

# AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Conservation and Integral Forms and Discontinuities



# Outline

- 1 Conservation Law Form
- 2 Integral Form
- 3 Relations at Discontinuities



## └ Conservation Law Form

- Definition: an equation (or set of equations) is said to be in conservation law form — or more precisely, in divergence form — if it is written as follows

$$\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(W) = S$$

- If  $S = 0$ , the equation is said to be in *strong* conservation law form
- For example, many of the equations presented in Chapter 2 are written in strong conservation form



## └ Conservation Law Form

- Recall that the transonic small disturbance equation discussed in Chapter 2 was written as

$$\left( 1 - M_{\infty}^2 - (\gamma + 1) M_{\infty}^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_{\infty}\|} \right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$



## Conservation Law Form

- Recall that the transonic small disturbance equation discussed in Chapter 2 was written as

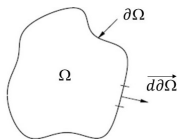
$$\left(1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_\infty\|}\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

This equation can be re-written in strong conservation form using

$$\vec{\mathcal{F}} = \left( \left[ (1 - M_\infty^2) \frac{\partial \phi}{\partial x} - (\gamma + 1)M_\infty^2 \frac{\frac{\partial \phi^2}{\partial x}}{2\|\vec{v}_\infty\|} \right] \quad \frac{\partial \phi}{\partial y} \quad \frac{\partial \phi}{\partial z} \right)^T$$



# Integral Form

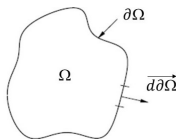


- The integration over an arbitrary stationary volume  $\Omega$  enclosed by the surface  $\partial\Omega$  of a generic equation written in conservation form can be written as

$$\int_{\Omega} \frac{\partial W}{\partial t} d\Omega + \int_{\partial\Omega} \vec{\nabla} \cdot \vec{F}(W) d\Omega = \int_{\Omega} S d\Omega$$



# Integral Form



- The integration over an arbitrary stationary volume  $\Omega$  enclosed by the surface  $\partial\Omega$  of a generic equation written in conservation form can be written as

$$\int_{\Omega} \frac{\partial W}{\partial t} d\Omega + \int_{\Omega} \vec{\nabla} \cdot \vec{\mathcal{F}}(W) d\Omega = \int_{\Omega} S d\Omega$$

- Dividing by  $\Omega$  and using the divergence (Gauss, or Ostrogradsky) theorem leads to

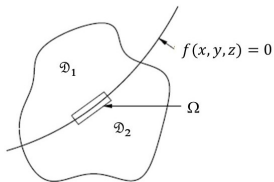
$$\begin{aligned} \frac{\partial \overline{W}}{\partial t} + \frac{1}{\Omega} \int_{\partial\Omega} \vec{\mathcal{F}} \cdot d\vec{\partial\Omega} &= \frac{1}{\Omega} \int_{\Omega} S d\Omega \\ \text{where } \overline{W} &= \frac{1}{\Omega} \int_{\Omega} W d\Omega \end{aligned} \quad (1)$$

- The above equation represents the rate of change of the mean value of  $W$  over the volume  $\Omega$  caused by the net flux of  $\vec{\mathcal{F}}$  crossing the surface  $\partial\Omega$  and the volume source  $S$



## └ Relations at Discontinuities

## └ Stationary Discontinuities



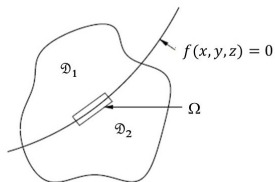
- Let  $f(x, y, z) = 0$  represent a surface located at a possible discontinuity within the fluid
- Assume that the flow is continuous within each of the two subdomains shown in the figure above
- Assume also that  $\Omega$  is placed **symmetrically** about an arbitrary point of the surface and is allowed to shrink to zero





## └ Relations at Discontinuities

## └ Stationary Discontinuities



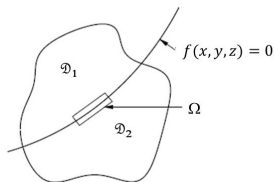
- Now, for the case of a steady flow, Eq. (1) becomes

$$\int_{\partial\Omega} \vec{\mathcal{F}} \cdot d\vec{\partial\Omega} = \int_{\Omega} S d\Omega$$



- Relations at Discontinuities

- Stationary Discontinuities



- Now, for the case of a steady flow, Eq. (1) becomes

$$\int_{\partial\Omega} \vec{\mathcal{F}} \cdot d\vec{\partial\Omega} = \int_{\Omega} S d\Omega$$

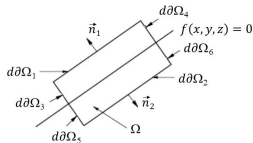
- As  $\Omega \rightarrow 0$ , the term on the right goes to zero at a faster rate than the surface integration term ( $h^3$  vs  $h^2$ , where  $h \approx \Omega^{\frac{1}{3}} = \partial\Omega^{\frac{1}{2}}$ )
- It follows that for an infinitesimal  $\Omega$

$$\int_{\partial\Omega} \vec{\mathcal{F}} \cdot d\vec{\partial\Omega} = 0$$



- Relations at Discontinuities

- Stationary Discontinuities

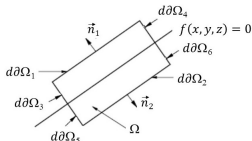


$$0 = \int_{\partial\Omega} \vec{\mathcal{F}} \cdot d\vec{\partial\Omega} = \sum_{i=1}^6 \vec{\mathcal{F}}_i \cdot \vec{n}_i d\partial\Omega_i, \quad \text{where} \quad \|\vec{n}_i\|_2 = 1, \quad i = 1, \dots, 6$$



Relations at Discontinuities

Stationary Discontinuities



$$0 = \int_{\partial\Omega} \vec{\mathcal{F}} \cdot \overrightarrow{d\partial\Omega} = \sum_{i=1}^6 \vec{\mathcal{F}}_i \cdot \vec{n}_i d\partial\Omega_i, \quad \text{where} \quad \|\vec{n}_i\|_2 = 1, \quad i = 1, \dots, 6$$

- Since the flow is continuous within each of subdomain  $\mathcal{D}_1$  and subdomain  $\mathcal{D}_2$ , in the limit when  $\partial\Omega \rightarrow 0$

$$\vec{\mathcal{F}}_3 \cdot \vec{n}_3 d\partial\Omega_3 + \vec{\mathcal{F}}_4 \cdot \vec{n}_4 d\partial\Omega_4 = 0 \quad \text{and} \quad \vec{\mathcal{F}}_5 \cdot \vec{n}_5 d\partial\Omega_5 + \vec{\mathcal{F}}_6 \cdot \vec{n}_6 d\partial\Omega_6 = 0$$

$$\Rightarrow \int_{\partial\Omega} \vec{\mathcal{F}} \cdot \overrightarrow{d\partial\Omega} = \vec{\mathcal{F}}_1 \cdot \vec{n}_1 d\partial\Omega_1 + \vec{\mathcal{F}}_2 \cdot \vec{n}_2 d\partial\Omega_2 = 0$$

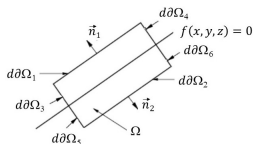
$$\Rightarrow (\vec{\mathcal{F}}_1 - \vec{\mathcal{F}}_2) \cdot \vec{n}_1 = 0 \quad \text{with} \quad \vec{n}_1 = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$$

$$\Rightarrow (\vec{\mathcal{F}}_1 - \vec{\mathcal{F}}_2) \cdot \vec{\nabla} f = 0$$



- Relations at Discontinuities

- Stationary Discontinuities



- The jump of  $\vec{\mathcal{F}}$  across the surface  $f$  is defined as and denoted by

$$(\vec{\mathcal{F}}_1 - \vec{\mathcal{F}}_2) = \llbracket \vec{\mathcal{F}} \rrbracket_1^2$$

$$\Rightarrow \llbracket \vec{\mathcal{F}} \rrbracket_1^2 \cdot \vec{\nabla} f = 0$$

which can also be written as

$$\llbracket \mathcal{F}_x \rrbracket_1^2 \frac{\partial f}{\partial x} + \llbracket \mathcal{F}_y \rrbracket_1^2 \frac{\partial f}{\partial y} + \llbracket \mathcal{F}_z \rrbracket_1^2 \frac{\partial f}{\partial z} = 0$$

- If  $\vec{\mathcal{F}}$  is the flux vector of the Euler equations, the above steady jump relations at surface  $f(x, y, z) = 0$  represent the **Rankine-Hugoniot** relations across a shock wave



# Relations at Discontinuities

## Moving Discontinuities

- Consider now the surface  $f(x, y, z, t) = 0$  representing a dynamic surface located at a possible moving discontinuity within a volume  $\Omega$  of a fluid
- Let

$$\vec{\nabla}^* = \left( \frac{\partial}{\partial t} \quad \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right)^T$$

and

$$\vec{\mathcal{F}}^*(W) = (W^T \mathcal{F}_x^T(W) \mathcal{F}_y^T(W) \mathcal{F}_z^T(W))^T$$

- Then  $\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(W) = S$  can be rewritten as  $\vec{\nabla}^* \cdot \vec{\mathcal{F}}^*(W) = S$
- Using the above notation, which includes time as a dimension, the previous discussion on stationary discontinuities can be generalized to obtain the following unsteady jump relations for moving discontinuities

$$\llbracket \vec{\mathcal{F}}^* \rrbracket_1^2 \cdot \vec{\nabla}^* f = \llbracket W \rrbracket_1^2 \frac{\partial f}{\partial t} + \llbracket \mathcal{F}_x \rrbracket_1^2 \frac{\partial f}{\partial x} + \llbracket \mathcal{F}_y \rrbracket_1^2 \frac{\partial f}{\partial y} + \llbracket \mathcal{F}_z \rrbracket_1^2 \frac{\partial f}{\partial z} = 0$$



## └ Relations at Discontinuities

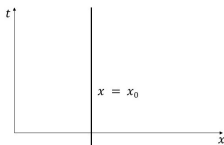
## └ Shock Waves

Simple Wave Equation

- Consider the model hyperbolic equation with constant wave speed  $c \neq 0$  and with scalar variable  $u$

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

- consider first the case of a stationary discontinuity surface of the form  $f(x) = x - x_0 = 0$



## Relations at Discontinuities

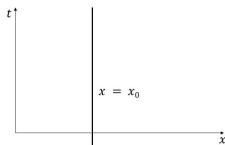
### Shock Waves

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$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

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- in this case,  $\vec{\mathcal{F}}^* = (u \quad cu)^T$  and  $\vec{n}_1 = \frac{\vec{\nabla}^* f}{\|\vec{\nabla}^* f\|} = (0 \ 1)^T$ , and therefore the jump relation is

$$\left[ \left[ \vec{\mathcal{F}}^* \right]_1 \right]^2 \cdot \vec{\nabla}^* f = \left[ [cu] \right]_1^2 = c(u_1 - u_2) = 0 \Leftrightarrow u_1 = u_2$$

- this implies that no stationary jump is possible, which is not surprising for a linear equation





## └ Relations at Discontinuities

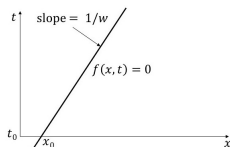
## └ Shock Waves

Simple Wave Equation

- Consider the model hyperbolic equation with constant wave speed  $c \neq 0$  and with scalar variable  $u$  (continue)

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

- consider next the case of a discontinuity surface moving at constant speed  $w$ ,  
 $f(x, t) = x - x_0 - w(t - t_0) = 0$



## Relations at Discontinuities

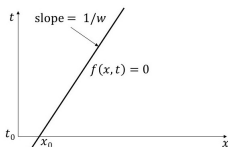
### Shock Waves

#### Simple Wave Equation

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- consider next the case of a discontinuity surface moving at constant speed  $w$ ,  $f(x, t) = x - x_0 - w(t - t_0) = 0$



- in this case,  $\vec{n}_1 = \frac{\vec{\nabla}^* f}{\|\vec{\nabla}^* f\|} = \frac{1}{\sqrt{1+w^2}}(-w \ 1)^T$ , and therefore the jump relation is

$$\begin{aligned} \left[ \vec{F}^* \right]_1^2 \cdot \vec{\nabla}^* f &= -w \left[ u \right]_1^2 + \left[ cu \right]_1^2 = -w(u_1 - u_2) + c(u_1 - u_2) = 0 \\ \Leftrightarrow (c - w)(u_1 - u_2) &= 0 \end{aligned}$$

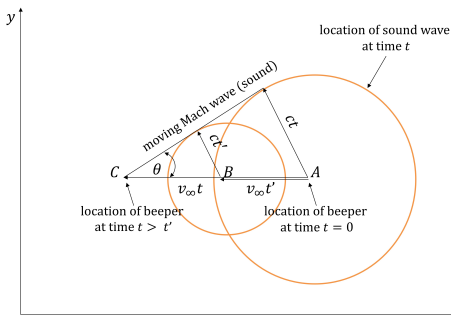
- this implies that any jump is possible, as long as it moves at the speed  $c$



## Relations at Discontinuities

### Shock Waves

#### Mach Waves



Mach wave: pressure wave traveling with the speed of sound caused by a slight change of pressure added to a compressible flow – these weak waves can combine in **supersonic** flow to become a shock wave if sufficient Mach waves are present at any location

$$\sin \theta = \frac{c}{v_{\infty}} = \frac{1}{M_{\infty}} \Rightarrow \tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{1}{\sqrt{M_{\infty}^2 - 1}}$$



- Relations at Discontinuities

- Shock Waves

### Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime

- Recall the linearized small-perturbation potential equation modeling a two-dimensional steady flow in either the subsonic or supersonic regime

$$(1 - M_\infty^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \Leftrightarrow \quad \vec{\nabla} \cdot \left( (1 - M_\infty^2) \frac{\partial \phi}{\partial x} \quad \frac{\partial \phi}{\partial y} \right)^T = 0$$

- For  $M_\infty > 1.2$ , this equation is hyperbolic and can describe purely supersonic flows with small perturbations about a supersonic free-stream with velocity  $\vec{v}_\infty = \|\vec{v}_\infty\| \vec{e}_x$

$$\Rightarrow \vec{v} = \vec{v}_\infty + \vec{\nabla} \phi = \left( \|\vec{v}_\infty\| + \frac{\partial \phi}{\partial x} \right) \vec{e}_x + \frac{\partial \phi}{\partial y} \vec{e}_y$$

- Consider as a possible stationary discontinuity surface  $f(x, y) = a(x - x_0) - b(y - y_0) = 0$ , where  $a$  and  $b$  are constants (stationary w.r.t the object generating it)



# Relations at Discontinuities

## Shock Waves

### Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime

- In this case,  $\vec{\mathcal{F}} = \left( (1 - M_\infty^2) \frac{\partial \phi}{\partial x} \quad \frac{\partial \phi}{\partial y} \right)^T$  and  $\vec{n}_1 = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|} = \frac{1}{\sqrt{a^2 + b^2}} (a \quad -b)^T$ , and therefore the jump relation is

$$\begin{aligned} \left[ \vec{\mathcal{F}} \right]_1^2 \cdot \vec{\nabla} f &= a (1 - M_\infty^2) \left[ \left[ \frac{\partial \phi}{\partial x} \right]_1^2 \right] - b \left[ \left[ \frac{\partial \phi}{\partial y} \right]_1^2 \right] = 0 \\ \Leftrightarrow a (1 - M_\infty^2) \left( \left. \frac{\partial \phi}{\partial x} \right|_1 - \left. \frac{\partial \phi}{\partial x} \right|_2 \right) - b \left( \left. \frac{\partial \phi}{\partial y} \right|_1 - \left. \frac{\partial \phi}{\partial y} \right|_2 \right) &= 0 \end{aligned}$$

- if  $a = 0$  or  $b = 0$ , there are no permissible jumps (why?)
- otherwise, a small perturbation jump can occur across a Mach line  $f(x, y)$  with angle  $\theta$ , in which case the slope of the discontinuity surface is  $\frac{a}{b} = \tan \theta = \frac{1}{\sqrt{M_\infty^2 - 1}}$

(recall that the Mach angle is given by  $\sin \theta = \frac{1}{M_\infty}$ ): along this Mach line, the jump

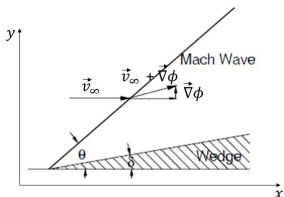
relation simplifies to  $-\sqrt{M_\infty^2 - 1} \left[ \left[ \frac{\partial \phi}{\partial x} \right]_1^2 \right] = \left[ \left[ \frac{\partial \phi}{\partial y} \right]_1^2 \right]$



- Relations at Discontinuities

- Shock Waves

## Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime



- (Continue)

- along the Mach line with the slope  $\frac{a}{b} = \tan \theta = \frac{1}{\sqrt{M_\infty^2 - 1}}$ , where

$-\sqrt{M_\infty^2 - 1} \left[ \frac{\partial \phi}{\partial x} \right]_1^2 = \left[ \frac{\partial \phi}{\partial y} \right]_1^2$ , the flow can turn through an angle  $\delta$  (small value because small perturbation) from the free-stream direction (see above figure, where  $\vec{\nabla} \phi|_1 = 0$ ) such that  $\frac{\partial \phi}{\partial x}|_2 = \frac{-\tan \delta}{\tan \delta + \sqrt{M_\infty^2 - 1}} \|\vec{v}_\infty\| \approx \frac{-\tan \delta}{\sqrt{M_\infty^2 - 1}} \|\vec{v}_\infty\|$  and

$$\frac{\partial \phi}{\partial y}|_2 = \frac{\tan \delta \sqrt{M_\infty^2 - 1}}{\tan \delta + \sqrt{M_\infty^2 - 1}} \|\vec{v}_\infty\| \approx \tan \delta \|\vec{v}_\infty\|$$

