

AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Conservation and Integral Forms and Discontinuities



Outline

1 Conservation Law Form

2 Integral Form

3 Relations at Discontinuities



└ Conservation Law Form

- Definition: an equation (or set of equations) is said to be in conservation law form — or more precisely, in divergence form — if it is written as follows

$$\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(W) = S$$

- If $S = 0$, the equation is said to be in *strong* conservation law form
- For example, many of the equations presented in Chapter 2 are written in strong conservation form



└ Conservation Law Form

- Recall that the transonic small disturbance equation discussed in Chapter 2 was written as

$$\left(1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_\infty\|}\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$



└ Conservation Law Form

- Recall that the transonic small disturbance equation discussed in Chapter 2 was written as

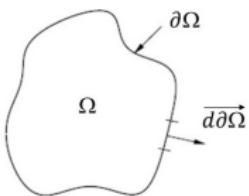
$$\left(1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \frac{\frac{\partial \phi}{\partial x}}{\|\vec{v}_\infty\|}\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

This equation can be re-written in strong conservation form using

$$\vec{\mathcal{F}} = \left(\left[(1 - M_\infty^2) \frac{\partial \phi}{\partial x} - (\gamma + 1)M_\infty^2 \frac{\frac{\partial \phi}{\partial x}^2}{2\|\vec{v}_\infty\|} \right] \quad \frac{\partial \phi}{\partial y} \quad \frac{\partial \phi}{\partial z} \right)^T$$



└ Integral Form

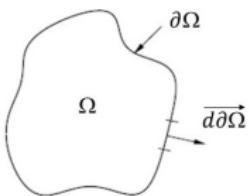


- The integration over an arbitrary stationary volume Ω enclosed by the surface $\partial\Omega$ of a generic equation written in conservation form can be written as

$$\int_{\Omega} \frac{\partial W}{\partial t} d\Omega + \int_{\Omega} \vec{\nabla} \cdot \vec{F}(W) d\Omega = \int_{\Omega} S d\Omega$$



└ Integral Form



- The integration over an arbitrary stationary volume Ω enclosed by the surface $\partial\Omega$ of a generic equation written in conservation form can be written as

$$\int_{\Omega} \frac{\partial W}{\partial t} d\Omega + \int_{\Omega} \vec{\nabla} \cdot \vec{F}(W) d\Omega = \int_{\Omega} S d\Omega$$

- Dividing by Ω and using the divergence (Gauss, or Ostrogradsky) theorem leads to

$$\frac{\partial \bar{W}}{\partial t} + \frac{1}{\Omega} \int_{\partial\Omega} \vec{F} \cdot \vec{d\partial\Omega} = \frac{1}{\Omega} \int_{\Omega} S d\Omega \quad (1)$$

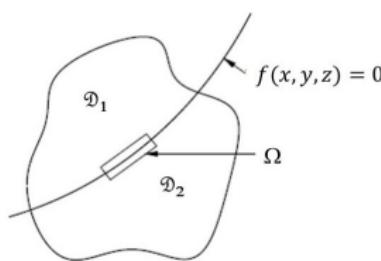
$$\text{where } \bar{W} = \frac{1}{\Omega} \int_{\Omega} W d\Omega$$

- The above equation represents the rate of change of the mean value of W over the volume Ω caused by the net flux of \vec{F} crossing the surface $\partial\Omega$ and the volume source S



└ Relations at Discontinuities

└ Stationary Discontinuities

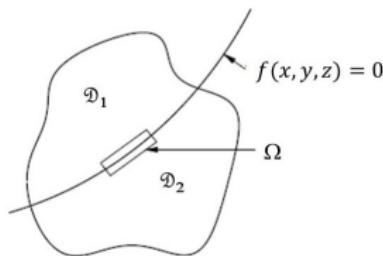


- Let $f(x, y, z) = 0$ represent a surface located at a possible discontinuity within the fluid
- Assume that the flow is continuous within each of the two subdomains shown in the figure above
- Assume also that Ω is placed **symmetrically** about an arbitrary point of the surface and is allowed to shrink to zero



└ Relations at Discontinuities

└ Stationary Discontinuities



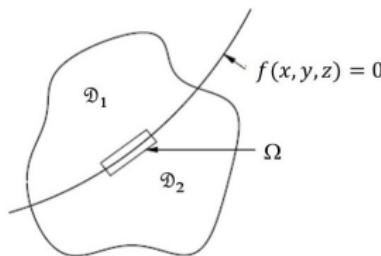
- Now, for the case of a steady flow, Eq. (1) becomes

$$\int_{\partial\Omega} \vec{F} \cdot d\partial\Omega = \int_{\Omega} S d\Omega$$



└ Relations at Discontinuities

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- Now, for the case of a steady flow, Eq. (1) becomes

$$\int_{\partial\Omega} \vec{F} \cdot \overrightarrow{d\partial\Omega} = \int_{\Omega} S \, d\Omega$$

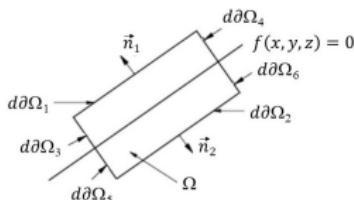
- As $\Omega \rightarrow 0$, the term on the right goes to zero at a faster rate than the surface integration term (h^3 vs h^2 , where $h \approx \Omega^{\frac{1}{3}} = \partial\Omega^{\frac{1}{2}}$)
- It follows that for an infinitesimal Ω

$$\int_{\partial\Omega} \vec{F} \cdot \overrightarrow{d\partial\Omega} = 0$$



└ Relations at Discontinuities

└ Stationary Discontinuities

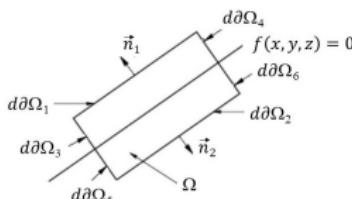


$$0 = \int_{\partial\Omega} \vec{F} \cdot d\vec{\Omega} = \sum_{i=1}^6 \vec{F}_i \cdot \vec{n}_i d\Omega_i, \quad \text{where} \quad \|\vec{n}_i\|_2 = 1, \quad i = 1, \dots, 6$$



└ Relations at Discontinuities

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$$0 = \int_{\partial\Omega} \vec{F} \cdot d\overrightarrow{\Omega} = \sum_{i=1}^6 \vec{F}_i \cdot \vec{n}_i d\Omega_i, \quad \text{where} \quad \|\vec{n}_i\|_2 = 1, \quad i = 1, \dots, 6$$

- Since the flow is continuous within each of subdomain \mathcal{D}_1 and subdomain \mathcal{D}_2 , in the limit when $\partial\Omega \rightarrow 0$

$$\vec{F}_3 \cdot \vec{n}_3 d\Omega_3 + \vec{F}_4 \cdot \vec{n}_4 d\Omega_4 = 0 \quad \text{and} \quad \vec{F}_5 \cdot \vec{n}_5 d\Omega_5 + \vec{F}_6 \cdot \vec{n}_6 d\Omega_6 = 0$$

$$\Rightarrow \int_{\partial\Omega} \vec{F} \cdot d\overrightarrow{\Omega} = \vec{F}_1 \cdot \vec{n}_1 d\Omega_1 + \vec{F}_2 \cdot \vec{n}_2 d\Omega_2 = 0$$

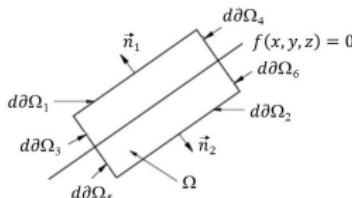
$$\Rightarrow (\vec{F}_1 - \vec{F}_2) \cdot \vec{n}_1 = 0 \quad \text{with} \quad \vec{n}_1 = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$$

$$\Rightarrow (\vec{F}_1 - \vec{F}_2) \cdot \vec{\nabla} f = 0$$



Relations at Discontinuities

Stationary Discontinuities



- The jump of $\vec{\mathcal{F}}$ across the surface f is defined as and denoted by

$$(\vec{\mathcal{F}}_1 - \vec{\mathcal{F}}_2) = [\![\vec{\mathcal{F}}]\!]_1^2$$

$$\implies [\![\vec{\mathcal{F}}]\!]_1^2 \cdot \vec{\nabla} f = 0$$

which can also be written as

$$[\![\mathcal{F}_x]\!]_1^2 \frac{\partial f}{\partial x} + [\![\mathcal{F}_y]\!]_1^2 \frac{\partial f}{\partial y} + [\![\mathcal{F}_z]\!]_1^2 \frac{\partial f}{\partial z} = 0$$

- If $\vec{\mathcal{F}}$ is the flux vector of the Euler equations, the above steady jump relations at surface $f(x, y, z) = 0$ represent the **Rankine-Hugoniot** relations across a shock wave

Relations at Discontinuities

└ Moving Discontinuities

- Consider now the surface $f(x, y, z, t) = 0$ representing a dynamic surface located at a possible moving discontinuity within a volume Ω of a fluid
- Let

$$\vec{\nabla}^* = \left(\frac{\partial}{\partial t} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right)^T$$

and

$$\vec{\mathcal{F}}^*(W) = (W^T \ \mathcal{F}_x^T(W) \ \mathcal{F}_y^T(W) \ \mathcal{F}_z^T(W))^T$$

- Then $\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{F}}(W) = S$ can be rewritten as $\vec{\nabla}^* \cdot \vec{\mathcal{F}}^*(W) = S$
- Using the above notation, which includes time as a dimension, the previous discussion on stationary discontinuities can be generalized to obtain the following unsteady jump relations for moving discontinuities

$$[\![\vec{\mathcal{F}}^\star]\!]_1^2 \cdot \vec{\nabla}^\star f = [\![W]\!]_1^2 \frac{\partial f}{\partial t} + [\![\mathcal{F}_x]\!]_1^2 \frac{\partial f}{\partial x} + [\![\mathcal{F}_y]\!]_1^2 \frac{\partial f}{\partial y} + [\![\mathcal{F}_z]\!]_1^2 \frac{\partial f}{\partial z} = 0$$



└ Relations at Discontinuities

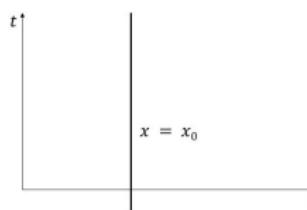
└ Shock Waves

Simple Wave Equation

- Consider the model hyperbolic equation with constant wave speed $c \neq 0$ and with scalar variable u

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

- consider first the case of a stationary discontinuity surface of the form $f(x) = x - x_0 = 0$



Relations at Discontinuities

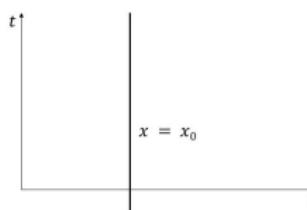
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- in this case, $\vec{F}^* = (u \ cu)^T$ and $\vec{n}_1 = \frac{\vec{\nabla}^* f}{\|\vec{\nabla}^* f\|} = (0 \ 1)^T$, and therefore the jump relation is

$$\left[\vec{F}^* \right]_1^2 \cdot \vec{\nabla}^* f = \llbracket cu \rrbracket_1^2 = c(u_1 - u_2) = 0 \Leftrightarrow u_1 = u_2$$

- this implies that no stationary jump is possible, which is not surprising for a linear equation



└ Relations at Discontinuities

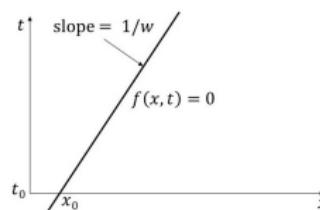
└ Shock Waves

Simple Wave Equation

- Consider the model hyperbolic equation with constant wave speed $c \neq 0$ and with scalar variable u (continue)

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

- consider next the case of a discontinuity surface moving at constant speed w ,
 $f(x, t) = x - x_0 - w(t - t^0) = 0$



Relations at Discontinuities

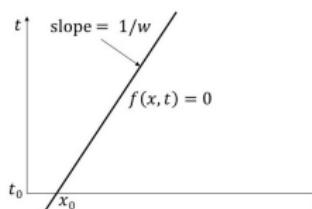
Shock Waves

Simple Wave Equation

- Consider the model hyperbolic equation with constant wave speed $c \neq 0$ and with scalar variable u (continue)

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

- consider next the case of a discontinuity surface moving at constant speed w ,
 $f(x, t) = x - x_0 - w(t - t_0) = 0$



- in this case, $\vec{n}_1 = \frac{\vec{\nabla}^* f}{\|\vec{\nabla}^* f\|} = \frac{1}{\sqrt{1+w^2}}(-w \ 1)^T$, and therefore the jump relation is

$$\begin{aligned} \left[\vec{\mathcal{F}}^* \right]_1^2 \cdot \vec{\nabla}^* f &= -w \left[u \right]_1^2 + \left[cu \right]_1^2 = -w(u_1 - u_2) + c(u_1 - u_2) = 0 \\ \Leftrightarrow (c - w)(u_1 - u_2) &= 0 \end{aligned}$$

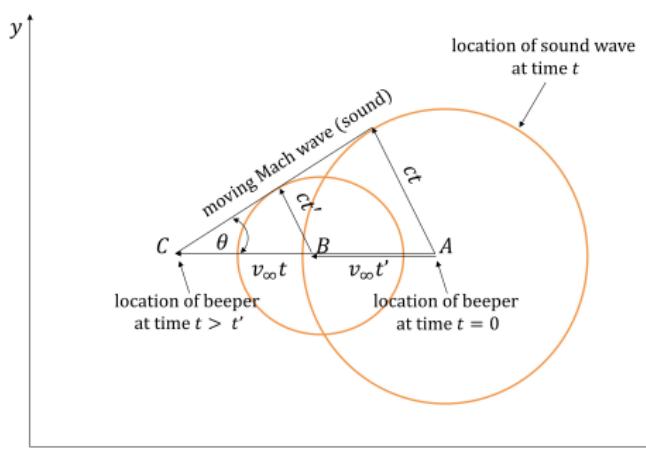
- this implies that any jump is possible, as long as it moves at the speed c



└ Relations at Discontinuities

└ Shock Waves

Mach Waves



Mach wave: pressure wave traveling with the speed of sound caused by a slight change of pressure added to a compressible flow – these weak waves can combine in **supersonic** flow to become a shock wave if sufficient Mach waves are present at any location

$$\sin \theta = \frac{c}{v_\infty} = \frac{1}{M_\infty} \Rightarrow \tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{1}{\sqrt{M_\infty^2 - 1}}$$



└ Relations at Discontinuities

└ Shock Waves

Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime

- Recall the linearized small-perturbation potential equation modeling a two-dimensional steady flow in either the subsonic or supersonic regime

$$(1 - M_\infty^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \Leftrightarrow \quad \vec{\nabla} \cdot \left((1 - M_\infty^2) \frac{\partial \phi}{\partial x} \quad \frac{\partial \phi}{\partial y} \right)^T = 0$$

- For $M_\infty > 1.2$, this equation is hyperbolic and can describe purely supersonic flows with small perturbations about a supersonic free-stream with velocity $\vec{v}_\infty = \|\vec{v}_\infty\| \vec{e}_x$
 $\Rightarrow \vec{v} = \vec{v}_\infty + \vec{\nabla} \phi = \left(\|\vec{v}_\infty\| + \frac{\partial \phi}{\partial x} \right) \vec{e}_x + \frac{\partial \phi}{\partial y} \vec{e}_y$
- Consider as a possible stationary discontinuity surface $f(x, y) = a(x - x_0) - b(y - y_0) = 0$, where a and b are constants (stationary w.r.t the object generating it)



Relations at Discontinuities

Shock Waves

Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime

- In this case, $\vec{\mathcal{F}} = \begin{pmatrix} (1 - M_\infty^2) \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix}^T$ and $\vec{n}_1 = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|} = \frac{1}{\sqrt{a^2+b^2}}(a - b)^T$, and therefore the jump relation is

$$\begin{aligned} \llbracket \vec{\mathcal{F}} \rrbracket_1^2 \cdot \vec{\nabla} f &= a \left(1 - M_\infty^2\right) \llbracket \frac{\partial \phi}{\partial x} \rrbracket_1^2 - b \llbracket \frac{\partial \phi}{\partial y} \rrbracket_1^2 = 0 \\ \Leftrightarrow a \left(1 - M_\infty^2\right) \left(\frac{\partial \phi}{\partial x} \big|_1 - \frac{\partial \phi}{\partial x} \big|_2 \right) - b \left(\frac{\partial \phi}{\partial y} \big|_1 - \frac{\partial \phi}{\partial y} \big|_2 \right) &= 0 \end{aligned}$$

- if $a = 0$ or $b = 0$, there are no permissible jumps (why?)
- otherwise, a small perturbation jump can occur across a Mach line $f(x, y)$ with angle θ , in which case the slope of the discontinuity surface is $\frac{a}{b} = \tan \theta = \frac{1}{\sqrt{M_\infty^2 - 1}}$

(recall that the Mach angle is given by $\sin \theta = \frac{1}{M_\infty}$): along this Mach line, the jump

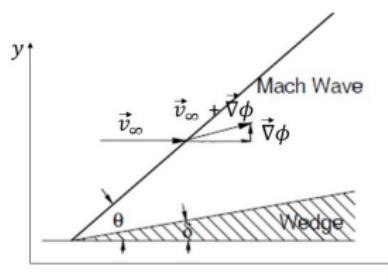
$$\text{relation simplifies to } -\sqrt{M_\infty^2 - 1} \llbracket \frac{\partial \phi}{\partial x} \rrbracket_1^2 = \llbracket \frac{\partial \phi}{\partial y} \rrbracket_1^2$$



Relations at Discontinuities

Shock Waves

Mach Waves: Linearized Small-Perturbation Potential Equation in the Supersonic Regime



■ (Continue)

- along the Mach line with the slope $\frac{a}{b} = \tan \theta = \frac{1}{\sqrt{M_\infty^2 - 1}}$, where $-\sqrt{M_\infty^2 - 1} \left[\left[\frac{\partial \phi}{\partial x} \right] \right]_1^2 = \left[\left[\frac{\partial \phi}{\partial y} \right] \right]_1^2$, the flow can turn through an angle δ (small value because small perturbation) from the free-stream direction (see above figure, where $\nabla \phi|_1 = 0$) such that $\frac{\partial \phi}{\partial x}|_2 = \frac{-\tan \delta}{\tan \delta + \sqrt{M_\infty^2 - 1}} \|\vec{v}_\infty\| \approx \frac{-\tan \delta}{\sqrt{M_\infty^2 - 1}} \|\vec{v}_\infty\|$ and $\frac{\partial \phi}{\partial y}|_2 = \frac{\tan \delta \sqrt{M_\infty^2 - 1}}{\tan \delta + \sqrt{M_\infty^2 - 1}} \|\vec{v}_\infty\| \approx \tan \delta \|\vec{v}_\infty\|$

