# AA214: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

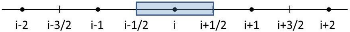
## The Finite Volume Method

These slides are partially based on the recommended textbook: Culbert B. Laney. "Computational Gas Dynamics," CAMBRIDGE UNIVERSITY PRESS, ISBN 0-521-62558-0

# Outline



Note: The material covered in this chapter equally applies to scalar conservation laws and the Euler equations, in one and multiple dimensions. To keep matters as simple as possible however, this material is presented primarily in one dimension then briefly extended to multiple dimensions.




- Recall that the integral form of a conservation law can be written as

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left[u(x, t^{n+1}) - u(x, t^n)\right] dx = -\int_{t^n}^{t^{n+1}} \left[f\left(u(x_{i+1/2}, t)\right) - f\left(u(x_{i-1/2}, t)\right)\right] dt$$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- This leads to the following numerical *conservation form*

$$\boxed{\Delta t \widehat{\left(\frac{\partial \bar{u}}{\partial t}\right)}_i^n = -\lambda(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n)} \qquad (1)$$

where

$$\bar{u}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n) dx, \quad \hat{f}_{i+1/2}^n \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u(x_{i+1/2}, t)\right) dt \qquad (2)$$

and the rest of the notation is the same as in the previous Chapter

- $\bar{u}_i^n$ is the spatial cell-integral average value of $u$ at time $t^n$ – that is, the average value of $u$ in the cell (or control volume) $[x_{i-1/2}, x_{i+1/2}]$ at time $t^n$
- $\hat{f}_{i+1/2}^n$ is the time-integral average of $f$ at the point $x_{i+1/2}$
- Note that for a uniform spatial discretization, $x_{i+1/2} - x_{i-1/2} = (x_{i+1} + x_i)/2 - (x_i + x_{i-1})/2 = (x_{i+1} - x_{i-1})/2 = \Delta x$ is constant, $x_{i\pm 1/2} = x_i \pm \Delta x/2$, and therefore

$$\boxed{\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} g(x)dx = g(x_i) + O(\Delta x^2)} \qquad (3)$$

- From (1)–(3), it follows that
  - if $u_i^n$ is replaced by $\bar{u}_i^n$, *ALL* concepts and results presented in the previous Chapter – that is, for the finite difference method – equally apply for the finite volume method presented in this Chapter
  - on a uniform spatial discretization, $\bar{u}_i^n = u_i^n + O(\Delta x^2) \Rightarrow$ first-order accuracy and second-order accuracy in space are not affected by identifying $\bar{u}_i^n$ with $u_i^n$ and vice-versa $\Rightarrow$ there is no need to distinguish between finite volume and finite difference when the order of spatial accuracy is less or equal to 2
  - the "bar" notation is often dropped in the remainder of this chapter, particularly when this should not create any confusion


$$\int_{x_{i-1/2}}^{x_{i+1/2}} [u(x,t^{n+1}) - u(x,t^n)]\,dx = -\int_{t^n}^{t^{n+1}} \left[f\left(u(x_{i+1/2},t)\right) - f\left(u(x_{i-1/2},t)\right)\right] dt$$

$$\Delta t \widehat{\left(\frac{\partial \bar{u}}{\partial t}\right)}_i^n = -\lambda(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n)$$

$$\bar{u}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t^n)dx, \quad \hat{f}_{i+1/2}^n \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u(x_{i+1/2},t)\right) dt$$

- Two-step finite volume design approach
  - spatial reconstruction
    - reconstruct $u(x, t^n)$ in each cell without necessarily accounting for the upwind direction
    - this step differentiates the finite volume method which forms $u$ then $f(u)$, from the finite difference method which forms directly $f(u)$
  - temporal evolution
    - approximate $u(x_{i+1/2}, t)$ for $t^n \leq t \leq t^{n+1} \Rightarrow \tilde{u}(x_{i+1/2}, t)$, then evaluate

$$\hat{f}^n_{i+1/2} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f\left(\tilde{u}(x_{i+1/2}, t)\right) dt$$

    - any reasonable approximation based on waves and characteristics naturally introduces the minimal amount of upwinding required by the CFL condition

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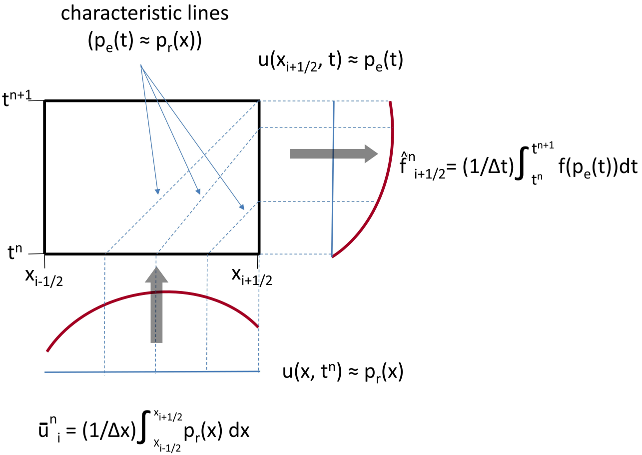
- Reconstruction-evolution methods are sometimes called *Godunov-type* methods or *MUSCL-type* methods


characteristic lines
(pe(t) ≈ pr(x))
u(xi+1/2, t) ≈ pe(t)
tn+1
tn
xi-1/2
xi+1/2
f̂ni+1/2 = (1/Δt)∫ tn tn+1 f(pe(t))dt
u(x, tn) ≈ pr(x)
ūni = (1/Δx)∫ xi-1/2 xi+1/2 pr(x) dx

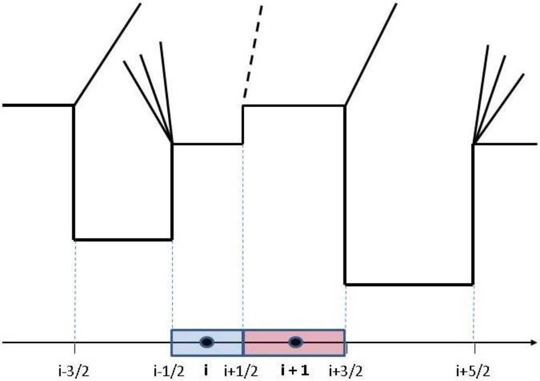
- Example: first-order reconstruction-evolution method for the linear advection equation
  - piecewise constant reconstruction in the cell $[x_{i-1/2}, x_{i+1/2}]$: $u(x, t^n) \approx p_r(x) = \bar{u}_i^n$
  - for the linear advection equation, $u(x, t) = u(x - a(t - t^n), t^n)$ and therefore

$$
\begin{aligned}
u(x_{i+1/2}, t) &\approx \tilde{u}(x_{i+1/2}, t) \\
&= p_{e,i+1/2}(t) = p_r\left(x_{i+1/2} - a(t - t^n)\right); \; -1 \leq \lambda a \leq 1 \\
&= \begin{cases} \bar{u}_i^n & \text{for} \quad 0 \leq \lambda a \leq 1 \\ \bar{u}_{i+1}^n & \text{for} \quad -1 \leq \lambda a \leq 0 \end{cases}
\end{aligned}
$$

  - then

$$
\begin{aligned}
\hat{f}_{i+1/2}^n &= \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(p_{e,i+1/2}(t)\right) dt = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} a p_{e,i+1/2}(t) dt \\
&= \begin{cases} a\bar{u}_i^n & \text{for} \quad 0 \leq \lambda a \leq 1 \\ a\bar{u}_{i+1}^n & \text{for} \quad -1 \leq \lambda a \leq 0 \end{cases}
\end{aligned}
$$


- What if the previous example was about a nonlinear conservation law instead of the linear advection equation?
  - piecewise constant reconstruction
    - gives rise to a Riemann problem at each cell edge, and the Riemann problem has an exact solution at $x/t = 0$
    - an approximate Riemann solver can be used instead of the true Riemann solver without changing the numerical solution (surprisingly)
  - higher-order reconstruction
    - the jump at each cell edge in the higher-order piecewise polynomial reconstruction gives rise to a problem that lacks a known exact solution
    - the exact solution can be approximated using Riemann solvers


i-3/2
i-1/2
i
i+1/2
i+1
i+3/2
i+5/2

- Suppose that the reconstruction is piecewise constant: Then, each cell edge gives rise to a Riemann problem
- The exact evolution of the piecewise constant reconstruction yields

$$u_i^{n+1} = u_i^n - \lambda(\hat{f}^n_{i+1/2} - \hat{f}^n_{i-1/2})$$

where

$$\boxed{\hat{f}^n_{i+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u_{Riemann}(x_{i+1/2}, t)\right) dt}$$

- Suppose that the reconstruction is piecewise constant: Then, each cell edge gives rise to a Riemann problem
- The exact evolution of the piecewise constant reconstruction yields

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where

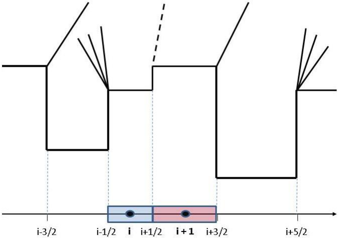
$$\boxed{\hat{f}_{i+1/2}^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f\left(u_{Riemann}(x_{i+1/2}, t)\right) dt}$$

- Since the solution of the Riemann problem is self-similar $\big($a function of $(x - x_{i+1/2})/t\big)$, $u_{Riemann}(x_{i+1/2}, t) = u\left((x_{i+1/2} - x_{i+1/2})/t\right)$ $= u(0)$ is constant for all time

$$\Longrightarrow \boxed{\hat{f}_{i+1/2}^n = f\left(u_{Riemann}(x_{i+1/2}, t)\right) = f\left(u_{Riemann}(0)\right)} \qquad (4)$$

where $f\left(u_{Riemann}(x_{i+1/2}, t)\right)$ is computed using any exact or approximate Riemann solver and $t$ is any arbitrary time $t > t^n$
- Many first-order upwind methods for scalar conservation laws are based on (4)

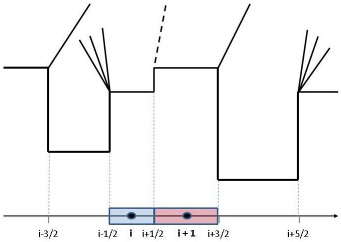
- Upwind methods based on $\hat{f}^n_{i+1/2} = f\left(u_{Riemann}(x_{i+1/2}, t)\right)$




$$\hat{f}^n_{i+1/2} = f\left(u_{Riemann}(x_{i+1/2}, t)\right) = f\left(u_{Riemann}(0)\right)$$

- above equation assumes that the waves from different cell edges do not interact (or at least that any interaction does not affect the solution at the cell edges)
  - $|\lambda a(u)| \leq 1/2 \Rightarrow$ waves originating at one cell edge cannot interact with those originating from any other cell edge
  - $|\lambda a(u)| \leq 1 \Rightarrow$ they can, but the interactions cannot reach the cell edges in one time-step

- Upwind methods based on $\hat{f}^n_{i+1/2} = f\left(u_{Riemann}(x_{i+1/2}, t)\right)$




- CFL condition: $|\lambda a(u)| \leq 1$
- conservative, consistent, converge when $\Delta x \to 0$ and $\Delta t \to 0$ and the CFL condition is satisfied
- explicit, finite volume
- linearly stable provided the CFL condition is satisfied
- satisfy all nonlinear stability conditions of previous Chapter
- formally $1^{st}$-order accurate in space and time (except possibly at sonic points)

- Upwind methods based on $\hat{f}^n_{i+1/2} = f\left(u_{Riemann}(x_{i+1/2}, t)\right)$ (continue)
  - if $|\lambda a(u)| \leq 1$, waves can interact only if there is a compressive sonic (sonic $\Rightarrow a(u) = 0$, compressive $\Rightarrow$ wave direction switches from right to left) point inside the cell[1]

compressive sonic point

    waves at the left cell edge are right-running, counterparts at the right cell edge are left-running
  - if the wave speeds are always positive, all waves originating from $x_{i+1/2}$ are right-running $\Rightarrow \hat{f}^n_{i+1/2} = f\left(u_{Riemann}(x_{i+1/2}, t)\right) = f(u^n_i) \Rightarrow$ FTBS
  - if the wave speeds are always negative, all waves originating from $x_{i+1/2}$ are left-running $\Rightarrow \hat{f}^n_{i+1/2} = f\left(u_{Riemann}(x_{i+1/2}, t)\right) = f(u^n_{i+1}) \Rightarrow$ FTFS
  - the above is true for the exact Riemann solver or any reasonable approximate Riemann solver $\Rightarrow$ all first-order upwind methods based on $\hat{f}^n_{i+1/2} = f\left(u_{Riemann}(x_{i+1/2}, t)\right)$ are FTBS or FTFS except near sonic points where the wave speeds change sign

[1] Compressive sonic points typically occur inside stationary or slowly moving shocks.

- Assuming that the reconstruction is piecewise constant, each cell gives rise to a Riemann problem and the exact evolution of the piecewise constant reconstruction yields

$$\boxed{\widehat{\mathcal{F}}^n_{x_{i+1/2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right) dt}$$

- Again, the solution of the Riemann problem being self-similar, $W_{Riemann}(x_{i+1/2}, t)$ is constant for all time

$$\Longrightarrow \boxed{\widehat{\mathcal{F}}^n_{x_{i+1/2}} = \mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right) = \mathcal{F}_x\left(W_{Riemann}(0)\right)} \quad (5)$$

where $\mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right)$ is computed using any exact or approximate Riemann solver and $t$ is any arbitrary time $t > t^n$

- Many first-order upwind methods for the Euler equations are based on (5)

$$\widehat{\mathcal{F}}^n_{x_{i+1/2}} = \mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right)$$

- Again, equation (5) assumes that the waves from different cell edges do not interact (or at least that any interaction does not affect the solution at the cell edges)
- If $\lambda\rho(A) \leq 1$, waves travel at most one grid cell per time-step: They can interact, but the interactions cannot reach the cell edges during a single time-step
- Unlike in the case of scalar conservation laws where a compressive point inside the cell must be present for waves to interact when $|\lambda a(u)| \leq 1$, waves in the subsonic Euler equations interact routinely for $1/2 \leq \lambda\rho(A) \leq 1$ (since there are always right- and left-running waves in subsonic flows)
- $\lambda\rho(A) \leq 1$ is also the CFL condition for all first-order upwind methods

- Upwind methods based on $\widehat{\mathcal{F}}^n_{x_{i+1/2}} = \mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right)$
  - if the wave speeds are always positive (as in some supersonic flows), all waves originating from $x_{i+1/2}$ are right-running
    $\Rightarrow \widehat{\mathcal{F}}^n_{x_{i+1/2}} = \mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right) = \mathcal{F}_x(W^n_i) \Rightarrow$ FTBS
  - if the wave speeds are always negative (as in some supersonic flows), all waves originating from $x_{i+1/2}$ are left-running
    $\Rightarrow \widehat{\mathcal{F}}^n_{x_{i+1/2}} = \mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right) = \mathcal{F}_x(W^n_{i+1}) \Rightarrow$ FTFS
  - the above is true for the exact Riemann solver or any reasonable approximate Riemann solver[2] $\Rightarrow$ all first-order upwind methods based on $\widehat{\mathcal{F}}^n_{x_{i+1/2}} = \mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right)$ are FTBS or FTFS *for supersonic flows*

[2] Recall that for Roe's approximate Riemann solver:

$$\widehat{\mathcal{F}}_x\left(W(0)\right) = \mathcal{F}_x(W_L) + A^-_{RL}(W_R - W_L) \qquad \left(\widehat{\frac{\partial \mathcal{F}_x}{\partial W}}\bigg|_{W_L}\right)$$

$$\widehat{\mathcal{F}}_x\left(W(0)\right) = \mathcal{F}_x(W_R) - A^+_{RL}(W_R - W_L) \qquad \left(\widehat{\frac{\partial \mathcal{F}_x}{\partial W}}\bigg|_{W_R}\right)$$

$\Longrightarrow$ also shows that the temporal evolution introduces the minimal amount of upwinding required by the CFL condition

- Most properties of first-order upwind methods for scalar conservation laws carry over to first-order upwind methods for the Euler equations
- The reverse is not necessarily true:
  - for example, the Riemann problem for the Euler equations is not monotonicity preserving but that for scalar conservation laws is monotonicity preserving $\Rightarrow$ first-order upwind methods for the Euler equations based on reconstruction-evolution sometimes produce spurious oscillations, especially at steady and slowly moving shocks

- Recall Roe's approximate Riemann solver for the Euler equations

$$\widehat{\mathcal{F}}_x\left(W(0)\right) = \frac{1}{2}\left(\mathcal{F}_x(W_R) + \mathcal{F}_x(W_L)\right) - \frac{1}{2}|A_{RL}|(W_R - W_L)$$

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- replace $W_L$ and $W_R$ by $W_i^n$ and $W_{i+1}^n$, respectively, and replace $A_{RL}$ by $A_{i+1/2}^n$
- replace also $t = 0$ by $t = t^n$ and $x = 0$ by $x = x_{i+1/2}$
- then, Roe's first-order upwind method for the Euler equations can be described by

$$\widehat{\mathcal{F}}^n_{x_{i+1/2}} = \frac{1}{2}\left(\mathcal{F}_x(W_i^n) + \mathcal{F}_x(W_{i+1}^n)\right) - \frac{1}{2}|A_{i+1/2}|(W_{i+1}^n - W_i^n)$$

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$$\widehat{\mathcal{F}}_{x_{i+1/2}}^n = \frac{1}{2}\left(\mathcal{F}_x(W_i^n) + \mathcal{F}_x(W_{i+1}^n)\right) - \frac{1}{2}|A_{i+1/2}|(W_{i+1}^n - W_i^n)$$

$$\boxed{\widehat{\mathcal{F}}_{x_{i-1/2}}^n = \underbrace{\frac{1}{2}\left(\mathcal{F}_x(W_{i-1}^n) + \mathcal{F}_x(W_i^n)\right)}_{\textit{centered}} - \underbrace{\frac{1}{2}|A_{i-1/2}|(W_i^n - W_{i-1}^n)}_{\substack{\textit{dissipation} \\ \textit{upwinding (due to approximate} \\ \textit{Riemann solver)}}}}$$

- Recall also that the Roe-average matrix must satisfy

$$\mathcal{F}_x(W_{i+1}^n) - \mathcal{F}_x(W_i^n) = A_{i+1/2}^n(W_{i+1}^n - W_i^n)$$

and that $A_{i+1/2}^{n^+} + A_{i+1/2}^{n^-} = A_{i+1/2}^n$ and $A_{i+1/2}^{n^+} - A_{i+1/2}^{n^-} = |A_{i+1/2}^n|$

[3] Recall that a method in flux split form is conservative if and only if $\hat{f}_{i+1/2}^n = g_i^n + \Delta\hat{f}_{i+1/2}^{-n}$ and $\hat{f}_{i-1/2}^n = g_{i-1}^n + \Delta\hat{f}_{i-1/2}^{-n}$.

- Recall also that the Roe-average matrix must satisfy

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$$\Longrightarrow \left\{ \begin{array}{l} \mathcal{F}_x(W_{i+1}^n) - \mathcal{F}_x(W_i^n) = \left(A_{i+1/2}^{n^+} + A_{i+1/2}^{n^-}\right)(W_{i+1}^n - W_i^n) \\ \\ \\ \\ \end{array} \right.$$

[3] Recall that a method in flux split form is conservative if and only if $\hat{f}_{i+1/2}^n = g_i^n + \Delta\hat{f}_{i+1/2}^{-^n}$ and $\hat{f}_{i-1/2}^n = g_{i-1}^n + \Delta\hat{f}_{i-1/2}^{-^n}$.

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$$\Longrightarrow \begin{cases} \mathcal{F}_x(W_{i+1}^n) - \mathcal{F}_x(W_i^n) = \left(A_{i+1/2}^{n^+} + A_{i+1/2}^{n^-}\right)(W_{i+1}^n - W_i^n) \\ \widehat{\mathcal{F}}_{x_{i+1/2}}^n = \underbrace{\frac{1}{2}\left(\mathcal{F}_x(W_i^n) + \mathcal{F}_x(W_{i+1}^n)\right) - \frac{1}{2}A_{i+1/2}^{n^+}(W_{i+1}^n - W_i^n)}_{g_i^n} + \underbrace{\frac{1}{2}A_{i+1/2}^{n^-}(W_{i+1}^n - W_i^n)}_{\Delta\hat{f}_{i+1/2}^{-n}} \\ \widehat{\mathcal{F}}_{x_{i-1/2}}^n = \underbrace{\frac{1}{2}\left(\mathcal{F}_x(W_{i-1}^n) + \mathcal{F}_x(W_i^n)\right) - \frac{1}{2}A_{i-1/2}^{n^+}(W_i^n - W_{i-1}i^n)}_{g_{i-1}^n} + \underbrace{\frac{1}{2}A_{i-1/2}^{n^-}(W_i^n - W_{i-1}^n)}_{\Delta\hat{f}_{i-1/2}^{-n}} \end{cases}$$

[3]Recall that a method in flux split form is conservative if and only if $\hat{f}_{i+1/2}^n = g_i^n + \Delta\hat{f}_{i+1/2}^{-n}$ and $\hat{f}_{i-1/2}^n = g_{i-1}^n + \Delta\hat{f}_{i-1/2}^{-n}$.

- Recall also that the Roe-average matrix must satisfy

$$\mathcal{F}_x(W_{i+1}^n) - \mathcal{F}_x(W_i^n) = A_{i+1/2}^n (W_{i+1}^n - W_i^n)$$

and that $A_{i+1/2}^{n^+} + A_{i+1/2}^{n^-} = A_{i+1/2}^n$ and $A_{i+1/2}^{n^+} - A_{i+1/2}^{n^-} = |A_{i+1/2}^n|$

$$\Longrightarrow \left\{ \begin{array}{l} \mathcal{F}_x(W_{i+1}^n) - \mathcal{F}_x(W_i^n) = \left(A_{i+1/2}^{n^+} + A_{i+1/2}^{n^-}\right)(W_{i+1}^n - W_i^n) \\ \widehat{\mathcal{F}}_{x_{i+1/2}}^n = \underbrace{\frac{1}{2}\left(\mathcal{F}_x(W_i^n) + \mathcal{F}_x(W_{i+1}^n)\right) - \frac{1}{2} A_{i+1/2}^{n^+}(W_{i+1}^n - W_i^n)}_{g_i^n} + \underbrace{\frac{1}{2} A_{i+1/2}^{n^-}(W_{i+1}^n - W_i^n)}_{\Delta \hat{f}_{i+1/2}^{-n}} \\ \widehat{\mathcal{F}}_{x_{i-1/2}}^n = \underbrace{\frac{1}{2}\left(\mathcal{F}_x(W_{i-1}^n) + \mathcal{F}_x(W_i^n)\right) - \frac{1}{2} A_{i-1/2}^{n^+}(W_i^n - W_{i-1}i^n)}_{g_{i-1}^n} + \underbrace{\frac{1}{2} A_{i-1/2}^{n^-}(W_i^n - W_{i-1}^n)}_{\Delta \hat{f}_{i-1/2}^{-n}} \end{array} \right.$$

- For this reason, Roe's first-order upwind method is sometimes called a *flux difference splitting method* [3]

$$\boxed{\left(\mathcal{F}_x(W_{i+1}^n) - \mathcal{F}_x(W_i^n)\right) = A_{i+1/2}^{n^+}(W_{i+1}^n - W_i^n) + A_{i+1/2}^{n^-}(W_{i+1}^n - W_i^n)}$$

[3] Recall that a method in flux split form is conservative if and only if $\hat{f}_{i+1/2}^n = g_i^n + \Delta \hat{f}_{i+1/2}^{-n}$ and $\hat{f}_{i-1/2}^n = g_{i-1}^n + \Delta \hat{f}_{i-1/2}^{-n}$.

- Higher-order accurate methods based on the *exact* Riemann solver
  - the spatial reconstruction is straightforward: Use any piecewise-linear, piecewise-quadratic, or higher-order piecewise polynomial reconstruction
  - the temporal evolution is more elaborate, as explained below
- Recall that the temporal evolution is given by

$$\widehat{\mathcal{F}}^n_{x_{i+1/2}} \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathcal{F}_x \left( W(x_{i+1/2}, t) \right) dt$$

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- Step 1: Compute a second- or even higher-order approximation of the above integral
  - for example, use the midpoint rule

$$\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathcal{F}_x\left(W(x_{i+1/2}, t)\right) dt = \underline{\mathcal{F}_x\left(W(x_{i+1/2}, t^{n+1/2})\right)} + O(\Delta t^2)$$

  - or the trapezoidal rule

$$\begin{aligned}\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathcal{F}_x\left(W(x_{i+1/2}, t)\right) dt &= \underline{\frac{1}{2}\mathcal{F}_x\left(W(x_{i+1/2}, t^{n+1})\right)} \\ &+ \frac{1}{2}\mathcal{F}_x\left(W(x_{i+1/2}, t^{n})\right) + O(\Delta t^2)\end{aligned}$$

- Step 2: Perform a Taylor series expansion about $t = t^n$ – for example

$$\begin{aligned}\mathcal{F}_x\left(W(x_{i+1/2}, t)\right) &= \mathcal{F}_x\left(W(x_{i+1/2}, t^n)\right) + \underline{\frac{\partial \mathcal{F}_x}{\partial t}\left(W(x_{i+1/2}, t^n)\right)}(t - t^n)\\ &+ O\left((t - t^n)^2\right)\end{aligned}$$

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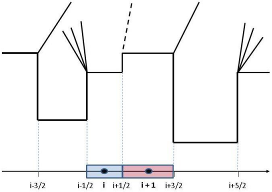
- Step 3: Express time derivatives in the Taylor series in terms of space derivatives (Cauchy-Kowalewski) – for example, the time derivative of the momentum flux can be obtained from the conservation of momentum

$$\frac{\partial}{\partial t}(\rho v_x) = -\rho v_x \underline{\frac{\partial v_x}{\partial x}} - v_x \underline{\frac{\partial}{\partial x}(\rho v_x)} - \underline{\frac{\partial p}{\partial x}}$$

$$\frac{\partial}{\partial t}(\rho v_x) = -\rho v_x \frac{\partial v_x}{\partial x} - v_x \frac{\partial}{\partial x}(\rho v_x) - \underline{\frac{\partial p}{\partial x}}$$




- Step 4: Differentiate the spatial reconstruction at time level $n$ to approximate the spatial derivative at $(x_{i+1/2}, t^n)$

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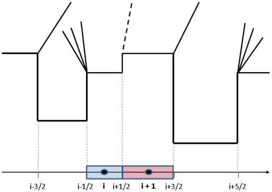


- Step 4: Differentiate the spatial reconstruction at time level $n$ to approximate the spatial derivative at $(x_{i+1/2}, t^n)$
  - note that in general, the reconstruction and/or its derivatives contain jump discontinuities at the cell edges at $x_{i+1/2}$
- Step 5: "Average" the left and right limits, $W_{i+1/2,L}(t)$ and $W_{i+1/2,R}(t)$, of the approximation of $W(x_{i+1/2}, t)$
  - the Riemann solver average is the only average that yields the exact solution in the case of piecewise constant reconstruction
  - for this purpose, use $W_{i+1/2,L}(t)$ and $W_{i+1/2,R}(t)$ as the left and right states in the exact solution of the Riemann problem

- Because the above four-step procedure is based on Taylor series and differential forms, it does not apply at shocks
- At shocks, all higher-order terms of the approximation should be eliminated and a return to first-order piecewise constant reconstruction becomes essential

- MUSCL: Monotonic Upwind Scheme for Conservation Laws
- More specifically, this section describes the method proposed in 1986 by Anderson, Thomas, and Van Leer who called it "the MUSCL" method, and not the original MUSCL method designed in 1979 by Van Leer using the approach summarized in the previous section
- Both of this method and the original MUSCL method are reconstruction-evolution methods

- Two-step approximation procedure
  - **Spatial Discretization**
    - time is frozen and space is discretized

$$\frac{\partial \mathcal{F}_x}{\partial x}(x_i, t) \approx \frac{\widehat{\mathcal{F}}_{x_{s,i+1/2}}(t) - \widehat{\mathcal{F}}_{x_{s,i-1/2}}(t)}{\Delta x}$$

    - the above equation is called the *semi-discrete* finite volume/difference approximation
    - $\widehat{\mathcal{F}}_{x_s}$ is called the *semi-discrete conservative numerical flux*
    - the semi-discrete approximation comprises a system of ordinary differential equations
    - in many cases, it is needed only at discrete time levels, in which case it can be written as

$$\frac{dW_i^n}{dt} \approx -\frac{\widehat{\mathcal{F}}^n_{x_{s,i+1/2}} - \widehat{\mathcal{F}}^n_{x_{s,i-1/2}}}{\Delta x} \qquad (6)$$

where $W_i^n = W_i(t^n)$ and $\widehat{\mathcal{F}}^n_{x_{s,i+1/2}} = \widehat{\mathcal{F}}_{x_{s,i+1/2}}(t^n)$

- Two-step approximation procedure (continue)

$$\frac{dW_i^n}{dt} \approx -\frac{\widehat{\mathcal{F}}^n_{x_{s,i+1/2}} - \widehat{\mathcal{F}}^n_{x_{s,i-1/2}}}{\Delta x}$$

  - **Temporal Discretization**
    - any ordinary differential equation solver (time-integration algorithm) can be used to solve Eq. (6)
    - in other words, space is frozen and time is discretized
    - the resulting approximation – for example using FT

$$\underbrace{\frac{W_i^{n+1} - W_i^n}{\Delta t}}_{x=x_i} = -\underbrace{\frac{\widehat{\mathcal{F}}^n_{x_{i+1/2}} - \widehat{\mathcal{F}}^n_{x_{i-1/2}}}{\Delta x}}_{t=t^n}$$

      is called the *fully discrete* finite difference approximation, and $\widehat{\mathcal{F}}_x$ is called the *fully discrete conservative numerical flux*

- The two-stage approximation procedure described above is sometimes called the *method of lines*, where the lines are the coordinate lines $x = cst$ and $t = cst$

- In the context of the (explicit) finite volume method

$$\widehat{\mathcal{F}}^n_{x_{s,i+1/2}} = \widehat{\mathcal{F}}_{x_{s,i+1/2}}(t^n)\left(\approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathcal{F}_x\left(W(x_{i+1/2},t)\right)\, dt\right) = \mathcal{F}_x\left(W(x_{i+1/2},t^n)\right)$$

- Using flux vector splitting – that is, assuming

$$\mathcal{F}_x(W) = \mathcal{F}_x^+(W) + \mathcal{F}_x^-(W)$$

- Leads to

$$\widehat{\mathcal{F}}^n_{x_{s,i+1/2}} \approx \mathcal{F}_x^+\left(W(x_{i+1/2},t^n)\right) + \mathcal{F}_x^-\left(W(x_{i+1/2},t^n)\right)$$

- Therefore, one needs to approximate $\mathcal{F}_x^+\left(W(x_{i+1/2},t^n)\right)$ – or equivalently, $W(x_{i+1/2},t^n)$ – with a leftward bias, and $\mathcal{F}_x^-\left(W(x_{i+1/2},t^n)\right)$ – or equivalently, $W(x_{i+1/2},t^n)$ – with a rightward bias

- First, $W(x_{i+1/2}, t^n)$ is approximated with a leftward bias, for use in $\mathcal{F}_x^+\left(W(x_{i+1/2}, t^n)\right)$
  - a first-order accurate reconstruction of the *primitive* variables $u$ of which $W$ is made leads to

$$u(x_{i+1/2}, t^n) \approx \bar{u}_i^n$$

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  - a second-order accurate linear reconstruction of the *primitive* variables $u$ of which $W$ is made leads to the following *extrapolation*

$$u(x_{i+1/2}, t^n) \approx \bar{u}_i^n + \frac{\bar{u}_i^n - \bar{u}_{i-1}^n}{\Delta x}(x_{i+1/2} - x_i) = \bar{u}_i^n + \frac{1}{2}(\bar{u}_i^n - \bar{u}_{i-1}^n)$$

- First, $W(x_{i+1/2}, t^n)$ is approximated with a leftward bias, for use in $\mathcal{F}_x^+\left(W(x_{i+1/2}, t^n)\right)$
  - a first-order accurate reconstruction of the *primitive* variables $u$ of which $W$ is made leads to

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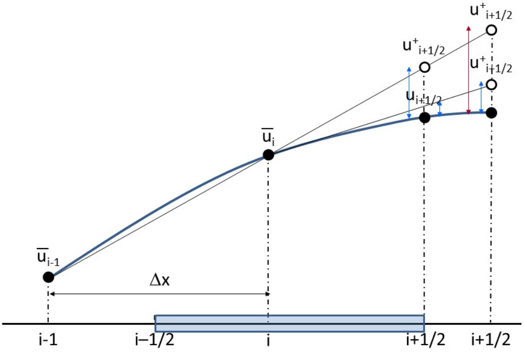
$$u(x_{i+1/2}, t^n) \approx \bar{u}_i^n + \frac{\bar{u}_i^n - \bar{u}_{i-1}^n}{\Delta x}(x_{i+1/2} - x_i) = \bar{u}_i^n + \frac{1}{2}(\bar{u}_i^n - \bar{u}_{i-1}^n)$$

  - instead of a pure constant or pure linear reconstruction, the following convex linear combination is considered $(0 \leq \theta \leq 1)$

$$\boxed{\begin{aligned} u_{i+1/2}^{n+} &= \theta_i^{n+}\bar{u}_i^n + (1-\theta_i^{n+})\left(\bar{u}_i^n + \frac{1}{2}(\bar{u}_i^n - \bar{u}_{i-1}^n)\right) \\ &= \bar{u}_i^n + \frac{1}{2}\phi_i^{n+}(\bar{u}_i^n - \bar{u}_{i-1}^n), \quad \text{where} \quad \phi_i^{n+} = 1 - \theta_i^{n+} \end{aligned}}$$

    which limits the effect of the extrapolation and achieves second-order accurate reconstruction $(\phi \approx 1)$ in the smooth regions of the flow

- Principle of a slope limiter




- What if the solution has an extremum in the vicinity of $x_i$?

- Next, $W(x_{i+1/2}, t^n)$ is approximated with a rightward bias, for use in $\mathcal{F}_x^-\left(W(x_{i+1/2}, t^n)\right)$
  - a first-order accurate reconstruction of the *primitive* variables of which $W$ is made leads to

$$u(x_{i+1/2}, t^n) \approx \bar{u}_{i+1}^n$$

  - a second-order accurate linear reconstruction of the *primitive* variables of which $W$ is made leads to the following *extrapolation*

$$u(x_{i+1/2}, t^n) \approx \bar{u}_{i+1}^n + \frac{\bar{u}_{i+2}^n - \bar{u}_{i+1}^n}{\Delta x}(x_{i+1/2} - x_{i+1}) = \bar{u}_{i+1}^n - \frac{1}{2}(\bar{u}_{i+2}^n - \bar{u}_{i+1}^n)$$

  - as for the approximation with a rightward bias, instead of a pure constant or pure linear reconstruction, the following convex linear combination is considered for limiting the effect of the extrapolation

$$\boxed{u_{i+1/2}^{n-} = \bar{u}_{i+1}^n - \frac{1}{2}\phi_{i+1}^{n-}(\bar{u}_{i+2}^n - \bar{u}_{i+1}^n)}$$

    where the superscript $n-$ refers to time $t^n$ and the rightward bias

- Applying the method of lines yields

$$\frac{dW_i^n}{dt} = -\frac{\widehat{\mathcal{F}}^n_{x_{s,i+1/2}} - \widehat{\mathcal{F}}^n_{x_{s,i-1/2}}}{\Delta x}$$

where

$$\begin{aligned}
\widehat{\mathcal{F}}^n_{x_{s,i+1/2}} &= \mathcal{F}_x^+\left(W(u^{n+}_{i+1/2})\right) + \mathcal{F}_x^-\left(W(u^{n-}_{i+1/2})\right) \\
&= \mathcal{F}_x^+\left(W\left(\bar{u}_i^n + \frac{1}{2}\phi_i^{n+}(\bar{u}_i^n - \bar{u}_{i-1}^n)\right)\right) + \mathcal{F}_x^-\left(W\left(\bar{u}_{i+1}^n - \frac{1}{2}\phi_{i+1}^{n-}(\bar{u}_{i+2}^n - \bar{u}_{i+1}^n)\right)\right) \\
\Rightarrow \frac{dW_i^n}{dt} &= -\frac{1}{\Delta x}\left(\mathcal{F}_x^+\left(W\left(\bar{u}_i^n + \frac{1}{2}\phi_i^{n+}(\bar{u}_i^n - \bar{u}_{i-1}^n)\right)\right) + \mathcal{F}_x^-\left(W\left(\bar{u}_{i+1}^n - \frac{1}{2}\phi_{i+1}^{n-}(\bar{u}_{i+2}^n - \bar{u}_{i+1}^n)\right)\right)\right) \\
&\quad + \frac{1}{\Delta x}\left(\mathcal{F}_x^+\left(W\left(\bar{u}_{i-1}^n + \frac{1}{2}\phi_{i-1}^{n+}(\bar{u}_{i-1}^n - \bar{u}_{i-2}^n)\right)\right) + \mathcal{F}_x^-\left(W\left(\bar{u}_i^n - \frac{1}{2}\phi_i^{n-}(\bar{u}_{i+1}^n - \bar{u}_i^n)\right)\right)\right) \\
&= -\frac{1}{\Delta x}\left(\underbrace{\mathcal{F}_x^+\left(W\left(\bar{u}_i^n + \frac{1}{2}\phi_i^{n+}(\bar{u}_i^n - \bar{u}_{i-1}^n)\right)\right) - \mathcal{F}_x^+\left(W\left(\bar{u}_{i-1}^n + \frac{1}{2}\phi_{i-1}^{n+}(\bar{u}_{i-1}^n - \bar{u}_{i-2}^n)\right)\right)}_{\Delta\hat{f}^{+n}_{i-1/2}}\right) \\
&\quad -\frac{1}{\Delta x}\left(\underbrace{\mathcal{F}_x^-\left(W\left(\bar{u}_{i+1}^n - \frac{1}{2}\phi_{i+1}^{n-}(\bar{u}_{i+2}^n - \bar{u}_{i+1}^n)\right)\right) - \mathcal{F}_x^-\left(W\left(\bar{u}_i^n - \frac{1}{2}\phi_i^{n-}(\bar{u}_{i+1}^n - \bar{u}_i^n)\right)\right)}_{\Delta\hat{f}^{-n}_{i+1/2}}\right)
\end{aligned}$$

- Many variants of the MUSCL method exist today
- The most popular ones use flux difference splitting along the lines of an approximate Riemann solver such as Roe's solver instead of flux vector splitting: In this case, convex linear combinations of the constant and linear reconstructions of $W_{i+1/2,L}(t)$ and $W_{i+1/2,R}(t)$ are used as the left and right states in the solution of the approximate Riemann problem
- Many slope limiters have been proposed in the literature and continue to be the subject of on-going research: One example is presented next
- Since at shocks, all higher-order terms of an approximation should be eliminated and a return to first-order piecewise constant reconstruction is essential, the value of a slope limiter must, by design, approach 1 away from shocks (and other discontinuities) and 0 near shocks (and other discontinuities)

- Ratios of solution differences

$$\boxed{r_i^{n+} = \frac{u_i^n - u_{i-1}^n}{u_{i+1}^n - u_i^n}, \qquad r_i^{n-} = \frac{u_{i+1}^n - u_i^n}{u_i^n - u_{i-1}^n} = \frac{1}{r_i^{n+}}}$$

- Note that
  - $r_i^{n\pm} \geq 0$ if the solution $u$ is monotone increasing or monotone decreasing
  - $r_i^{n\pm} \leq 0$ if the solution $u$ has a maximum or a minimum around $x_i$
  - $|r_i^{n+}|$ is large and $|r_i^{n-}|$ is small if the solution differences decrease dramatically from left to right or if $u_{i+1}^n \approx u_i^n$
  - $|r_i^{n+}|$ is small and $|r_i^{n-}|$ is large if the solution differences increase dramatically from left to right or if $u_{i-1}^n \approx u_i^n$

- Very large or very small ratios $|r_i^{n\pm}|$ sometimes signal shocks, but not always
  - for example, if $u_{i+1}^n - u_i^n = 0$ and $u_i^n - u_{i-1}^n \neq 0$, then $|r_i^{n+}| = \infty$ regardless of whether the solution is smooth or shocked
  - in general, because there is only limited information contained in solution samples, no completely reliable way to distinguish shocks from smooth regions exists
  - consequently, slope-limited methods do not even attempt to identify shocks: Instead, they regulate maxima and minima – whether or not they are associated with shocks – using the nonlinear stability conditions (for example, TVD)

- Van Leer's slope limiter can be described as

$$\phi(r) = \begin{cases} \dfrac{2r}{1+r} & \text{for} \quad r \geq 0 \quad \text{limits the slope in monotone regions of the flow} \\ 0 & \text{for} \quad r < 0 \quad \text{reverses to constant (first-order) reconstruction elsewhere} \end{cases}$$

- Note that for a uniform mesh, $\phi(1) = 1$ ($r = 1 \Rightarrow u_i = (u_{i-1} + u_{i+1})/2 \Rightarrow$ solution behaves locally as an affine (linear) function $\Rightarrow$ no slope limiting is needed)
- For a nonuniform mesh, a similar result is obtained for $r$ equal to the ratio of two consecutive space increments
- Note also that $\lim_{r \to \infty} \phi(r) = 2$
- Equipping the scheme described in the previous pages with

$$\boxed{\phi_i^{n\pm} = \phi\left(r_i^{n-/+}\right)}$$

(explain why in this case it is $\phi(r_i^{n-/+})$ and not $\phi(r_i^{n\pm})$) makes it:
  - TVD and therefore nonlinearly stable
  - second-order accurate in monotone regions ($r \geq 0$) of the flow (proof given in class)
  - in practice, between first-order and second-order accurate at extrema

- Recall that flux splitting is defined as

$$\begin{aligned}\mathcal{F}_x(W) &= \mathcal{F}_x^+(W) + \mathcal{F}_x^-(W)\\ \frac{d\mathcal{F}_x^+}{dW} &\geq 0, \qquad \frac{d\mathcal{F}_x^-}{dW} \leq 0\end{aligned}$$

- Hence, the *flux split form* of the Euler equations is

$$\frac{\partial W}{\partial t} + \frac{\partial \mathcal{F}_x^+}{\partial x} + \frac{\partial \mathcal{F}_x^-}{\partial x} = 0$$

- Then, $\dfrac{\partial \mathcal{F}_x^+}{\partial x}$ can be discretized conservatively using at least one point to the left – for example, using BS – and $\dfrac{\partial \mathcal{F}_x^-}{\partial x}$ can be discretized conservatively using at least one point to the right – for example, using FS – thus obtaining conservation and satisfaction of the CFL condition

- Recall the related concept of wave speed splitting

$$\begin{aligned} A(W) &= A^+(W) + A^-(W) \\ A^+(W) &\geq 0, \qquad A^-(W) \leq 0 \end{aligned}$$

- Then, the *vector* conservation law can be written in *wave speed split form* as

$$\frac{\partial W}{\partial t} + A^+(W)\frac{\partial W}{\partial x} + A^-(W)\frac{\partial W}{\partial x} = 0$$

where the matrices $A^+(W)$ and $A^-(W)$ are usually obtained by splitting the eigenvalues of $A(W)$ into positive and negative parts

$$\lambda_i(W) = \lambda_i^+(W) + \lambda_i^-(W), \qquad \lambda_i^+(W) \geq 0,\ \lambda_i^-(W) \leq 0$$

$$\Longrightarrow \Lambda(W) = \Lambda^+(W) + \Lambda^-(W), \qquad \Lambda^+(W) \geq 0,\ \Lambda^-(W) \leq 0$$

- For the 1D Euler equations written in conservation form

$$
\begin{aligned}
Q(W)A(W)Q^{-1}(W) &= \Lambda(W) \quad \Rightarrow A(W) = Q^{-1}(W)\Lambda(W)Q(W) \\
\text{where} \quad \lambda_1 &= v_x \quad \lambda_2 = v_x + c \quad \lambda_3 = v_x - c \\
\text{and} & \\
Q^{-1}(W) &= \begin{pmatrix} 1 & \frac{\rho}{2c} & -\frac{\rho}{2c} \\ v_x & \frac{\rho}{2c}(v_x + c) & -\frac{\rho}{2c}(v_x - c) \\ \frac{v_x^2}{2} & \frac{\rho}{2c}\left(\frac{v_x^2}{2} + \frac{c^2}{\gamma - 1} + c v_x\right) & -\frac{\rho}{2c}\left(\frac{v_x^2}{2} + \frac{c^2}{\gamma-1} - c v_x\right) \end{pmatrix} \\
Q(W) &= \frac{\gamma - 1}{\rho c} \begin{pmatrix} \frac{\rho}{c}\left(-\frac{v_x^2}{2} + \frac{c^2}{\gamma - 1}\right) & \frac{\rho}{c} v_x & -\frac{\rho}{c} \\ \frac{v_x^2}{2} - \frac{c v_x}{\gamma - 1} & -v_x + \frac{c}{\gamma - 1} & 1 \\ -\frac{v_x^2}{2} - \frac{c v_x}{\gamma - 1} & v_x + \frac{c}{\gamma - 1} & -1 \end{pmatrix} \\
A(W) &= A^+(W) + A^-(W), \qquad A^+(W) \geq 0, \qquad A^-(W) \leq 0 \\
A(W)^+ &= Q^{-1}(W)\Lambda^+(W)Q \geq 0, \quad A^-(W) = Q^{-1}(W)\Lambda^-(W)Q(W) \leq 0
\end{aligned}
$$

- For the Euler equations in conservation form and a perfect gas, $\mathcal{F}_x$ is homogeneous function of $W$ of degree 1
  - Steger and Warming (1981)[4]

$$\begin{aligned} \mathcal{F}_x(W) &= \left(\frac{d\mathcal{F}_x}{dW}(W)\right) W = A(W)W \qquad \text{(Euler's theorem)} \\ \Rightarrow \mathcal{F}_x^{\pm}(W) &= A^{\pm}(W)W = \left[Q^{-1}(W)\Lambda^{\pm}(W)Q(W)\right] W \qquad (7) \end{aligned}$$

$$\begin{aligned} \mathcal{F}_x^{\pm} &= \frac{\gamma-1}{\gamma}\rho\lambda_1^{\pm}\begin{bmatrix} 1 \\ v_x \\ \frac{1}{2}v_x^2 \end{bmatrix} \\ &+ \frac{\rho}{2\gamma}\lambda_2^{\pm}\begin{bmatrix} 1 \\ v_x + c \\ \frac{1}{2}v_x^2 + \frac{c^2}{\gamma-1} + cv_x \end{bmatrix} \\ &+ \frac{\rho}{2\gamma}\lambda_3^{\pm}\begin{bmatrix} 1 \\ v_x - c \\ \frac{1}{2}v_x^2 + \frac{c^2}{\gamma-1} - cv_x \end{bmatrix} \end{aligned}$$

[4] Lerat (1983): $d\mathcal{F}_x^+/dW \geq 0$ and $d\mathcal{F}_x^-/dW \leq 0$

- Practical implementation
  - compute at each grid point (constant reconstruction) or each edge of each computational cell (linear reconstruction)

$$\lambda_i^+ = \max(0, \lambda_i) = \frac{1}{2}(\lambda_i + |\lambda_i|), \quad \lambda_i^- = \min(0, \lambda_i) = \frac{1}{2}(\lambda_i - |\lambda_i|) \tag{8}$$

  - compute at each grid point (constant reconstruction) or each edge of each computational cell (linear reconstruction) the Mach number $M = \frac{v_x}{c} \in ]-\infty, -1] \cup [-1, 0] \cup [0, 1] \cup [1, \infty[$

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    - if $M \leq -1$ ($\Rightarrow \lambda_1 < 0,\ \lambda_2 \leq 0,\ \lambda_3 < 0$)

$$\mathcal{F}_x^+ = 0, \qquad \mathcal{F}_x^- = \mathcal{F}_x$$

- Practical implementation
  - compute at each grid point (constant reconstruction) or each edge of each computational cell (linear reconstruction)

$$\lambda_i^+ = \max(0, \lambda_i) = \frac{1}{2}(\lambda_i + |\lambda_i|), \quad \lambda_i^- = \min(0, \lambda_i) = \frac{1}{2}(\lambda_i - |\lambda_i|) \tag{8}$$

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    - if $M \leq -1$ ($\Rightarrow \lambda_1 < 0,\ \lambda_2 \leq 0,\ \lambda_3 < 0$)

$$\mathcal{F}_x^+ = 0, \qquad \mathcal{F}_x^- = \mathcal{F}_x$$

    - if $-1 < M \leq 0$ ($\Rightarrow \lambda_1 \leq 0,\ \lambda_2 > 0,\ \lambda_3 < 0$)

$$\mathcal{F}_x^+ = \frac{\rho}{2\gamma}(v_x + c)\begin{bmatrix} 1 \\ v_x + c \\ \frac{1}{2}v_x^2 + \frac{c^2}{\gamma-1} + cv_x \end{bmatrix}$$

$$\mathcal{F}_x^- = \frac{\gamma-1}{\gamma}\rho v_x \begin{bmatrix} 1 \\ v_x \\ \frac{1}{2}v_x^2 \end{bmatrix} + \frac{\rho}{2\gamma}(v_x - c)\begin{bmatrix} 1 \\ v_x - c \\ \frac{1}{2}v_x^2 + \frac{c^2}{\gamma-1} - cv_x \end{bmatrix}$$

- Practical implementation (continue)
  - continue
  - continue
    - if $0 < M < 1$ ($\Rightarrow \lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 < 0$)

$$
\begin{aligned}
\mathcal{F}_x^+ &= \frac{\gamma-1}{\gamma}\rho v_x \begin{bmatrix} 1 \\ v_x \\ \frac{1}{2}v_x^2 \end{bmatrix} + \frac{\rho}{2\gamma}(v_x + c) \begin{bmatrix} 1 \\ v_x + c \\ \frac{1}{2}v_x^2 + \frac{c^2}{\gamma-1} + c v_x \end{bmatrix} \\
\mathcal{F}_x^- &= \frac{\rho}{2\gamma}(v_x - c) \begin{bmatrix} 1 \\ v_x - c \\ \frac{1}{2}v_x^2 + \frac{c^2}{\gamma-1} - c v_x \end{bmatrix}
\end{aligned}
$$

- Practical implementation (continue)
  - continue
  - continue
    - if $0 < M < 1$ ($\Rightarrow \lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 < 0$)

$$\mathcal{F}_x^+ = \frac{\gamma-1}{\gamma}\rho v_x \begin{bmatrix} 1 \\ v_x \\ \frac{1}{2}v_x^2 \end{bmatrix} + \frac{\rho}{2\gamma}(v_x + c)\begin{bmatrix} 1 \\ v_x + c \\ \frac{1}{2}v_x^2 + \frac{c^2}{\gamma-1} + cv_x \end{bmatrix}$$

$$\mathcal{F}_x^- = \frac{\rho}{2\gamma}(v_x - c)\begin{bmatrix} 1 \\ v_x - c \\ \frac{1}{2}v_x^2 + \frac{c^2}{\gamma-1} - cv_x \end{bmatrix}$$

    - if $M \geq 1$ ($\Rightarrow \lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 \geq 0$)

$$\mathcal{F}_x^+ = \mathcal{F}_x, \qquad \mathcal{F}_x^- = 0$$

- Problem: At the sonic points ($M = 0$ for the entropy waves and $M = \pm 1$ for the acoustic waves), the wave speed splitting (8) is not differentiable – the first derivative of the split wave speeds is discontinuous at the sonic points because the function "absolute value" is discontinuous at zero
- Consequently, the wave speed splitting (8) may experience numerical difficulties at these sonic points – for example, when an implicit scheme is used; and the Jacobian of (7) is evaluated and thus $|\lambda_i|$ is differentiated
- Solution: Regularization (or "entropy fix", or "rounding the corner")

$$\boxed{\lambda_i^{\pm} = \frac{1}{2}\left(\lambda_i \pm \left|\sqrt{\lambda_i^2 + \epsilon^2}\right|\right)}$$

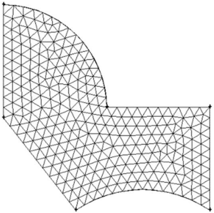
where $\epsilon$ is a small user-adjustable parameter

- The extension to multiple dimensions of the material covered in this chapter is straightforward
- The expressions of the Euler equations in 2D and 3D can be obtained from Chapter 2 (as particular cases of the expression of the Navier-Stokes equations in 3D)
- Furthermore, given that Chapter 6 discusses the extension of the finite difference method to multiple dimensions on structured grids, this chapter discusses – for the sake of variety – the extension of the finite volume method to multiple dimensions on unstructured grids
- For simplicity, the focus is set here on the 2D Euler equations

$$\frac{\partial W}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x}(W) + \frac{\partial \mathcal{F}_y}{\partial y}(W) = 0 \tag{9}$$

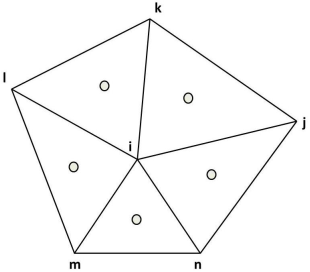
and on unstructured triangular meshes

- 2D *unstructured triangular mesh*



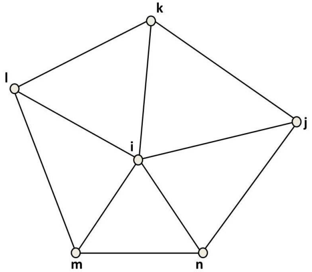
- Euler-Poincaré theorem: Asymptotically, there are about two times more triangles than nodes in a 2D triangular mesh

- 2D *cell-centered grid*



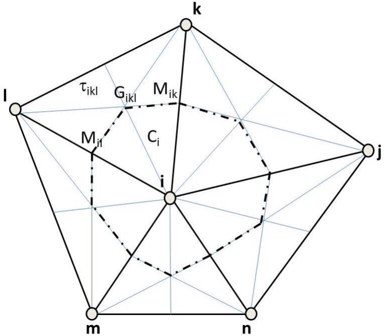

  - simple implementation: each triangle is at the same time a control volume and a *primal cell*
  - three numerical fluxes per triangle only, but more flow variables than necessary (recall the Euler-Poincaré theorem)

- 2D *vertex-based grid*



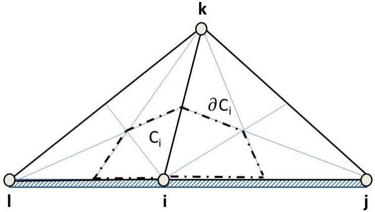

- deemed more accurate than the cell-centered approach for stretched/skewed grids
- more memory efficient than comparable cell-centered techniques (recall the Euler-Poincaré theorem)
- requires however the construction of an associated control volume or *dual* cell

- 2D control volume or *dual cell* for the vertex-based approach



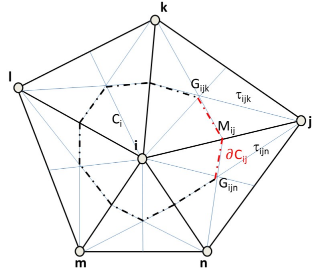

- Sample algorithm for constructing a dual cell $C_i$ attached to vertex $i$
  - for each triangle $\tau_{ikl}$ connected to $i$:
    - determine the point of intersection of the three *medians*, $G_{ikl}$
    - connect the point $G_{ikl}$ to the midpoints $M_{il}$ and $M_{ik}$ of the edges $il$ and $ik$, respectively

- 2D control volume or *dual cell* for the vertex-based approach (continue)




  - at the wall boundaries and at the far-field boundaries, one obtains “half” dual cells
  - the union of all full and half dual cells covers exactly the union of all triangles $\tau_{ikl}$ and therefore covers exactly the given triangular mesh

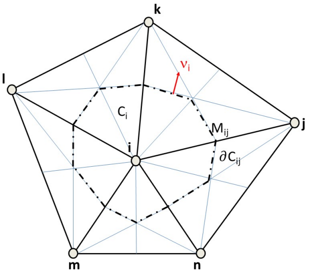
- 2D *cell edge* for the vertex-based approach




- the boundary $\partial C_i$ of each dual cell $C_i$ attached to vertex $i$ is the union of *cell edges* $\partial C_{i\star}$

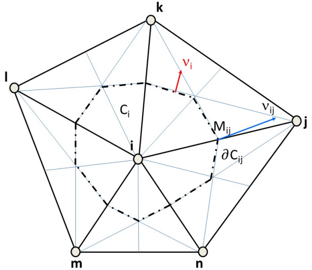
$$\partial C_i = \partial C_{ij} \cup \partial C_{ik} \cup \partial C_{il} \cup \partial C_{im} \cup \partial C_{in}$$

- the cell edge $\partial C_{ij}$, which is attached to the edge $ij$, is itself the union of two segments $M_{ij}G_{ijk}$ and $M_{ij}G_{ijn}$




- Consider again the 2D Euler equations (9): From the results of Chapter 3, it follows that the integration over the dual cell $C_i$ of these equations can be written (after scaling by $1/||C_i||$ and usage of the divergence theorem) as

$$\frac{\partial \overline{W}_i}{\partial t} + \frac{1}{\|C_i\|}\int_{\partial C_i} \vec{\mathcal{F}} \cdot \vec{\nu}_i d\Gamma_i = 0, \quad \text{where} \quad \overline{W}_i = \frac{1}{\|C_i\|}\int_{C_i} W_i d\Omega \tag{10}$$




- Integrating Eq. (10) between $t^n$ and $t^{n+1}$ and re-arranging gives

$$\begin{aligned}\int_{t^n}^{t^{n+1}} \frac{\partial \overline{W}_i}{\partial t} dt &= -\int_{t^n}^{t^{n+1}} \left( \frac{1}{\| C_i \|} \int_{\partial C_i} \vec{\mathcal{F}} \cdot \vec{\nu}_i d\Gamma_i \right) dt \\ &= -\sum_{\star} \int_{t^n}^{t^{n+1}} \left( \frac{1}{\| C_i \|} \int_{\partial C_{i\star}} \vec{\mathcal{F}} \cdot \vec{\nu}_{i\star} d\Gamma_i \right) dt \quad (11)\end{aligned}$$

- Eq. (11) can be re-written as

$$\int_{t^n}^{t^{n+1}} \frac{\partial \overline{W}_i}{\partial t} dt = -\sum_{\star} \frac{\Delta t}{\| C_i \|} \left( \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \left( \int_{\partial C_{i\star}} \vec{\mathcal{F}} \cdot \vec{\nu}_{i\star} d\Gamma_i \right) dt \right)$$

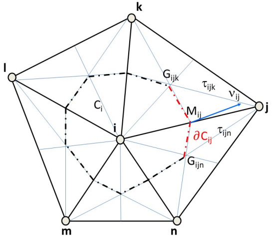
which leads to

$$\boxed{\Delta t \widehat{\left( \frac{\partial \overline{W}}{\partial t} \right)}_i^n = -\sum_{\star} \frac{\Delta t}{\| C_i \|} \widehat{\mathcal{F}}_{i\star}^n} \tag{12}$$

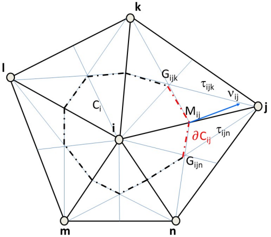
where

$$\overline{W}_i^n \approx \frac{1}{\| C_i \|} \int_{C_i} W_i(x, y, t^n) d\Omega, \quad \widehat{\mathcal{F}}_{i\star}^n \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \left( \int_{\partial C_{i\star}} \vec{\mathcal{F}} \cdot \vec{\nu}_{i\star} d\Gamma_i \right) dt \tag{13}$$

- Eq. (12) is the 2D extension of Eq. (1); and definitions (13) are the 2D extensions of definitions (2)



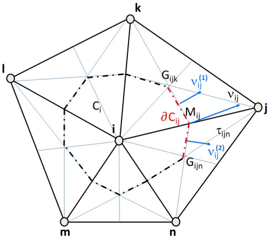

- Hence, $\widehat{\mathcal{F}}^n_{i\star} \approx \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \overbrace{\left(\int_{\partial C_{i\star}} \vec{\mathcal{F}}(W)\cdot\vec{\nu}_{i\star}\, d\Gamma_i\right)}^{I} dt$




- Hence, $\widehat{\mathcal{F}}^n_{i\star} \approx \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \overbrace{\left(\int_{\partial C_{i\star}} \vec{\mathcal{F}}(W)\cdot\vec{\nu}_{i\star}\,d\Gamma_i\right)}^{I} dt$
- A popular approximation of $I$ is $I \approx \vec{\mathcal{F}}\left(W(M_{i\star}, t)\right)\cdot\displaystyle\int_{\partial C_{i\star}} \vec{\nu}_{i\star}\,d\Gamma_i$ (1 quadrature point)

  $\Longrightarrow \widehat{\mathcal{F}}^n_{i\star} \approx \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \left(\vec{\mathcal{F}}\left(W(M_{i\star}, t)\right)\cdot\int_{\partial C_{i\star}} \vec{\nu}_{i\star}\,d\Gamma_i\right) dt$ is constructed in 2D similarly

  to how $\widehat{\mathcal{F}}^n_{x_{i+1/2}} \approx \dfrac{1}{\Delta t}\displaystyle\int_{t^n}^{t^{n+1}} \mathcal{F}_x\left(W(x_{i+1/2}, t)\right) dt$ in constructed in 1D




- In particular, upwind methods based on

$$\widehat{\mathcal{F}}^n_{i\star}\left(W_{Riemann}(0)\right) = \overrightarrow{\mathcal{F}}\left(W_{Riemann}(M_{i\star}, t)\right) \cdot \int_{\partial C_{i\star}} \vec{\nu}_{i\star}\, d\Gamma_i$$
$$= \overrightarrow{\mathcal{F}}\left(W_{Riemann}(M_{i\star}, t)\right) \cdot \left(l^{(1)}\vec{\nu}^{(1)}_{i\star} + l^{(2)}\vec{\nu}^{(2)}_{i\star}\right)$$

  are the 2D extensions of 1D upwind methods based on
  $\widehat{\mathcal{F}}^n_{x_{i+1/2}} = \mathcal{F}_x\left(W_{Riemann}(x_{i+1/2}, t)\right) = \mathcal{F}_x\left(W_{Riemann}(0)\right)$

- Let $\tilde{\vec{\nu}}_{i\star} = \left(l^{(1)}\vec{\nu}^{(1)}_{i\star} + l^{(2)}\vec{\nu}^{(2)}_{i\star}\right)$ and $\tilde{\vec{\nu}}^{\circ}_{ij} = \dfrac{\tilde{\vec{\nu}}_{ij}}{\|\tilde{\vec{\nu}}_{ij}\|_2}$

- Recall that in three dimensions

$$\left(\mathcal{F}_x(W)\ \mathcal{F}_y(W)\ \mathcal{F}_z(W)\right) = \begin{pmatrix} \rho\vec{v}^{\,T} \\ \rho v_x \vec{v}^{\,T} + p\vec{e}_x^{\,T} \\ \rho v_y \vec{v}^{\,T} + p\vec{e}_y^{\,T} \\ \rho v_z \vec{v}^{\,T} + p\vec{e}_z^{\,T} \\ (E+p)\vec{v}^{\,T} \end{pmatrix}$$

$$\left(\mathcal{R}_x(W)\ \mathcal{R}_y(W)\ \mathcal{R}_z(W)\right) = \begin{pmatrix} \vec{0}^{\,T} \\ (\tau \cdot \vec{e}_x)^{\,T} \\ (\tau \cdot \vec{e}_y)^{\,T} \\ (\tau \cdot \vec{e}_z)^{\,T} \\ (\tau \cdot \vec{v} + \kappa \nabla T)^{\,T} \end{pmatrix}$$

$$\vec{e}_x^{\,T} = (1\ 0\ 0), \quad \vec{e}_y^{\,T} = (0\ 1\ 0), \quad \vec{e}_z^{\,T} = (0\ 0\ 1), \quad \vec{0}^{\,T} = (0\ 0\ 0)$$

- Method #1 for computing $\widehat{\mathcal{F}}_{ij}\left(W_{Riemann}(0)\right)$ using Roe's flux: extension of Roe's one-dimensional (1D) approximate Riemann solver to 3D
  - consider the multidimensional flux in the direction of $\tilde{\vec{\nu}}_{ij}^{\circ}$

$$\underbrace{\mathcal{F}_{\tilde{\vec{\nu}}_{ij}^{\circ}}}_{5\times 1} = \underbrace{\vec{\mathcal{F}}}_{5\times 3} \cdot \underbrace{\tilde{\vec{\nu}}_{ij}^{\circ}}_{3\times 1} = \left(\rho v_n \;\; \rho v_x v_n + p\tilde{\nu}_{ij_x}^{\circ} \;\; \rho v_y v_n + p\tilde{\nu}_{ij_y}^{\circ} \; \rho v_z v_n + p\tilde{\nu}_{ij_z}^{\circ} \;\; (E+p)v_n\right)^T$$

where $v_n = \vec{v} \cdot \tilde{\vec{\nu}}_{ij}^{\circ} = v_x\,\tilde{\vec{\nu}}_{ij_x}^{\circ} + v_y\,\tilde{\vec{\nu}}_{ij_y}^{\circ} + v_z\,\tilde{\vec{\nu}}_{ij_z}^{\circ}$

  - Roe's averages are based in this case on the secant approximation defined by

$$\mathcal{F}_{\tilde{\vec{\nu}}_{ij}^{\circ}}(W_j) - \mathcal{F}_{\tilde{\vec{\nu}}_{ij}^{\circ}}(W_i) = \underbrace{A_{\tilde{\vec{\nu}}_{ij}^{\circ}}}_{5\times 5}(W_{ij})(\underbrace{W_j}_{5\times 1} - \underbrace{W_i}_{5\times 1}) \tag{14}$$

where $A_{\tilde{\vec{\nu}}_{ij}^{\circ}} = \dfrac{\partial \mathcal{F}_{\tilde{\vec{\nu}}_{ij}^{\circ}}}{\partial W}$ is given by

$$\begin{bmatrix} 0 & \tilde{\vec{\nu}}_{ij_x}^{\circ} & \tilde{\vec{\nu}}_{ij_y}^{\circ} & \tilde{\vec{\nu}}_{ij_z}^{\circ} & 0 \\ (\gamma-1)q\tilde{\vec{\nu}}_{ij_x}^{\circ} - v_x v_n & v_n - (\gamma-2)v_x\tilde{\vec{\nu}}_{ij_x}^{\circ} & v_x\tilde{\vec{\nu}}_{ij_y}^{\circ} - (\gamma-1)v_y\tilde{\vec{\nu}}_{ij_x}^{\circ} & v_x\tilde{\vec{\nu}}_{ij_z}^{\circ} - (\gamma-1)v_z\tilde{\vec{\nu}}_{ij_x}^{\circ} & (\gamma-1)\tilde{\vec{\nu}}_{ij_x}^{\circ} \\ (\gamma-1)q\tilde{\vec{\nu}}_{ij_y}^{\circ} - v_y v_n & v_y\tilde{\vec{\nu}}_{ij_x}^{\circ} - (\gamma-1)v_x\tilde{\vec{\nu}}_{ij_y}^{\circ} & v_n - (\gamma-2)v_y\tilde{\vec{\nu}}_{ij_y}^{\circ} & v_y\tilde{\vec{\nu}}_{ij_z}^{\circ} - (\gamma-1)v_z\tilde{\vec{\nu}}_{ij_y}^{\circ} & (\gamma-1)\tilde{\vec{\nu}}_{ij_y}^{\circ} \\ (\gamma-1)q\tilde{\vec{\nu}}_{ij_z}^{\circ} - v_z v_n & v_z\tilde{\vec{\nu}}_{ij_x}^{\circ} - (\gamma-1)v_x\tilde{\vec{\nu}}_{ij_z}^{\circ} & v_z\tilde{\vec{\nu}}_{ij_y}^{\circ} - (\gamma-1)v_y\tilde{\vec{\nu}}_{ij_z}^{\circ} & v_n - (\gamma-2)v_z\tilde{\vec{\nu}}_{ij_z}^{\circ} & (\gamma-1)\tilde{\vec{\nu}}_{ij_z}^{\circ} \\ ((\gamma-1)q - h)\,v_n & h\tilde{\vec{\nu}}_{ij_x}^{\circ} - (\gamma-1)v_x v_n & h\tilde{\vec{\nu}}_{ij_y}^{\circ} - (\gamma-1)v_y v_n & h\tilde{\vec{\nu}}_{ij_z}^{\circ} - (\gamma-1)v_z v_n & \gamma v_n \end{bmatrix}$$

$q = \|\vec{v}\|_2^2/2$ and $h = H/\rho = e + q + p/\rho$

- From (14), it follows that

$$
\begin{aligned}
v_{x_{ij}} &= \frac{\sqrt{\rho_i} v_{x_i} + \sqrt{\rho_j} v_{x_j}}{\sqrt{\rho_i} + \sqrt{\rho_j}} \quad v_{y_{ij}} = \frac{\sqrt{\rho_i} v_{y_i} + \sqrt{\rho_j} v_{y_j}}{\sqrt{\rho_i} + \sqrt{\rho_j}} \quad v_{z_{ij}} = \frac{\sqrt{\rho_i} v_{z_i} + \sqrt{\rho_j} v_{z_j}}{\sqrt{\rho_i} + \sqrt{\rho_j}} \\
h_{ij} &= \frac{\frac{H_i}{\sqrt{\rho_i}} + \frac{H_j}{\sqrt{\rho_j}}}{\sqrt{\rho_i} + \sqrt{\rho_j}} = \frac{\sqrt{\rho_i} h_i + \sqrt{\rho_j} h_j}{\sqrt{\rho_i} + \sqrt{\rho_j}} \quad \rho_{ij} = \sqrt{\rho_i \rho_j} \\
\Longrightarrow c_{ij} &= \sqrt{(\gamma - 1)\left(h_{ij} - \frac{1}{2}(v_{x_{ij}}^2 + v_{y_{ij}}^2 + v_{z_{ij}}^2)\right)} \quad \Rightarrow W_{ij}
\end{aligned}
$$

- the eigenvalues of $A_{\tilde{\nu}^{\circ}_{ij}}$ are: $v_{\tilde{\nu}^{\circ}_{ij}} \pm c$, and $v_{\tilde{\nu}^{\circ}_{ij}}$ with multiplicity 3

- Method #1 for computing $\widehat{\mathcal{F}}_{ij}\left(W_{Riemann}(0)\right)$ using Roe's flux
  - extension of Roe's 1D approximate Riemann solver to 3D (continue)
    - formulate Roe's multidimensional approximate Riemann problem:
      $\underbrace{\dfrac{\partial W}{\partial t}}_{5\times 1} + \underbrace{A_{\tilde{\vec{\nu}}^{\,\circ}_{ij}}(W_{ij})}_{5\times 5}\underbrace{\dfrac{\partial W}{\partial s}}_{5\times 1} = 0$, where $s$ is the abscissa along $\vec{ij}$, with uniform initial conditions on an infinite spatial domain, except for a single jump discontinuity
    - compute Roe's flux

$$\boxed{\underbrace{\widehat{\mathcal{F}}_{ij}\left(W_{Riemann}(0)\right)}_{5\times 1} = \left(\frac{1}{2}\left(\underbrace{\mathcal{F}_{\tilde{\vec{\nu}}^{\,\circ}_{ij}}(W_i)}_{5\times 1} + \underbrace{\mathcal{F}_{\tilde{\vec{\nu}}^{\,\circ}_{ij}}(W_j)}_{5\times 1}\right) - \frac{1}{2}\underbrace{\left|A_{\tilde{\vec{\nu}}^{\,\circ}_{ij}}(W_{ij})\right|}_{5\times 5}\left(\underbrace{W_j}_{5\times 1} - \underbrace{W_i}_{5\times 1}\right)\right)\|\tilde{\vec{\nu}}_{ij}\|_2}$$

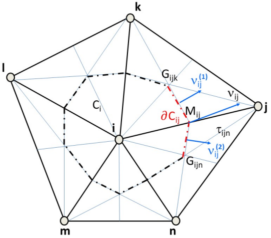
- Method #2 for computing $\widehat{\mathcal{F}}_{ij}\left(W_{Riemann}(0)\right)$ using Roe's flux
  - along the edge $ij$, "compress" the conservative fluid state vector to obtain $\underbrace{W^{comp}}_{3\times 1} = \left(\rho \quad \rho v_n \quad \rho e + \frac{1}{2}\rho v_n^2\right)^T$,
    where $v_n = \vec{v}\cdot\tilde{\vec{\nu}}_{ij}^{\circ}$, $\vec{v}$ denotes the fluid velocity vector, and *comp* stands for "compressed"
  - along the edge $ij$, formulate Roe's 1D approximate Riemann problem using $W_i^{comp}$ and $W_j^{comp}$, and solve this problem to obtain $W_{Riemann}^{comp}(M_{ij}, t) = W_{Riemann}^{comp} = \left(\rho_{Riemann} \quad (\rho v_n)_{Riemann} \quad \left(\rho e + \frac{1}{2}\rho v_n^2\right)_{Riemann}\right)^T$
  - now, given that the fluid velocity vector $\vec{v}$ can be decomposed as $\vec{v} = \underbrace{v_n\tilde{\vec{\nu}}_{ij}^{\circ}}_{normal\ direction} + \underbrace{(\vec{v} - v_n\tilde{\vec{\nu}}_{ij}^{\circ})}_{tangential\ direction}$, it follows that the solution of (Roe's) 1D approximate Riemann problem along the edge $ij$ averages at $M_{ij}$ only the normal component $v_n$ of the velocity vector $\vec{v}$ along $\tilde{\vec{\nu}}_{ij}^{\circ} \Rightarrow v_{n_{Riemann}}$
  - next, "expand" $\vec{v}_{Riemann}$ as

$$\vec{v}_{Riemann}^{exp} = v_{n_{Riemann}}\tilde{\vec{\nu}}_{ij}^{\circ} + (\vec{v} - v_n\tilde{\vec{\nu}}_{ij}^{\circ})_{average} = v_{x_{Riemann}}^{exp}\vec{e}_x + v_{y_{Riemann}}^{exp}\vec{e}_y + v_{z_{Riemann}}^{exp}\vec{e}_z$$

    where *exp* stands for "expanded" and *average* designates a standard (non-Riemann solver) average
  - then, compute the "expanded" vector $\underbrace{W_{Riemann}}_{5\times 1}$

$$W_{Riemann}^{exp} = \Big(\rho_{Riemann} \quad \rho_{Riemann}v_{x_{Riemann}}^{exp} \quad \rho_{Riemann}v_{y_{Riemann}}^{exp} \quad \rho_{Riemann}v_{z_{Riemann}}^{exp} \quad \rho_{Riemann}e_{Riemann} + \frac{1}{2}\rho_{Riemann}\left\|\vec{v}_{Riemann}^{exp}\right\|^2\Big)^T$$

  - finally, compute $\underbrace{\widehat{\mathcal{F}}_{ij}\left(W_{Riemann}(0)\right)}_{5\times 1}$ as $\underbrace{\widehat{\mathcal{F}}_{ij}}_{5\times 1}\left(\underbrace{W_{Riemann}^{exp}(0)}_{5\times 1}\right)$




- Note that

$$\widehat{\mathcal{F}}_{ij}^n = \vec{\mathcal{F}}\left(W_{Riemann}(M_{ij}, t)\right) \cdot \left(l^{(1)}\vec{\nu}_{ij}^{(1)} + l^{(2)}\vec{\nu}_{ij}^{(2)}\right)$$

$$\widehat{\mathcal{F}}_{ji}^n = \vec{\mathcal{F}}\left(W_{Riemann}(M_{ij}, t)\right) \cdot \left(l^{(1)}\vec{\nu}_{ji}^{(1)} + l^{(2)}\vec{\nu}_{ji}^{(2)}\right)$$

- Since $\vec{\nu}_{ji} = -\vec{\nu}_{ij}$, it follows that $\widehat{\mathcal{F}}_{ji}^n = -\widehat{\mathcal{F}}_{ij}^n \Rightarrow \widehat{\mathcal{F}}_{ij}^n + \widehat{\mathcal{F}}_{ji}^n = 0$
$\Longrightarrow \widehat{\mathcal{F}}_{ij}$ is a conservative numerical flux (explain using telescoping property)