AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Linearization and Characteristic Relations

Outline

- Consider the following equation written in conservation law form

$$\frac{\partial W}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{F}}(W) = S$$

where $\overrightarrow{\mathcal{F}}(W) = \left(\mathcal{F}_x^T(W)\ \mathcal{F}_y^T(W)\ \mathcal{F}_z^T(W)\right)^T$

- In three dimensions, this equation can be re-written as follows

$$\frac{\partial W}{\partial t} + \frac{\partial \mathcal{F}_x(W)}{\partial W}\frac{\partial W}{\partial x} + \frac{\partial \mathcal{F}_y(W)}{\partial W}\frac{\partial W}{\partial y} + \frac{\partial \mathcal{F}_z(W)}{\partial W}\frac{\partial W}{\partial z} = S$$

or

$$\boxed{\frac{\partial W}{\partial t} + A\frac{\partial W}{\partial x} + B\frac{\partial W}{\partial y} + C\frac{\partial W}{\partial z} = S} \qquad (1)$$

where

$$A = A(W) = \frac{\partial \mathcal{F}_x}{\partial W}(W),\ B = B(W) = \frac{\partial \mathcal{F}_y}{\partial W}(W),\ C = C(W) = \frac{\partial \mathcal{F}_z}{\partial W}(W)$$

are called the Jacobians of $\mathcal{F}_x$, $\mathcal{F}_y$, and $\mathcal{F}_z$ with respect to W, respectively

- For example, for the Euler equations in two dimensions, each of the Jacobians is a 4×4 matrix
- In general for m-dimensional vectors $W = \begin{pmatrix} W_1 & \cdots & W_m \end{pmatrix}^T$ and $\mathcal{F} = \begin{pmatrix} \mathcal{F}_1 & \cdots & \mathcal{F}_m \end{pmatrix}^T$

$$\boxed{\frac{\partial \mathcal{F}}{\partial W} = \begin{pmatrix} \dfrac{\partial \mathcal{F}_1}{\partial W_1} & \cdots & \dfrac{\partial \mathcal{F}_1}{\partial W_m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \mathcal{F}_m}{\partial W_1} & \cdots & \dfrac{\partial \mathcal{F}_m}{\partial W_m} \end{pmatrix}}$$

- If $W = W(V)$, Eq. (1) can be transformed as follows

$$\boxed{\frac{\partial V}{\partial t} + A'\frac{\partial V}{\partial x} + B'\frac{\partial V}{\partial y} + C'\frac{\partial V}{\partial z} = S'} \qquad (2)$$

where

$$A' = T^{-1}AT, \qquad B' = T^{-1}BT, \qquad C' = T^{-1}CT, \qquad S' = T^{-1}S$$

and

$$T = \frac{\partial W}{\partial V}, \qquad T^{-1} = \frac{\partial V}{\partial W}$$

represents the Jacobian of W with respect to V

- The Jacobians with respect to W are then given by

$$\frac{\partial}{\partial W} = \frac{\partial}{\partial V}\frac{\partial V}{\partial W} = \frac{\partial}{\partial V}T^{-1}$$

- Definition: $\mathcal{G}(W_1, \cdots, W_m)$ is said to be a homogeneous function of degree p, where p is an integer, if

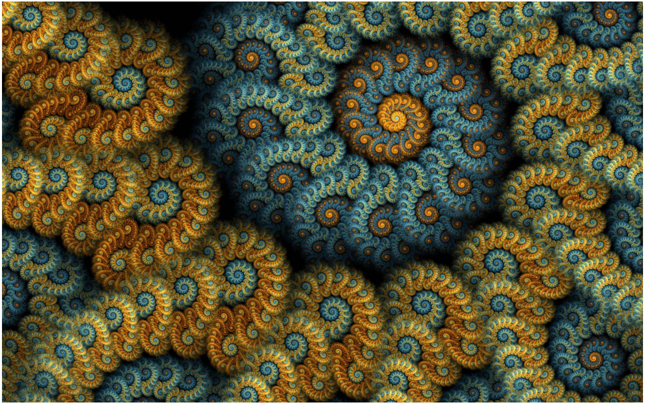
$$\forall s > 0 \quad \mathcal{G}(sW_1, \cdots, sW_m) = s^p \mathcal{G}(W_1, \cdots, W_m)$$

- Example: A linear function is a homogeneous function of degree 1

$$\forall s > 0, \quad \mathcal{G}(sW_1, \cdots, sW_m) = s\mathcal{G}(W_1, \cdots, W_m)$$

- Exercise: Show that for a perfect gas, the fluxes $\mathcal{F}_x$, $\mathcal{F}_y$, and $\mathcal{F}_z$ of the Euler equations written in conservation form are homogeneous functions (of W) of degree 1 (see TA Session)
- A homogeneous function of degree p has *scale invariance* – that is, it has some properties that remain constant when looking at them either at different length- or time-scales and thus represent a universality
- In mathematics, scale invariance usually refers to an invariance of individual functions or curves: A closely related concept is self-similarity, where a function or curve is invariant under a discrete subset of dilations (transformations that change the size of a geometric figure but not its shape)

- Example: Fractals are scale-invariant – more precisely, self-similar (in the figure below, the same drawing is repeated within itself at smaller and smaller scales)



- Theorem 1 (Euler's theorem): A differentiable function $\mathcal{G}(W_1, \cdots, W_m)$ is a homogeneous function of degree p if and only if

$$\boxed{\sum_{i=1}^{m} \frac{\partial \mathcal{G}}{\partial W_i}(W_1, \cdots, W_m) W_i = p\mathcal{G}(W_1, \cdots, W_m)}$$

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- Proof: ($\Rightarrow$) differentiate definition with respect to s and set $s = 1$
  ($\Leftarrow$) define $g(s) = \mathcal{G}(sW_1, \cdots, sW_m) - s^p\mathcal{G}(W_1, \cdots, W_m)$, differentiate $g(s)$ to get an ordinary differential equation in $g(s)$, note that $g(1) = 0$, and conclude that $g(s) = cst = 0$

- Theorem 2: If $\mathcal{G}(W_1, \cdots, W_m)$ is differentiable and homogeneous of degree p, then each of its partial derivatives $\dfrac{\partial \mathcal{G}}{\partial W_i}$ (for $i = 1, \cdots, m$) is a homogeneous function of degree $p - 1$

$$\boxed{\forall s > 0, \quad \frac{\partial \mathcal{G}}{\partial W_i}(sW_1, \cdots, sW_m) = s^{p-1} \frac{\partial \mathcal{G}}{\partial W_i}(W_1, \cdots, W_m)}$$

- Proof: Straightforward (differentiate both sides of the definition with respect to W_i)

- Linearization can be either physically relevant (small perturbations), convenient for analysis, or useful for constructing a linear model problem – in either case, it leads to a linear problem
- For the purpose of constructing a linear model version of Eq. (1), the coefficient matrices A, B, and C of this equation are often simply "frozen" to their values at a local flow condition designated by the subscript $_o$ and represented by the fluid state vector W_o, which leads to

$$\boxed{\frac{\partial W}{\partial t} + A_o \frac{\partial W}{\partial x} + B_o \frac{\partial W}{\partial y} + C_o \frac{\partial W}{\partial z} = S_o} \qquad (3)$$

- The above linear equation can be insightful for the construction or analysis of a CFD scheme

- On the other hand, the "genuine" linearization of

$$\boxed{\frac{\partial W}{\partial t} + A\frac{\partial W}{\partial x} + B\frac{\partial W}{\partial y} + C\frac{\partial W}{\partial z} = S}$$

(with S dependent on W) about a flow equilibrium condition W_o – which is physically more relevant – leads to the following equation where the perturbation δW around W_0 has been renamed W for simplicity

$$\boxed{\begin{aligned} \frac{\partial W}{\partial t} \quad &+ \quad A_o\frac{\partial W}{\partial x} + B_o\frac{\partial W}{\partial y} + C_o\frac{\partial W}{\partial z} - \left.\frac{\partial S}{\partial W}\right|_o W \\ &+ \quad \left.\frac{\partial A}{\partial W}\right|_o W\frac{\partial W_o}{\partial x} + \left.\frac{\partial B}{\partial W}\right|_o W\frac{\partial W_o}{\partial y} + \left.\frac{\partial C}{\partial W}\right|_o W\frac{\partial W_o}{\partial z} = 0 \end{aligned}}$$

$$\begin{aligned}\frac{\partial W}{\partial t} \quad &+ \quad A_o \frac{\partial W}{\partial x} + B_o \frac{\partial W}{\partial y} + C_o \frac{\partial W}{\partial z} - \left.\frac{\partial S}{\partial W}\right|_o W \\ &+ \quad \left.\frac{\partial A}{\partial W}\right|_o W \frac{\partial W_o}{\partial x} + \left.\frac{\partial B}{\partial W}\right|_o W \frac{\partial W_o}{\partial y} + \left.\frac{\partial C}{\partial W}\right|_o W \frac{\partial W_o}{\partial z} = 0\end{aligned} \tag{4}$$

- Hence, the following remarks are noteworthy:
  - in a genuine linearization around a dynamic equilibrium condition, the source term does not contribute a “frozen” right hand-side
  - in general, Eq. (4) and Eq. (3) are different
  - however, if the linearization is performed about a uniform flow condition W_o and S is independent of W (or $S = 0$), Eq. (4) and Eq. (3) become identical

- Consider here the linear model equation (3)

$$\frac{\partial W}{\partial t} + A_o \frac{\partial W}{\partial x} + B_o \frac{\partial W}{\partial y} + C_o \frac{\partial W}{\partial z} = S_o$$

- Linear equations such as the above equation have exact solutions
- Let $W(x, y, z, t^0)$ denote an initial value for W at time t^0: This initial condition can be expanded by Fourier decomposition with wave numbers k_{x_j}, k_{y_j}, and k_{z_j} as follows

$$W(x, y, z, t^0) = I(x, y, z) = \sum_j c_j e^{i(k_{x_j} x + k_{y_j} y + k_{z_j} z)}$$

- In this case, the exact solution of Eq. (3) for $t > t^0$ is

$$\boxed{W(x, y, z, t) = \underbrace{\sum_j e^{-i(t-t^0)(k_{x_j} A_o + k_{y_j} B_o + k_{z_j} C_o)} c_j e^{i(k_{x_j} x + k_{y_j} y + k_{z_j} z)}}_{\textit{homogeneous solution}} + \underbrace{(t - t^0) S_o}_{\textit{particular solution}}}$$

$$W(x,y,z,t) = \sum_j e^{-i(t-t^0)(k_{x_j}A_o + k_{y_j}B_o + k_{z_j}C_o)} c_j e^{i(k_{x_j}x + k_{y_j}y + k_{z_j}z)} + (t-t^0)S_o$$

- Hence, the solution of Eq. (3) has both a linear growth term and, depending on the eigenvalues of the matrix

$$M_j = k_{x_j}A_o + k_{y_j}B_o + k_{z_j}C_o$$

  a possible exponential growth in time components

- Consider the following equation

$$\frac{\partial G}{\partial x_\alpha} + \frac{\partial H}{\partial x_1} = 0 \tag{5}$$

- For example, for the unsteady Euler equations in one dimension

$$x_\alpha = t, \quad x_1 = x, \quad G = W = (\rho \;\; \rho v_x \;\; E)^T$$

$$H = \mathcal{F}_x = \left(\rho v_x \;\; \rho v_x^2 + p \;\; (E+p)v_x\right)^T$$

- For the steady Euler equations in two dimensions

$$x_\alpha = x, \quad x_1 = y$$

$$G = \mathcal{F}_x \left(\rho v_x \;\; \rho v_x^2 + p \;\; \rho v_x v_y \;\; (E+p)v_x\right)^T$$

$$H = \mathcal{F}_y = \left(\rho v_y \;\; \rho v_x v_y \;\; \rho v_y^2 + p \;\; (E+p)v_y\right)^T$$

- Let
$$A = \frac{\partial H}{\partial G}$$
and let $\Lambda = \text{diag}(\lambda_1, \; \cdots, \; \lambda_m)$ be the diagonal matrix of eigenvalues $\lambda_1, \; \cdots, \; \lambda_m$ of A
- Eq. (5) is hyperbolic if
    - (1) λ_k is real for each $k = 1, \; \cdots, \; m$
    - (2) A has a complete set of eigenvectors $\Leftrightarrow$ A is diagonalizable – that is
$$\exists Q \, / \, A = \frac{\partial H}{\partial G} = Q^{-1} \Lambda Q$$
- In the general multidimensional case $\big($see Eq. (1)$\big)$, the system is hyperbolic if the matrix
$$M = k_x A + k_y B + k_z C$$
has only real eigenvalues and a complete set of eigenvectors, for all sets of real numbers (k_x, k_y, k_z)

- In mathematics, the "method" of characteristics is a technique for solving partial differential equations
- Essentially, it reduces a *partial differential equation* to a **family** of *ordinary differential equations* along which the solution can be integrated from some initial data given on a suitable **hypersurface**
- It is applicable to any hyperbolic partial differential equation, but has been developed mostly for first-order hyperbolic partial differential equations
- Characteristic "theory" is pertinent to the treatment of boundary conditions and CFD schemes such as flux split schemes (Steger and Warming) and flux difference splitting schemes (Roe)

- Consider the following unsteady homogeneous hyperbolic equations written in non conservation form

$$\frac{\partial W}{\partial t} + A\frac{\partial W}{\partial x} = 0, \qquad A = \frac{\partial \mathcal{F}}{\partial W} = A(W) \tag{6}$$

- A is diagonalizable and therefore

$$A = Q^{-1}\Lambda Q \tag{7}$$

where $\Lambda = \text{diag}\,(\lambda_1, \ \cdots, \ \lambda_m) = \Lambda(W)$ and $Q = Q(W)$

- Let r_i denote the i-th column of Q^{-1}:
$AQ^{-1} = Q^{-1}\Lambda \Rightarrow Ar_i = \lambda_i r_i \Rightarrow r_i$ is A's i-th *right* eigenvector
- Let ℓ_i denote the i-th column of Q^T which is the i-th row of Q:
$QA = \Lambda Q \Rightarrow A^T Q^T = Q^T \Lambda \Rightarrow A^T \ell_i = \lambda_i \ell_i$ (or $\ell_i^T A = \lambda_i \ell_i^T$) $\Rightarrow \ell_i$ is A's i-th *left* eigenvector

- Substituting Eq. (7) into Eq. (6) and pre-multiplying by Q leads to the so-called *characteristic form* of Eq. (6)

$$Q\frac{\partial W}{\partial t} + \Lambda Q\frac{\partial W}{\partial x} = 0$$

- The *characteristic variables* $\xi = (\xi_1 \ \cdots \ \xi_m)^T$ are defined as follows (note the differential form)

$$\boxed{d\xi = Q(W)dW}$$

- Substituting in the characteristic form of the governing equations leads to

$$\frac{\partial \xi}{\partial t} + \Lambda\frac{\partial \xi}{\partial x} = 0 \qquad (8)$$

  which is also called the characteristic form of the governing equations and which **decouples** the characteristic variables

- Each characteristic equation within Eq. (8) can be written as $\left[\frac{\partial \xi_i}{\partial t} \; \frac{\partial \xi_i}{\partial x}\right]^T \cdot (1 \;\lambda_i)^T = \overrightarrow{\nabla}^{\star}\xi_i \cdot (1 \;\lambda_i)^T = 0, \;\; i = 1, \;\cdots, \; m$, which shows and states that in the $x - t$ plane,
  - $\frac{\partial \xi_i}{\partial t} + \lambda_i \frac{\partial \xi_i}{\partial x}$ is a *directional derivative* [1] – in the direction $(1 \;\lambda_i)^T$
  - there is no change in the solution ξ_i in the direction of $(1 \;\lambda_i)^T$

- Now, consider a curve $x = x(t)$ that is everywhere tangent to $(1 \;\lambda_i)^T$ in the $x - t$ plane: Then, the slope of the vector $(1 \;\lambda_i)^T$ is the slope of the curve $x = x(t)$ and is given by

$$\frac{dx}{dt} = \lambda_i$$

[1] The directional derivative $\overrightarrow{\nabla}_u f(x_0, y_0, z_0)$ is the rate at which the function $f(x, y, z)$ changes at a point (x_0, y_0, z_0) in the direction $\vec{u}$. It can be defined as: $\overrightarrow{\nabla}_u f = \overrightarrow{\nabla} f \cdot \vec{u}/\|\vec{u}\| = \lim_{h \to 0} (f(X + hu) - f(X))/h$.

- Then, Eq. (8) is equivalent to

$$d\xi_i = 0 \; (\,\text{or}\, \xi_i = cst) \quad \text{on} \quad \frac{dx}{dt} = \lambda_i, \;\; i = 1, \;\cdots, \;\; m$$

- This is a wave solution: The eigenvalues λ_i are wave speeds, and the wavefronts $\dfrac{dx}{dt} = \lambda_i$ are sometimes also called *characteristic curves* (or simply *characteristics*)

$$\frac{\partial W}{\partial t}+\frac{\partial \mathcal{F}_x}{\partial x}=0, \quad W=\left(\rho \;\; \rho v_x \;\; E\right)^T, \quad \mathcal{F}_x=\left(\rho v_x \;\; \rho v_x^2+p \;\; (E+p)v_x\right)^T$$

with $p=(\gamma-1)\left(E-\rho\frac{v_x^2}{2}\right)$ and the speed of sound c given by

$$c^2=\gamma\frac{p}{\rho}$$

- Choose $V=\left(\rho \;\; v_x \;\; p\right)^T$ as the fluid state vector (with primitive variables) and re-write the governing equations in non conservation form (see Eq. (1) and Eq. (2))

$$\frac{\partial V}{\partial t}+A'\frac{\partial V}{\partial x}=0, \qquad A'=\begin{pmatrix} v_x & \rho & 0 \\ 0 & v_x & \frac{1}{\rho} \\ 0 & \rho c^2 & v_x \end{pmatrix}$$

- Diagonalize the resulting hyperbolic equations

$$A' = Q^{-1}\Lambda Q \Leftrightarrow QA'Q^{-1} = \Lambda$$

$$\Lambda = \begin{pmatrix} v_x & 0 & 0 \\ 0 & v_x + c & 0 \\ 0 & 0 & v_x - c \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & -\frac{1}{c^2} \\ 0 & 1 & \frac{1}{\rho c} \\ 0 & 1 & -\frac{1}{\rho c} \end{pmatrix} \qquad Q^{-1} = \begin{pmatrix} 1 & \frac{\rho}{2c} & -\frac{\rho}{2c} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\rho c}{2} & -\frac{\rho c}{2} \end{pmatrix} \quad (9)$$

- Let $\xi = (\xi_0 \;\; \xi_+ \;\; \xi_-)^T$ denote the characteristic variables
- The three characteristic equations are

$$\begin{aligned}
\frac{\partial \xi_0}{\partial t} + v_x \frac{\partial \xi_0}{\partial x} &= 0 \\
\frac{\partial \xi_+}{\partial t} + (v_x + c) \frac{\partial \xi_+}{\partial x} &= 0 \\
\frac{\partial \xi_-}{\partial t} + (v_x - c) \frac{\partial \xi_-}{\partial x} &= 0
\end{aligned}$$

with in this case $\boxed{d\xi = Q(V)dV}$ and $V = (\rho \;\; v_x \;\; p)^T$

- From (9), it follows that the above equations are equivalent to

$$d\xi_0 = d\rho - \frac{dp}{c^2} = ds = 0 \quad \text{for} \quad dx = v_x dt \quad (s \text{ denotes here the entropy})$$

$$d\xi_+ = dv_x + \frac{dp}{\rho c} = 0 \quad \text{for} \quad dx = (v_x + c)dt$$

$$d\xi_- = dv_x - \frac{dp}{\rho c} = 0 \quad \text{for} \quad dx = (v_x - c)dt$$

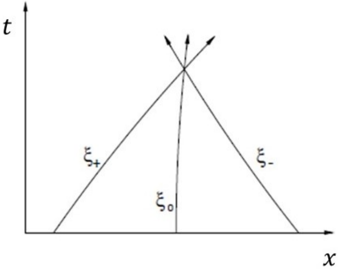
- The solution of these characteristic equations can be written as

$$
\begin{array}{llll}
\xi_0 = s = cst & \text{on} & dx = v_x dt & \text{(entropy wave)} \\
\xi_+ = v_x + \displaystyle\int \frac{dp}{\rho c} = cst & \text{on} & dx = (v_x + c)dt & \text{(acoustic wave)} \\
\xi_- = v_x - \displaystyle\int \frac{dp}{\rho c} = cst & \text{on} & dx = (v_x - c)dt & \text{(acoustic wave)}
\end{array}
\tag{10}
$$

- Notice that in this case, only the first characteristic equation is fully analytically integrable (but not its corresponding characteristic curve $dx = v_x dt$)

- For this and other reasons, characteristics are important conceptually, but not of too great importance quantitatively

- Note that



$\Rightarrow$ the state (ξ_0, ξ_+, ξ_-) at a point in the $x - t$ plane can be fully determined by walking along each of the three corresponding characteristic curves

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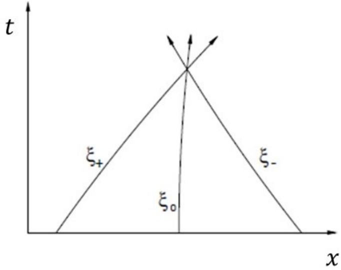

$\Rightarrow$ the state (ξ_0, ξ_+, ξ_-) at a point in the $x - t$ plane can be fully determined by walking along each of the three corresponding characteristic curves

- Recall that $d\xi = Q(V)dV \Leftrightarrow dV = Q^{-1}(V)d\xi$
$\Rightarrow$ the corresponding state V can be fully determined accordingly, as shown next

- *Integral curves of the characteristic family*

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  - recall that the i-th column of Q^{-1} ($i = 1, 2, 3$), denoted here by r_i, is the i-th right eigenvector of the Jacobian matrix (here A') associated with the i-th eigenvalue λ_i defining the characteristic curve $\dfrac{dx}{dt} = \lambda_i$: It depends entirely and only on the state $V = (\rho \quad v_x \quad p)^T = (V_1 \quad V_2 \quad V_3)^T$ and therefore defines a vector field

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  - since $d\xi = Q(V)dV \Leftrightarrow dV = Q^{-1}(V)d\xi$, it follows that

$$dV = Q^{-1}(V)d\xi = \sum_{i=1}^{3} r_i(V)d\xi_i \qquad (11)$$

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  - hence, one can look for a set of states $V(\eta)$ that connect to some starting state V_0 through integration along one of the vector fields r_i: These constitute *integral curves of the characteristic family*
  - two states V_a and V_b belong to the same j-characteristic integral curve if they are connected via the integral

$$\boxed{V_b = V_a + \int_a^b r_j(V)d\xi_j} \tag{12}$$

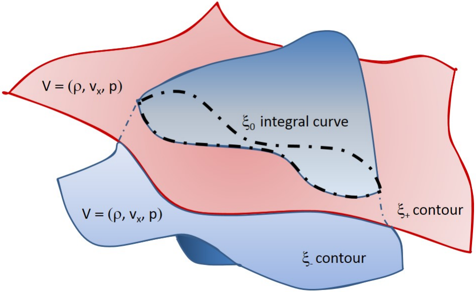
- *Integral curves of the characteristic family* (continue)
  - consider now the case of a linear hyperbolic equation with a constant advection matrix A'
    - the state vector V can be decomposed in eigen components as follows

$$V(x,t) = Q^{-1}\xi(x,t) = \sum_{i=1}^{3} r_i \xi_i(x,t)$$

    - a j-characteristic integral curve in state-space is a set of states for which only the component ξ_j along the eigenvector r_j varies, while the components along the other eigenvectors may be non zero but should be non varying
  - for a nonlinear hyperbolic equation, the above decomposition of V is no longer a useful concept, but the integral curves are the nonlinear equivalent of this idea

- *Riemann invariants*
  - one can express integral curves not only as integrals along the eigenvectors of the Jacobian (as in Eq. (12)), but also curves on which some *special scalars* are constant (as in Eq. (11), with only one $d\xi_j \neq 0$ and thus two $d\xi_i = 0 \Rightarrow$ see Eqs. (10))
  - in the 3D parameter space of $V = (V_1, V_2, V_3) = (\rho, v_x, p)$ – but otherwise 1D Euler equation – each curve is defined by two of such scalars
  - such scalar fields are called *Riemann invariants* of the characteristic family
    - here, ξ_+ and ξ_- are the Riemann invariants of the 1-characteristic integral curve
    - ξ_0 and ξ_- are the Riemann invariants of the 2-characteristic integral curve
    - ξ_0 and ξ_+ are the Riemann invariants of the 3-characteristic integral curve
    - the 2- and 3-characteristic integral curves represent here acoustic waves which, if they do not topple to become shocks, preserve entropy: Hence, entropy (ξ_0) is a Riemann invariant of these two families

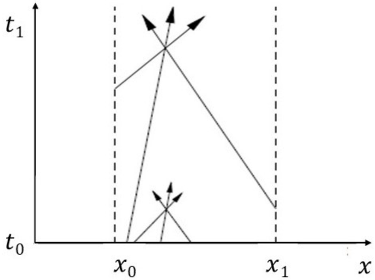
- *Riemann invariants* (continue)
  - hence, one can regard an integral curve as the crossing line between two contour curves of two Riemann invariants



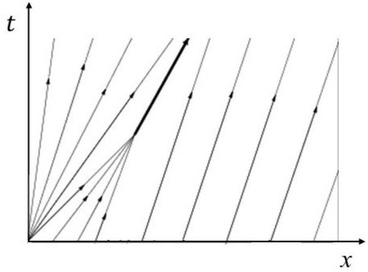
  - the value of each of the two Riemann invariants identifies this characteristic integral curve

- *Riemann invariants* (continue)
  - in summary, the Riemann invariants
    - arise from mathematical transformations made on a system of first-order partial differential equations to make them more easily solvable
    - are constant along characteristic integral curves of the partial differential equation

- *Simple waves*
  - note that if the Riemann invariants are constant along the characteristic curve $\dfrac{dx}{dt} = \lambda_i$, all flow properties are constant along this characteristic curve
  - by definition, a wave is called a *simple wave* if all states along the wave lie on the same integral curve of one of the characteristic families
  - hence, one can say that a simple wave is a pure wave in only one of the eigenvectors
  - examples
    - a simple wave in the 1-characteristic family ($dV = r_1 d\xi_0$) is a wave (or region of the flow) in which $v_x = cst$ and $p = cst$ but the entropy s can vary
    - a simple wave in the 3-characteristic family ($dV = r_3 d\xi_-$) is for example an infinitesimally weak acoustic wave in one direction
  - in Chapter 5, situations will be encountered where a *contact discontinuity* and a *rarefaction wave* are simple waves



- The characteristic relations coming to or from the boundaries determine the number and nature of the required boundary conditions for solving a given hyperbolic problem



- In general, characteristic curves *of the same family* do not intersect: If they do, they originate from a point to form an expansion fan or merge into a shock