the loss coefficient for the symmetric planar waveguide with \( \Delta_x = \Delta_y = 0.002 \) as shown in Fig. 5, we find that the loss in the fiber resembles closely the loss in the planar waveguide. However, \( \alpha_y \) is smaller, which can be expected from the smaller \( r \) in the fiber for fixed \( V_r \), and from the expression for \( \alpha_y \) given by Eq. (27), and the locations of the maxima and the zeros are shifted. It can also be shown that, for fixed waveguide parameters \( \Delta, V (\Delta_x = \Delta_y = \Delta, V_x = V_y = V) \), the envelope of the loss in the fiber is smaller than that in the symmetric waveguide when \( V < 1.1 \), is larger when \( V > 1.1 \), and is approximated by \( 1.17g \) when \( V \gg 1 \), where \( g \) is the envelope function for the loss in a symmetric planar waveguide as given by Eq. (25).

As was the case for the planar waveguide, the loss coefficient for the anisotropic fiber becomes extremely large when \( Q_x \to 0 \) and the first antisymmetric mode is near cutoff. This behavior is a result of the factor \( |p| \) in Eqs. (33) and (64a), where \( p_i \), defined in Eq. (22b), is proportional to \( 1/F_{ij} (V) \) and hence diverges when \( V \to 2.405 \), which is the cutoff condition for the first antisymmetric mode. This divergence in \( |p| \) leads to a divergence in the loss coefficient unless \( \epsilon_{xy} = 0 \).

### D. Symmetric Linear-Profile Waveguide

Thus far we have compared several types of waveguide, but all have had step profiles. In order to determine the effect of grading the refractive-index profile, we consider an analytically tractable case, the symmetric linear-index profile, illustrated in Fig. 10. This profile is defined by

\[
f(x) = \begin{cases} 
|x|/\rho & |x| < \rho, \\
1 & |x| > \rho,
\end{cases}
\]

where \( i = x, y, \) and \( \rho \) is the half-thickness of the waveguide. The LP modes for this profile are presented in Appendix B. In order to have a closed-form expression for the loss, we assume here that \( \Delta_1 = \Delta_3 \), and hence \( \Delta_2 = \Delta_4 \).

For this profile, the overlap integral in Eq. (23), \( \langle \phi_f | \partial/\partial x | \phi_0 \rangle = \int 0 e^{i \phi_0} e^{i \phi_f} dx \), reduces to \( \int 0 e^{i \phi_0} e^{i \phi_f} dx \). Rather than carry out this complicated integration, we transform the coupling coefficient to a simpler form that involves inte-
Fig. 10. Symmetric linear-index profile, where the half-thickness of the guiding layer is \( \rho \).

...tion only over the cladding region. Since \( \epsilon_{(y)}^{(0)} \) and \( \epsilon_{(x)}^{(0)} \) are degenerate, it can be derived from Eqs. (18) that

\[
(\epsilon_{xx} - \epsilon_{yy})\epsilon_{(x)}^{(0)}\epsilon_{(y)}^{(0)} = 0.
\]

(36)

As we have assumed that \( \Delta_1 = \Delta_3 \), \( \epsilon_{xx} - \epsilon_{yy} \) is independent of \( x \) and can be taken out of the integral in Eq. (36), it follows that

\[
\big\langle \epsilon_{(y)}^{(0)} \epsilon_{(x)}^{(0)} \big\rangle = 0.
\]

(37)

Therefore the integral in the guiding layer is equal in magnitude and opposite in sign to the integral in the substrate, which is easier to evaluate because of the simpler form of the fields in the latter region. With this approach, and using the LP modes given in Eqs. (B30) and (B31), the expression for \( \beta_{\text{lin}}^{\text{im}} \) given in Eq. (23) becomes

\[
\beta_{\text{lin}}^{\text{im}} = (4\beta_{(y,0)}\Delta_0^2 \cot^2 \alpha) \left[ \frac{s_i(1)}{2\epsilon_{(y)}^{(0)}} \right] \left( \frac{Q_x}{U_x^2 - U_y^2} \right)^2 \left( \frac{s_i(1) + W_{(y,0)} s_i(1)p_i(1)}{s_i(1) + Q_{(y,0)}^2 s_i(1)} \right),
\]

(38a)

where \( s_i, i = x, y \) are combinations of the Airy functions defined by Eq. (B33), \( \epsilon_{(y)}^{(0)} \epsilon_{(x)}^{(0)} \) is given in Eqs. (B37), and \( s_i' \) is the derivative of the function \( s_i \).

Similarly to those in Eq. (24) for the symmetric slab waveguide, the first three factors of Eq. (38a) behave like the envelope function \( g \) given by Eq. (25), and the last factor is comparable with the oscillatory function \( h \) given by Eq. (26). To compare the two envelope functions, we use the relation between \( U_x^2 \) and \( U_y^2 \) given in Table 1 and find that the ratio of the envelope function for linear profiles to the envelope function for step profiles is

\[
\frac{\delta_{\text{lin}}}{\delta_{\text{step}}} = \frac{U_x^{\text{step}}}{(U_x^2 - U_y^2)^{\text{step}}} \sin \alpha \left( \frac{\Delta_x}{V_x^2 \delta_{\text{lin}, \alpha} \sin \alpha} \right),
\]

(38b)

where we neglect the difference in the amplitudes of the bound-mode fields at \( R = 1 \), i.e., we take \( s_i(1)/2\epsilon_{(y)}^{(0)} \approx (U_x^2/V_x^2)/(1 + 1/W_x) \). The difference comes from the overlap integral in Eq. (23), which is equal to the value of \( \epsilon_{(y)}^{(0)} \epsilon_{(x)}^{(0)} \) at the interface in the cases of a step profile but is a weighted average of this product when the profile is graded. From Eq. (38b), we conclude that waveguides with linear profiles are less leaky than waveguides with step profiles when \( V^2 \delta_{\text{lin}}/\Delta \gg 1 \), such as for Ti:LiNbO_3 waveguides; but, if \( V^2 \delta_{\text{lin}}/\Delta \) is of order unity, as is the case in a proton-exchanged waveguide, the sizes of the losses in waveguides with linear profiles and step profiles are comparable. The above statements are verified by Figs. 11 and 12, which show the plots of loss versus \( \alpha \) for linear and step-profile waveguides with the same \( \Delta \) and \( V \), where \( V(\delta_{(y)}/\Delta)^{1/2} \) is 14 in Fig. 11 and 0.95 in Fig. 12. Although the above conclusions are drawn for the case of linear profiles, we expect a similar trend for all graded profiles.

The linear graded profile again shows high loss when \( Q_x \rightarrow 0 \) and \( s_i(1) = 0 \); the latter can be shown to be the cutoff condition for the first antisymmetric mode.

5. EXTENSIONS OF THE METHOD

A. Positive Uniaxial Waveguides

In the previous analysis in Subsection 4.D, we have assumed that the media are negative uniaxial. For a
negative uniaxial waveguide as shown in Fig. 1, only the $y$-polarized modes are leaky, and the loss appears when $\alpha$ exceeds a threshold angle $\alpha_{th}$. Analogously, for positive uniaxial waveguides, only the $x$-polarized modes are leaky, and the corresponding threshold angle for leakage is the same as the one in negative media given by Eq. (27). All the expressions for the leakage rate that are derived with negative uniaxial guides can also be used for positive uniaxial guides, provided that the coordinates $x$ and $y$ are interchanged. Thus, the loss in positive uniaxial waveguides behaves in the same way as in negative uniaxial waveguides with comparable anisotropy and index profiles.

B. Strongly Asymmetric Waveguides

The results in Sections 3 and 4 were derived under the assumptions of weak guidance and weak anisotropy. In many cases of practical interest one or both of these assumptions are violated to some degree. It is interesting to investigate the possibility of extending the analysis to those cases. In this section we discuss the case of a strongly asymmetric planar waveguide for which the weak-guidance assumption is violated at one interface, and then we touch on the question of strong birefringence.

A common type of waveguide is constructed by diffusing a dopant into a thin layer at the surface of a substrate, with air, or perhaps a low-index-of-refraction buffer, serving as the superstrate. Such a waveguide is shown schematically in Fig. 13. The thickness of the waveguide is $\rho$, and the profile heights at the interface between the guiding layer and the substrate, $\Delta_{x2}$ and $\Delta_{y2}$, result from diffusion and are thus small; the sizes of $\Delta_{x1}$ and $\Delta_{y1}$, the profile heights at $x = 0$ (the air-film interface), are of order unity. The material anisotropy, as in the previous cases, is assumed to be weak everywhere. In this case the only deviation from our previous model is the relaxation of the weak guidance assumption at one interface.

While $\Delta \ll 1$ is used in the previous analysis as a sufficient condition for the validity of the perturbation expansion (except for special points where the guided mode is degenerate with near-cutoff radiation modes), a necessary condition for valid approximation is only that the coupling terms on the RHS’s of Eqs. (10) be small. For the planar waveguide shown in Fig. 13, the coupling terms given by Eqs. (11) reduce to

$$P_{zz} \tilde{e}_z = \frac{\partial}{\partial x} \left( 1 - \varepsilon_{zz} \frac{\partial \varphi_{zz}}{\partial x} + \varepsilon_{zz} \frac{\partial}{\partial x} \ln \varepsilon_{zz} \right),$$

$$P_{zz} \tilde{e}_y = \varepsilon_{zz} \frac{\partial}{\partial x} (\varepsilon_{zz} \tilde{e}_z),$$

$$P_{zz} \tilde{e}_x = \frac{k^2}{i \beta} \frac{\partial}{\partial x} (\varepsilon_{zz} \tilde{e}_z),$$

$$P_{zz} \tilde{e}_y = 0.$$  

Examining Eqs. (39), we find that all coupling terms are small except the second term on the RHS of Eq. (39a), the expression for $P_{zz} \tilde{e}_z$. Therefore this term should be added to the RHS of Eq. (14a) for the zeroth-order $x$-polarized mode. The zeroth-order modes of a step profile obtained in this way are the ordinary TM modes of an isotropic waveguide, given in Eqs. (B17) and (B20).

After moving the second $P_{zz} \tilde{e}_z$ term to the zeroth-order equations, we can repeat the derivation leading from Eqs. (18) to Eq. (23) and find the second-order correction to $\beta_{y1}$ to be

$$\beta_{y1}^{(2)} = \beta_{y1}^{(0)} \pi \rho^2 \Delta_{y2} \cot^2 \alpha \frac{\varepsilon_{xx} \varepsilon_{zz} \frac{\partial \varphi_{zz}}{\partial x}}{\varepsilon_{xx} \varepsilon_{zz} \frac{\partial \varphi_{zz}}{\partial x} \varepsilon_{yy} \varepsilon_{zz} \Psi_{y1}^{(0)} \Psi_{y2}^{(0)}} N_{z2} Q_{z2},$$  

(40a)

where $N_{z2}$ is defined by

$$N_{z2} (Q_{z2} \varphi_{zz} (Q_{z2} \varphi_{zz} (Q_{zz} \varphi_{zz}))) = N_{z2} \delta (Q_{zz} - Q_{zz}).$$  

(40b)

Note that, with this definition, $f(x)$ is no longer of order unity, having a maximum value $\Delta_{x2}/\Delta_{x2}$. In deriving Eq. (40a) we have not assumed any particular form for the index profile, and hence it can be applied to graded as well as to step profiles. Comparing Eq. (40a) with Eq. (23), the loss for $\Delta \ll 1$, we find that the only difference is that the overlap integral is modified by $\varepsilon_{xx}, \varepsilon_{zz}$ in Eq. (40a).

We need to check the criteria for the validity of the perturbation eigenvalues, i.e., $|\beta_{y1}^{(1)}|, |\beta_{y2}^{(1)}| \ll |\beta_{y1}^{(0)}|$. It can be shown that the first-order correction to $\beta_{y1}, \beta_{y2}$ proportional to $\varepsilon_{zz}^{(0)} P_{zz} e_{zz}^{(0)}$, and therefore vanishes in this case because $P_{zz} e_{zz}^{(0)} = 0$. The real and imaginary parts of $\beta_{y1}^{(1)}$ are of the same order, and thus we check only the imaginary part. From Eq. (40a) we see that, in order to prove that $|\beta_{y2}^{(1)}| \ll |\beta_{y1}^{(0)}|$, we need to demonstrate that the product $\Delta_{x} \frac{\partial f_{zz}}{\partial x} e_{zz}^{(0)} \Psi_{z2}^{(0)}$ in the overlap integral is much smaller than unity. On the small $\Delta$ side the product is small, while on the other side, where $\Delta$ is of order unity, if it can be shown that the transverse fields are small in regions where the index of refraction is changing, then the product is small. We now specialize to the case of step-profile waveguides. For planar waveguides with step-index profiles, the index of refraction varies only at the interfaces between the guiding layer and the substrates or superstrates. In addition, because the guide is strongly asymmetric, $U_{2} \ll V_{1}, U_{2} \ll V_{1}$, and hence $W_{2}$ and $W_{1}$ can be approximated by $V_{2}$ and $V_{1}$. In these limits it can be shown from Eqs. (B6) and (B20) that, at fixed $V$, and $V_{2}$, the magnitude of the transverse fields, $e_{zz}^{(0)}$ at $x = 0$ (where $\Delta$ is of order unity), is proportional to $(\Delta_{y2}/\Delta_{x})^{1/2}$ (notice that $e_{zz}^{(0)}$ has been set to be of order unity at the small $\Delta$)}

![Fig. 13. Index profile for a strongly asymmetric planar waveguide, where the superstrate is air. $\Delta_{x1}, \Delta_{x2} \ll 1$, but $\Delta_{x1}$ and $\Delta_{y1}$ are of the order unity.](image-url)
interface), so that \( \Delta_\nu (\partial f_v/\partial x) e^{(0)}_{V_1}(x) e^{(0)}_{V_2}(x) \) is approximately proportional to \((\Delta_{\nu 2}\Delta_{\nu 1})^{1/2}\) and is small there.

So far we have proved that the perturbation analysis can be applied in the case of strongly asymmetric step-index waveguides, with one extra term retained in the zeroth-order equations for the x-polarized LP modes. To calculate the loss we use Eqs. (B6) and (B20) for \( e^{(0)}_{V_1} \) and \( e^{(0)}_{V_2} \) and use the normalization constants given by Eqs. (B17) and (B27) in Eq. (40a), which becomes

\[
\beta_{im}(\nu) = (4\beta_{im} \Delta_{\nu 2}^2 \cot^2 \alpha) \left[ \frac{U_2^2 W_2}{V_2^2 (1 + W_2)} \right] \eta^2 
\times \frac{(1 - \eta^{-1} \sin U_1)^2}{Q_1[1 + (V_2^2/Q_1^2) \cos^2 U_1]}, \tag{41a}
\]

where

\[
\eta = \frac{1}{2} \left( \frac{\varepsilon_{xx,c}}{\varepsilon_{xx cl}} + \frac{\varepsilon_{xx,ll}}{\varepsilon_{xx,co}} \right) \frac{V_2 U_1}{V_2^2}, \tag{41b}
\]

As was the case for the weakly guiding waveguide to first order in \( \Delta_{\nu 2} \) and \( \delta_{31} \), the results of the perturbation analysis agree with the exact results from Appendix C.

Now we compare Eq. (41a) with its counterpart for symmetric planar waveguides, Eq. (24). Equation (41a) looks similar to the loss for a symmetric waveguide with profile heights equal to \( \Delta_{\nu 2} \) and \( \Delta_{\nu 1} \) and half-thickness equal to \( \rho \), except for the last oscillatory factor and the factor \( \eta^2 \) in the envelope. \( \eta \) is of order unity at \( \alpha = 0 \) and increases monotonically with \( \alpha \). When \( \delta_{31} \gg \Delta_{\nu 2}, \Delta_{\nu 1}, \eta = (\Delta_{\nu 2} \delta_{31})^{1/2} \sin \alpha \Delta_{\nu 2} \gg 1 \), so that the loss coefficient of the strongly asymmetric waveguide can be considerably larger than that of a weakly asymmetric waveguide.

The envelope becomes flat in strongly asymmetric guides because \( \eta \) increases with \( \alpha \) and hence lifts the envelope. \( \eta \) is independent of \( \Delta_{\nu 2} \) and \( \Delta_{\nu 1} \) in the limit \( \Delta_{\nu 1}, \Delta_{\nu 1} \gg \Delta_{\nu 2}, \Delta_{\nu 2} \), since, from the definition of \( U_1 \) given in Table 2 and the definitions of \( V_2 \) and \( \Delta \), in Table 1, it can be shown that \( U_1 \) does not depend on \( \Delta_{\nu 1} \) and \( \Delta_{\nu 2} \) in that limit and \( V_2^2/\Delta_{\nu 2} \) is always independent of \( \Delta_{\nu 2} \) and \( \Delta_{\nu 1} \). The other factors in the envelope of the loss given by Eq. (41a) are also independent of \( \Delta_{\nu 2} \) and \( \Delta_{\nu 1} \); thus the envelope of the loss does not depend on the large profile heights. \( \eta^2 \) is also the ratio of the envelope of the loss given in Eq. (41a) to \( g \), the envelope function for symmetric waveguides given by Eq. (25). It has been mentioned before that this ratio is proportional to \( \delta_{31} \sin^2 \alpha \Delta_{\nu 2} \), which is of order unity when the anisotropy is comparable with the small profile heights, but it is much larger than unity when \( \delta_{31} \gg \Delta_{\nu 2} \), unless \( \alpha \) is small.

There are two terms in the numerator of the last oscillatory factor in Eq. (41a); the first one is a constant and arises from the integral over the region near \( x = 0 \), where \( \Delta \) is of order unity, and the second one is an oscillatory function of \( \alpha \) with a magnitude \( \eta^{-1} \) and comes from the integral over the region near the small \( \Delta \) interface. At small \( \alpha \) the two terms are comparable in magnitude. As \( \alpha \) increases, the first term becomes increasingly important and the oscillation becomes weaker, until, if \( \delta_{31} \gg \Delta_{\nu 2} \), the oscillation disappears. Figure 14 shows the loss coefficient of a strongly asymmetric waveguide, with air as the superstrate. We see that the oscillations decrease as \( \alpha \) increases, and the envelope is quite flat as a function of \( \alpha \), compared with the loss for symmetric waveguides given in Fig. 4.

We can reach the following conclusions: For a strongly asymmetric waveguide, the loss is insensitive to the large profile height. When the anisotropy \( \delta \) is of the same order as the small profile height \( \Delta \), the size of the loss for a strongly asymmetric waveguide is comparable with that of a symmetric waveguide with a profile height equal to \( \Delta \). When \( \delta \gg \Delta \), the leakage rate for the strongly asymmetric waveguide is larger than that of the symmetric waveguide by a factor of \( \delta \sin^2 \alpha / \Delta \). In the latter case burying the waveguide with a buffer layer between the air and the guiding layer will reduce the loss substantially.

For a general waveguide with one large index step, we would expect the behavior of the coupling to be similar to that of a step-index waveguide. The results of perturbation calculations should be accurate for asymmetric planar waveguides if \( \Delta \) is small at least on one side, if Eq. (23) for the coupling coefficient is replaced with the modified form, Eq. (40).

Waves with highly birefringent cores could also occur in practice. In the case of strong anisotropy but weak guidance, the perturbation method is still useful. The modal fields and the propagation constant can be evaluated by methods similar to those that lead to Eq. (40), incorporating modified forms of the \( P_i e_i \) terms. These results will be discussed in a future publication.

5. SUMMARY

Guided modes of anisotropic waveguides are often leaky because of coupling to degenerate orthogonally polarized radiation modes. We have extended the calculation of these losses that was first presented in Ref. 1 to include waveguides with arbitrarily oriented dielectric tensors. When a principal axis is not aligned with the waveguide axis, the loss arising from the off-axis components of the dielectric tensor is much larger than the structural loss present in two-dimensional anisotropic waveguides that have a principal axis aligned with the waveguide axis. We have derived a simplified expression for the loss, which shows that the coupling occurs in regions where the refractive index is spatially varying. Application of these results to step-index planar waveguides, for which exact
solutions are available, shows that the approximate results are accurate in the limit of small anisotropy even for strongly asymmetric waveguides, as long as one of the interfaces is weakly guiding.

Examples drawn from planar and fiber waveguides illustrate general trends for the dependence of the loss on the profile height $\Delta$, the angle between the optical axis and the waveguide axis $\alpha$, and the anisotropy $\delta$. For symmetric planar waveguides, the loss is proportional to $\Delta^2 \cot^2 \alpha$. For strongly asymmetric waveguides, the loss is insensitive to the large-profile height and proportional to the square of the small-profile height $\Delta$. Compared with a symmetric waveguide that has the same $\Delta$, the loss of a strongly asymmetric waveguide is much larger when $\Delta \ll \delta$ but is of the same order when $\Delta = \delta$. The loss for a birefringent fiber with $\alpha \neq 0$ is of the order $\Delta^{-1}$ larger than that of a fiber with $\alpha = 0$ and is comparable with that of a symmetric planar waveguide that has similar profile height and anisotropy. The loss of a planar waveguide with a linearly graded refractive-index profile is much smaller than that of a step profile when $2\pi r b \Delta^2 \alpha / \lambda \gg 1$ but is comparable when this quantity is of order unity.

**APPENDIX A: INTEGRAL EXPRESSIONS FOR THE COUPLED FIELDS AND THE PROPAGATION CONSTANT**

Snyder and Ruhl\(^1\) approximate the $j$ component of a predominantly $i$-polarized leaky mode ($i$ and $j$ are set to be $y$ and $x$, respectively) by the integral

$$
e_{(i,j)}^{(1)} = \int_{0}^{\Phi_{\beta}} a_{\beta} e_{(i,j)}^{(0)}(Q_{j}) dQ_{j},$$  \hspace{1cm} (A1)

where $e_{(i,j)}^{(0)}(Q_{j})$ are the continuum solutions to the homogeneous Eq. (14a). To find the modal coefficients $a_{\beta}$, one first substitutes the form given in Eq. (A1) for $e_{(i,j)}^{(1)}$ in Eq. (14b), then multiplies Eq. (14b) by $e_{(i,j)}^{(0)*}(Q_{j})$. With the orthogonality relation

$$\langle e_{(i,j)}^{(0)*}(Q_{j}) e_{(i,j)}^{(0)}(Q_{j}) \rangle = N_{e} \delta(Q_{j} - Q_{j}'),$$  \hspace{1cm} (A2)

$a_{\beta}$ is obtained as

$$a_{\beta} = \frac{\langle e_{(i,j)}^{(0)*}(Q_{j}) P_{\beta} e_{(i,j)}^{(0)}(Q_{j}) \rangle}{N_{e} (\beta_{j,\alpha \beta}^{2} - \beta_{(i,j)}^{2})},$$  \hspace{1cm} (A3)

and the $j$ component of the leaky mode becomes

$$e_{(i,j)}^{(1)} = -\int_{0}^{\Phi_{\beta}} a_{\beta} e_{(i,j)}^{(0)}(Q_{j}) dQ_{j} = -\rho^{2} \beta_{j,\alpha \beta}^{2} e_{(i,j)}^{(0)}(Q_{j}) dQ_{j},$$  \hspace{1cm} (A4)

where the relation $dQ_{j} = -\rho^{2} \beta_{j,\alpha \beta}^{2} dQ_{j} / \beta_{j,\alpha \beta}$ has been used. In the expression for $\beta_{i,j}^{2}$ given by Eq. (15), only the third term can result in an imaginary component. Thus, by taking $e_{(i,j)}^{(1)}$ in the form given in (A4), one is led to

$$\beta_{i,j}^{2} = \frac{1}{2 \beta_{j,\alpha \beta}^{2} e_{(i,j)}^{(0)}(Q_{j})} \left[ \int_{0}^{\Phi_{\beta}} \rho^{2} \beta_{(i,j)}^{2} e_{(i,j)}^{(0)*}(Q_{j}) P_{\beta} e_{(i,j)}^{(0)}(Q_{j}) dQ_{j} \right],$$  \hspace{1cm} (A5)

where the relation

$$\langle e_{(i,j)}^{(0)*}(Q_{j}) P_{\beta} e_{(i,j)}^{(0)}(Q_{j}) \rangle = \langle e_{(i,j)}^{(0)*} P_{\beta} e_{(i,j)}^{(0)}(Q_{j}) \rangle \rangle$$  \hspace{1cm} (A6)

is used. Equation (A6) results from the requirement that the perturbative modes be orthogonal to each other and is correct to lowest order in $\Delta$ and $\delta$. Therefore the function to be integrated in Eq. (A5) is real. If $\beta_{i,j}^{2} > k_{n_{\beta}}$, the integral in Eq. (A5) is real since the contour is on the real axis only, while for $\beta_{i,j}^{2} < k_{n_{\beta}}$, deformation of the contour around the pole $\beta_{j,\alpha \beta}^{2} = \beta_{i,j}^{2}$ gives rise to an imaginary part of the integral. Thus we can define a threshold value for $\beta_{i,j}^{2}$, $\beta_{th}$, as

$$\beta_{th} = k_{n_{\beta}}^{2} \cot^{2} \alpha = k_{n_{\beta}} \alpha,$$  \hspace{1cm} (A7)

such that for $\beta_{i,j}^{2} > \beta_{th}$ the imaginary part of the propagation constant vanishes, while for $\beta_{i,j}^{2} < \beta_{th}$ the imaginary part of the propagation constant is given by

$$\beta_{i,j}^{2} = \frac{\rho^{2} \beta_{j,\alpha \beta}^{2} \beta_{j,\alpha \beta}^{2} e_{(i,j)}^{(0)*}(Q_{j}) P_{\beta} e_{(i,j)}^{(0)}(Q_{j})}{2N_{e} Q_{j} e_{(i,j)}^{(0)}(Q_{j})},$$  \hspace{1cm} (A8)

The correction to the modal field that comes from the coupling between $e_{(i,j)}^{(0)}$ and the $i$-polarized radiation modes can be obtained in the same way, and the result is same as Eq. (A4) with $j$ replaced by $i$. However, this $e_{(i,j)}^{(0)}$ will not introduce the loss to the mode. To see this, we can go through a similar process from Eqs. (A5)–(A8) and find that there is no pole in the contour of the integral in Eq. (A5) if $j$ is replaced by $i$, because no $i$-polarized radiation mode can be degenerate with $e_{(i,j)}^{(0)}$.

**APPENDIX B: LP MODES**

The LP modes $e_{(y)}^{(0)}$ and $e_{(x)}^{(0)}$ are the solutions of Eq. (14a). In this appendix we present the expressions for LP modes of several model-guiding structures. In these expressions a normalized transverse coordinate, $\bar{x}$, is defined by $\bar{x} = x/\rho$ for planar waveguides, and $R$ is defined by $R = r/\rho$ for waveguides of circular cross section, where $\rho$ is a characteristic dimension of the guiding region that is defined below.

1. **Symmetric Step-Profile Planar Waveguide**

A symmetric step-profile planar waveguide is shown in Fig. 2. An even-guided LP mode of the $y$ polarization has the form\(^15\)

$$e_{(y)}^{(0)} = \frac{\cos(U_{x} \bar{x})}{\cos(U_{x}) \exp[-W_{x}(|\bar{x}| - 1)]} - \frac{\sin(U_{x} \bar{x})}{\sin(U_{x}) \sin(|\bar{x}| - 1 + \phi_{x})},$$  \hspace{1cm} (B1)

and an odd $x$-polarized radiation mode is given by

$$e_{(x)}^{(0)} = \frac{\sin(U_{x} \bar{x})}{\bar{x}^{2} \sin(U_{x}) \sin(|\bar{x}| - 1 + \phi_{x})},$$  \hspace{1cm} (B2)

where $\phi_{x}$ satisfies the relation $\cos(\phi_{x}) = U_{x} \cot(U_{x})/Q_{x}$ and the eigenvalue equation for the guided mode is

$$\tan(U_{x}) = W_{x} \cot(U_{x}).$$  \hspace{1cm} (B3)

The normalization integral and constant are

$$\langle e_{(y)}^{(0)*} e_{(y)}^{(0)} \rangle = \frac{\rho}{2} \left[ 1 + \frac{1}{W_{x}} \right],$$  \hspace{1cm} (B4)

$$N_{e} = \frac{\rho}{2} \left[ 1 + \frac{V_{x}^{2}}{Q_{x}^{2}} \cos^{2}(U_{x}) \right],$$  \hspace{1cm} (B5)

where $\rho$ is the half-thickness of the symmetric waveguide.
2. Asymmetric Step-Profile Planar Waveguide

For an asymmetric planar waveguide as illustrated in Fig. 6, the guided LP mode \( \varepsilon_{(0)}^{(2)}\) is given by\(^{16} \)

\[
\varepsilon_{(0)}^{(2)} = \begin{bmatrix}
\frac{V_x}{V_y} \\
\frac{V_y}{V_x} \\
\frac{V_z}{V_x} \\
\frac{V_y}{V_z}
\end{bmatrix} \exp(-W_x \bar{x}) \begin{cases}
\bar{x} > 0 \\
\bar{x} < -1
\end{cases}
\]

where \( \phi_x = \cos^{-1}(U_x/V_{x1}) \), and the normalization integral \( F_{1,2} \)

\[
F_{1,2} = \frac{V_x^2}{2} \cos^2(2U_x) + \frac{(Q_x/Q_{x1})V_{x1}^2}{2} \left[ \frac{V_x^4}{2} + 2(Q_x/Q_{x1})V_{x1}^2V_x^2 \cos(2U_x) + (Q_{x1}^2/Q_{x1}^4) V_{x1}^4 \cos^2(2U_x) \right]
\]

is given by

\[
\langle \varepsilon_{(0)}^{(2)} \rangle = \frac{\rho}{2} \frac{V_x^2}{U_x^2} \left[ 1 + \frac{1}{W_x} + \frac{1}{W_z} \right]
\]

where \( \rho \) is the thickness of the asymmetric waveguide. The eigenvalue equation for this mode is given by

\[
\cos(U_x) = \frac{U_x^2 - W_x W_z}{V_{x1} V_{x2}}
\]

3. Strongly Asymmetric Step-Profile Planar Waveguide

In strongly asymmetric waveguides the profile height \( h \) is small on one side only, as is shown in Fig. 13. The guided LP modes of the TE polarization can be deduced directly from those for weakly guiding waveguides by taking the limit \( \Delta_x, \Delta_z \ll \Delta_{x1}, \Delta_{z1} \). The expression for the modal field, Eq. (B6), remains unchanged, while the normalization integral, Eq. (B9), becomes

\[
\langle \varepsilon_{(0)}^{(2)} \rangle = \frac{\rho}{2} \frac{V_x^2}{U_x^2} \left[ 1 + \frac{1}{W_{x1}} \right]
\]

and the eigenvalue equation, Eq. (B10), becomes

\[
\cos(U_x) = -\frac{W_x}{V_{x2}}
\]

The LP radiation modes of the TM polarization, \( \varepsilon_{(0)}^{(2)} \), are not solutions of Eq. (18b) anymore; instead they satisfy

\[
(V_x^2 + k^2 \varepsilon_{xx} - \beta_{\varepsilon}(k)^2) \varepsilon_{(0)}^{(2)} \sin(U_x) = \frac{\partial}{\partial \bar{x}} \left( \varepsilon_{(0)}^{(1)} \frac{\partial \ln \varepsilon_{xx}}{\partial \bar{x}} \right)
\]

as is discussed in Section 5. Using \( U_x, W_x, V_x \ll V_{x1} \), and

\[
\tan(\phi_x') = \frac{U_{x1} - \varepsilon_{xx}}{Q_{x2}^2} \tan(U_x + \phi_{\varepsilon})
\]

and \( \phi_{\varepsilon} = \cos^{-1}(1/(1 + F_{m}^2 k^2)) \). There are two types of modes, with \( F_{m}, m = 1, 2 \) representing two orthogonal solutions chosen in such a way that in the limit of a symmetric waveguide even and odd radiation modes result. The above requirement yields the following expressions for \( F_1 \) and \( F_2 \):

\[
F_{1,2} = \frac{V_x^2}{2} \cos^2(2U_x) + \frac{(Q_x/Q_{x1})V_{x1}^2}{2} \left[ \frac{V_x^4}{2} + 2(Q_x/Q_{x1})V_{x1}^2V_x^2 \cos(2U_x) + (Q_{x1}^2/Q_{x1}^4) V_{x1}^4 \cos^2(2U_x) \right]
\]

where for \( F_1 \) the positive sign is used and for \( F_2 \) the negative sign is used. The corresponding normalization constants are

\[
N_{c2} = \frac{\pi \rho}{2} \left[ 1 + \frac{V_x^2 Q_{x1}^2}{Q_{x1}^4} \sin^2(U_x + \phi_{\varepsilon}) \right]
\]

\[
+ \frac{Q_{x1}}{Q_{x2}} \left( 1 + \frac{V_x^2 Q_{x2}^2}{Q_{x1}^4 (1 + F_{m}^2)} \right)
\]

\section{Conclusion}

The guided LP modes of the TM polarization, \( \varepsilon_{(0)}^{(2)} \), are not solutions of Eq. (18b) anymore; instead they satisfy

\[
(V_x^2 + k^2 \varepsilon_{xx} - \beta_{\varepsilon}(k)^2) \varepsilon_{(0)}^{(2)} \sin(U_x) = \frac{\partial}{\partial \bar{x}} \left( \varepsilon_{(0)}^{(1)} \frac{\partial \ln \varepsilon_{xx}}{\partial \bar{x}} \right)
\]

as is discussed in Section 5.
5. Symmetric Linear-Profile Planar Waveguide

A symmetric linear-index profile is shown in Fig. 10. An even-guided mode of the y polarization for waveguides with such a profile has the field expression

\[ e_{(0)_{y}}^{(0)} = \begin{cases} 
0, & |\bar{x}| < 1 \\
\frac{1}{\sin(\phi_x)} \sin(Q_x(\bar{x} - |\bar{x}|) + \phi_x), & |\bar{x}| > 1
\end{cases} \] (B30)

and the modal solution for an odd x-polarized radiation mode is

\[ e_{(0)_{x}}^{(0)} = \begin{cases} 
\frac{1}{\sin(\phi_x)} \sin(Q_x(\bar{x} - |\bar{x}|) + \phi_x), & |\bar{x}| < 1 \\
0, & |\bar{x}| > 1
\end{cases} \] (B31)

where

\[ \cot(\phi_x) = \frac{s_x(1)}{Q_x S_x(1)}, \] (B32)

\[ s_j(\bar{x}) = Ai \left[ \left( \frac{\bar{x} - U_j^2}{V_j^2} \right) V_j^{3/2} \right] + d_j Bi \left[ \left( \frac{\bar{x} - U_j^2}{V_j^2} \right) V_j^{3/2} \right], \] (B33)

and Ai(x) and Bi(x) are Airy functions. The eigenvalue equation for \( e_{(0)_{y}}^{(0)} \) is given by

\[ Ai \left( \frac{V_x^2}{V_y^{4/3}} \right) + d_x Bi \left( \frac{V_x^2}{V_y^{4/3}} \right) \]

\[ = \frac{W_x}{V_x^{4/3}} \left[ Ai \left( \frac{V_x^2}{V_y^{4/3}} \right) + d_x Bi \left( \frac{V_x^2}{V_y^{4/3}} \right) \right]. \] (B34)

The coefficients \( d_j \) are given by

\[ d_y = -\frac{Ai\left(-U_x^2/V_x^{4/3}\right)}{Bi\left(-U_x^2/V_x^{4/3}\right)}, \] (B35)

\[ d_x = -\frac{Ai\left(-U_x^2/V_x^{4/3}\right)}{Bi\left(-U_x^2/V_x^{4/3}\right)}. \] (B36)

The normalization integral \((e_{(0)_{y}}^{(0)}, e_{(0)_{y}}^{(0)})\) is

\[ \langle e_{(0)_{y}}^{(0)}, e_{(0)_{y}}^{(0)} \rangle = \rho \left( \frac{W_x^2}{V_x^2} s_x(1) + \frac{U_x^2}{V_x^2} s_y(0) - \frac{1}{V_x^2} s_y(1) \right) + \rho \frac{s_x^2(1)}{2 W_x s_y(1)} \] (B37)

and the normalization constant of the radiation mode \( e_{(0)_{x}}^{(0)} \) is

\[ N_x = \frac{\pi \rho}{2} \left( s_x^2(1) + \frac{s_x^2(1)}{Q_x^2} \right). \] (B38)

APPENDIX C: EXACT SOLUTION FOR THE PROPAGATION CONSTANT OF AN ASYMMETRIC PLANAR WAVEGUIDE

Marcuse has given the exact solution for the propagation constants and the modal fields of symmetric waveguides. Following his procedure, we can derive the exact solution for asymmetric waveguides. Here we list only the result for the propagation constant, which satisfies the following equation:

\[ AD - CB = 0, \] (C1)

where
\[ A = - \frac{h_1 \cot^2 \alpha}{F_i \varepsilon_{xx,co}} \left( \cos U_o - \frac{G \, W_{o1}}{F_i \varepsilon_{xx,co}} \sin U_o \right) + \frac{h_2 \cot^2 \alpha}{F_i \varepsilon_{xx,co}} \left( \cos U_e - \frac{\varepsilon_{xx,cl} \, G \, W_{o1}}{F_i \varepsilon_{xx,co}} \sin U_e \right) \]  
\[ B = - \left( \frac{W_{o1}}{F_i} + \frac{W_{o2}}{F_i} \right) G \, U_o \cos U_c + \left( \frac{G^2 \, W_{o1} W_{o2}}{F_i F_2} - 1 \right) \sin U_o - \frac{\rho^2 k^2 h_1 h_2 \cot^2 \alpha}{\varepsilon_{xx,co} F_i F_2} \sin U_e \]  
\[ C = \frac{\rho^2 k^2 h_1 h_2 \cot^2 \alpha}{\varepsilon_{xx,co} F_i F_2} \sin U_o \frac{U_o}{U_c} + \frac{\left( \varepsilon_{xx,cl} G \, W_{o1} F_i + \frac{\varepsilon_{xx,cl} G}{\varepsilon_{xx,co} F_2} W_{o2} \right) \cos U_c + \left( 1 - \frac{G^2 \, \varepsilon_{xx,cl} \, W_{o1} W_{o2}}{F_i F_2} \right) \sin U_e}{\varepsilon_{xx,co} F_i F_2} \]  
\[ D = \frac{\rho^2 k^2 h_2}{F_i U_o} \left( \frac{G \, W_{o1}}{F_i} \sin U_o + \cos U_o \right) + \frac{\rho^2 k^2 h_1}{F_i U_o} \left( \cos U_e - \frac{\varepsilon_{xx,cl} \, W_{o1}}{\varepsilon_{xx,co}} \sin U_e \right) \]  

\[ \rho \text{ is the thickness of the waveguide; } k \text{ is the free-space propagation constant; and } F_i, G, W_{o1}, W_{o2}, U_c, \text{ and } U_e \text{ are defined as in Marcuse's paper:} \]

\[ U_o^2 = \rho^2 \left[ 1 + \frac{\varepsilon_{xx,cl}}{\varepsilon_{xx,co}} \right], \]  
\[ W_{o1}^2 = \rho^2 \left[ \beta^2 - k^2 \varepsilon_{xx,cl} \right], \]  
\[ U_e^2 = \rho^2 \left[ \beta^2 \left( \sin^2 \alpha + \frac{\varepsilon_{xx,cl}}{\varepsilon_{xx,co}} \cos^2 \alpha \right) - k^2 \varepsilon_{xx,cl} \right], \]  
\[ W_{o2}^2 = \rho^2 \left[ \beta^2 \left( \sin^2 \alpha + \frac{\varepsilon_{xx,cl}}{\varepsilon_{xx,co}} \cos^2 \alpha \right) - k^2 \varepsilon_{xx,cl} \right], \]  
\[ F_i = W_{o1}^2 \cot^2 \alpha - \rho^2 k^2 \varepsilon_{xx,cl}, \]  
\[ G = U_e^2 \cot^2 \alpha + \rho^2 k^2 \varepsilon_{xx,co}, \]  

and \( h_i \) is defined as

\[ h_i = \varepsilon_{xx,co} W_{o1}^2 + \varepsilon_{xx,cl} U_o^2. \]  

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