AA214B: NUMERICAL METHODS FOR COMPRESSIBLE FLOWS

Hierarchy of Mathematical Models
Outline

1 Preliminaries
2 Nomenclature
3 Equations Hierarchy
4 Navier-Stokes Equations
5 Euler Equations
6 Full Potential Equation
7 Linearized Small-Perturbation Potential Equation
Throughout this chapter — and as a matter of fact, this entire course — the flow is assumed to be compressible and the fluid is assumed to be a perfect gas.

The Equation Of State (EOS) of a perfect gas is

\[ p = \rho RT \Rightarrow p = p(\rho, T) \text{ or } T = T(p, \rho) \]

where \( p \) denotes the gas pressure, \( \rho \) its density, \( T \) its temperature, and \( R \) is the specific gas constant (in SI units, \( R = 287.058 \text{ m}^2/\text{s}^2/\text{K} \)).

From the thermodynamics of a perfect (calorically perfect) gas, it follows that

\[ e = C_v T = \frac{R}{\gamma - 1} T \Rightarrow e = e(T) \text{ or } T = T(e) \]

where \( C_v \) denotes the heat capacity at constant volume of the gas and \( \gamma \) denotes the ratio of its heat capacities (\( C_p/C_v \), where \( C_p \) denotes the heat capacity at constant pressure).
### Nomenclature

<table>
<thead>
<tr>
<th>Character</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>$\gamma$</td>
<td>heat capacity ratio</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature</td>
</tr>
<tr>
<td>$T^T$</td>
<td>transpose</td>
</tr>
<tr>
<td>$\vec{v}$</td>
<td>velocity vector</td>
</tr>
<tr>
<td>$e$</td>
<td>internal energy per unit mass</td>
</tr>
<tr>
<td>$E$</td>
<td>total energy per unit volume</td>
</tr>
<tr>
<td>$H$</td>
<td>total enthalpy per unit volume</td>
</tr>
<tr>
<td>$\tau$</td>
<td>(deviatoric) viscous stress tensor/matrix</td>
</tr>
<tr>
<td>$\mu$</td>
<td>(laminar) dynamic (absolute) molecular viscosity</td>
</tr>
<tr>
<td>$\nu$</td>
<td>(laminar) kinematic molecular viscosity, $\mu/\rho$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>$I$</td>
<td>identity tensor/matrix</td>
</tr>
<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$Re$</td>
<td>Reynolds number, $\rho|\vec{v}|L_c/\mu = |\vec{v}|L_c/\nu$</td>
</tr>
<tr>
<td>$L_c$</td>
<td>characteristic length</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$\vec{t}$</td>
<td>unitary axis for the time dimension</td>
</tr>
<tr>
<td>subscript $t$</td>
<td>turbulence eddy quantity</td>
</tr>
<tr>
<td>subscripts $x$, $y$, $z$ (or occasionally $i$, $j$)</td>
<td>components in the $x$, $y$, and $z$ directions</td>
</tr>
<tr>
<td>$\vec{e}_x$ (or $\vec{e}_y$, or $\vec{e}_z$)</td>
<td>unitary axis in the $x$ ($y$, or $z$) direction</td>
</tr>
<tr>
<td>subscript $\infty$</td>
<td>free-stream quantity</td>
</tr>
</tbody>
</table>
Equations Hierarchy

- Navier-Stokes equations
  - Reynolds-averaged Navier-Stokes equations (RANS)
  - large eddy simulation (LES)
- Euler equations
- Full potential equation
- Linearized Small-Perturbation Potential Equation
  - subsonic and supersonic regimes
  - transonic regime
Navier-Stokes Equations

Assumptions

- The fluid of interest is a continuum
- The fluid of interest is not moving at relativistic velocities
- The fluid stress is the sum of a pressure term and a diffusing viscous term proportional to the gradient of the velocity

\[
\sigma = -pI + \tau = -pI + 2\mu \left[ \frac{1}{2} (\nabla + \nabla^T) \vec{v} - \frac{1}{3} (\vec{\nabla} \cdot \vec{v}) I \right]
\]

where

\[
\vec{v} = (v_x \ v_y \ v_z)^T, \quad \nabla \vec{v} = \left( \begin{array}{ccc}
\frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\
\frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\
\frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z}
\end{array} \right)
\]

\[
\vec{\nabla} = \left( \frac{\partial}{\partial x} \ \frac{\partial}{\partial y} \ \frac{\partial}{\partial z} \right)^T
\]
Navier-Stokes Equations

- Eulerian setting
- Dimensional form

\[
\frac{\partial W}{\partial t} + \nabla \cdot \vec{F}(W) = \nabla \cdot \vec{R}(W)
\]

\[
W = (\rho, \rho \vec{v}^T, E)^T
\]

\[
\vec{F}(W) = (\mathcal{F}_x^T(W), \mathcal{F}_y^T(W), \mathcal{F}_z^T(W))^T
\]

\[
\vec{R}(W) = (\mathcal{R}_x^T(W), \mathcal{R}_y^T(W), \mathcal{R}_z^T(W))^T
\]

- One continuity equation, three momentum equations and one energy equation \(\Rightarrow\) five equations
- Closed system \((\rho, \vec{v}, e, T = T(e), p = p(\rho, T))\)
Navier-Stokes Equations

\[
(\mathcal{F}_x(W) \mathcal{F}_y(W) \mathcal{F}_z(W)) = \begin{pmatrix}
\rho \vec{v}^T \\
\rho v_x \vec{v}^T + p \vec{e}_x^T \\
\rho v_y \vec{v}^T + p \vec{e}_y^T \\
\rho v_z \vec{v}^T + p \vec{e}_z^T \\
(E + p)\vec{v}^T
\end{pmatrix}
\]

\[
(\mathcal{R}_x(W) \mathcal{R}_y(W) \mathcal{R}_z(W)) = \begin{pmatrix}
\vec{0}^T \\
(\tau \cdot \vec{e}_x)^T \\
(\tau \cdot \vec{e}_y)^T \\
(\tau \cdot \vec{e}_z)^T \\
(\tau \cdot \vec{v} + \kappa \nabla T)^T
\end{pmatrix}
\]

\[
\vec{e}_x^T = (1 0 0), \quad \vec{e}_y^T = (0 1 0), \quad \vec{e}_z^T = (0 0 1), \quad \vec{0}^T = (0 0 0)
\]
The Navier-Stokes equations are named after Claude-Louis Navier (French engineer) and George Gabriel Stokes (Irish mathematician and physicist).

They are generally accepted as an adequate description for aerodynamic flows at standard temperatures and pressures.

Because of mesh resolution requirements however, they are practically useful “as is” only for laminar viscous flows, and low Reynolds number turbulent viscous flows.

Today, mathematicians have not yet proven that in three dimensions solutions always exist, or that if they do exist, then they are smooth.

The above problem is considered one of the seven most important open problems in mathematics: the Clay Mathematics Institute offers $1,000,000 prize for a solution or a counter-example.
Consider the flow graphically depicted in the figure below:

- an oblique shock wave impinges on a boundary layer
- the adverse pressure gradient \((dP/ds > 0)\) produced by the shock can propagate upstream through the subsonic part of the boundary layer and, if sufficiently strong, can separate the flow forming a circulation within a separation bubble
- the boundary layer thickens near the incident shock wave and then necks down where the flow reattaches to the wall, generating two sets of compression waves bounding a rarefaction fan, which eventually form the reflected shockwave
Consider the flow graphically depicted in the figure below (continue)

- the Navier-Stokes equations describe well this problem
- but at Reynolds numbers of interest to aerodynamics, their practical discretization cannot capture adequately the inviscid-viscous interactions described above
- today, this problem and most turbulent viscous flow problems of interest to aerodynamics (high $R_e$) require turbulence modeling to represent scales of the flow that are not resolved by practical grids
- the Reynolds-Averaged Navier-Stokes (RANS) equations are one approach for modeling a class of turbulent flows
Navier-Stokes Equations

Reynolds-Averaged Navier-Stokes Equations

Approach

- The RANS equations are time-averaged equations of motion for fluid flow

\[ W \rightarrow \overline{W} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} W \, dt \]

- The main idea is to decompose an instantaneous quantity into time-averaged and fluctuating components

\[ W = \overline{W} + W' \]

- The substitution of this decomposition (first proposed by the Irish engineer Osborne Reynolds) into the Navier-Stokes equations, the averaging of the resulting equations and the injection in them of various approximations based on knowledge of the properties of flow turbulence lead to a closure problem induced by the arising non-linear Reynolds stress term

\[ R_{ij} = -\overline{v'_i v'_j} \]

- Additional modeling of \( R_{ij} \) is therefore required to close the RANS equations, which has led to many different turbulence models
Many of these turbulence models are based on:

- the Boussinesq assumption $R_{ij} = R_{ij}(\nu_t)$ — that is, on assuming that the additional turbulence stresses are given by augmenting the laminar molecular viscosity $\mu$ with a (turbulence) eddy viscosity $\mu_t$ (which leads to augmenting the laminar kinematic molecular viscosity $\nu$ with a (turbulence) kinematic eddy viscosity $\nu_t$)
- a parameterization $\nu_t = \nu_t(\chi_1, \cdots, \chi_m)$
- additional transport equations similar to the Navier-Stokes equations for modeling the dynamics of the parameters $\chi_1, \cdots, \chi_m$
In any case, whatever RANS turbulence model is chosen, $W$ is augmented by the $m$ parameters of the chosen turbulence model (usually, $m = 1$ or $2$)

$$W_{aug} \leftarrow (\rho \, \rho \vec{v}^T \, E \, \chi_1 \, \cdots \, \chi_m)^T$$

and the standard Navier-Stokes equations are transformed into the RANS equations which have the same form but are written in terms of $\overline{W}$ and feature a source term $S$ that is turbulence model dependent

$$\frac{\partial \overline{W}}{\partial t} + \nabla \cdot \overline{\mathcal{F}(\overline{W})} = \nabla \cdot \overline{\mathcal{R}(\overline{W})} + S(\overline{W} \, \chi_1 \, \cdots \, \chi_m)$$
The fluid of interest is inviscid (or the viscous effects are negligible)

There are no thermal conduction effects (or they are negligible)

\[ \Rightarrow \mu = \kappa = 0 \]
Euler Equations

- Eulerian setting
- Dimensional form

\[
\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{F}(W) = (0 \ 0 \ 0)^T
\]

- One continuity equation, three momentum equations and one energy equation
The Euler equations are named after Leonhard Euler (Swiss mathematician and physicist).

Historically, only the continuity and momentum equations have been derived by Euler around 1757, and the resulting system of equations was underdetermined except in the case of an incompressible fluid.

The energy equation was contributed by Pierre-Simon Laplace (French mathematician and astronomer) in 1816 who referred to it as the adiabatic condition.

The Euler equations are nonlinear hyperbolic equations and their general solutions are waves.

Waves described by the Euler equations can break and give rise to shock waves.
Euler Equations

Some Noteworthy Facts

- Mathematically, this is a nonlinear effect and represents the solution becoming multi-valued.
- Physically, this represents a breakdown of the assumptions that led to the formulation of the differential equations.
- Weak solutions are then formulated by working with jumps of flow quantities (density, velocity, pressure, entropy) using the Rankine-Hugoniot shock conditions.
- In real flows, these discontinuities are smoothed out by viscosity.
- Shock waves with Mach numbers just ahead of the shock greater than 1.3 are usually strong enough to cause boundary layer separation and therefore require using the Navier-Stokes equations.
- Shock waves described by the Navier-Stokes equations would represent a jump as a smooth transition — of length equal to a few mean free paths — between the same values given by the Euler equations.
Flow is isentropic

\[ \Rightarrow \text{flow containing weak (or no) shocks and with peak Mach numbers below 1.3} \]

\[ \vec{v} = \nabla \Phi, \text{where } \Phi \text{ is referred to as the velocity potential} \]

\[ \Rightarrow \begin{cases} \nabla \times \nabla \Phi = 0 \\ \nabla \times \vec{v} = 0 \end{cases} \]

\[ \Rightarrow \text{irrotational flow} \]

\[ \Rightarrow \text{not suitable in flow regions where vorticity is known to be important (for example, wakes and boundary layers)} \]
Steady flow (but the potential flow approach equally applies to unsteady flows)

from the isentropic flow conditions \( (p/\rho^\gamma = \text{cst}) \) and \( \vec{v} = \nabla \Phi \), it follows that

\[
T = T_\infty \left[ 1 - \frac{\gamma - 1}{2} M_\infty^2 \left( \frac{\partial \Phi^2}{\partial x} + \frac{\partial \Phi^2}{\partial y} + \frac{\partial \Phi^2}{\partial z} \right) \right]^{\frac{\gamma}{\gamma - 1}}
\]

\[
p = p_\infty \left[ 1 - \frac{\gamma - 1}{2} M_\infty^2 \left( \frac{\partial \Phi^2}{\partial x} + \frac{\partial \Phi^2}{\partial y} + \frac{\partial \Phi^2}{\partial z} \right) \right]^{\frac{\gamma}{\gamma - 1}}
\]

\[
\rho = \rho_\infty \left[ 1 - \frac{\gamma - 1}{2} M_\infty^2 \left( \frac{\partial \Phi^2}{\partial x} + \frac{\partial \Phi^2}{\partial y} + \frac{\partial \Phi^2}{\partial z} \right) \right]^{\frac{1}{\gamma - 1}}
\]
Steady flow (continue)

- non-conservative form (see later)

\[
\left(1 - M_x^2\right) \frac{\partial^2 \Phi}{\partial x^2} + \left(1 - M_y^2\right) \frac{\partial^2 \Phi}{\partial y^2} + \left(1 - M_z^2\right) \frac{\partial^2 \Phi}{\partial z^2} \\
- 2M_x M_y \frac{\partial^2 \Phi}{\partial x \partial y} - 2M_y M_z \frac{\partial^2 \Phi}{\partial y \partial z} - 2M_z M_x \frac{\partial^2 \Phi}{\partial z \partial x} = 0
\]

where

\[
M_x = \frac{1}{c} \frac{\partial \Phi}{\partial x}, \quad M_y = \frac{1}{c} \frac{\partial \Phi}{\partial y}, \quad M_z = \frac{1}{c} \frac{\partial \Phi}{\partial z}
\]

are the local Mach components and

\[
c = \sqrt{\frac{\gamma p}{\rho}}
\]

is the local speed of sound
Steady flow (continue)

conservative form (see later)

\[
\frac{\partial (\rho \frac{\partial \phi}{\partial x})}{\partial x} + \frac{\partial (\rho \frac{\partial \phi}{\partial y})}{\partial y} + \frac{\partial (\rho \frac{\partial \phi}{\partial z})}{\partial z} = 0
\]

where

\[
\rho = \rho_\infty \left[ 1 - \frac{\gamma - 1}{2} M_\infty^2 \left( \frac{\partial \phi}{\partial x}^2 + \frac{\partial \phi}{\partial y}^2 + \frac{\partial \phi}{\partial z}^2 \right) \right]^{\frac{1}{\gamma - 1}}
\]
Linearized Small-Perturbation Potential Equation

Transonic Regime

Additional Assumptions

- Uniform free-stream flow *near Mach one* (say $0.8 \leq M_\infty \leq 1.2$)
- Thin body and small angle of attack

Flow slightly perturbed from the uniform free-stream condition

$$\vec{v} = \|\vec{v}_\infty\|\hat{e}_x + \nabla \phi$$

where $\phi$ – *which is not to be confused with* $\Phi$ – is referred to as the small-perturbation velocity potential

$$v_x = \|\vec{v}_\infty\| + \frac{\partial \phi}{\partial x}, \quad v_y = \frac{\partial \phi}{\partial y}, \quad v_z = \frac{\partial \phi}{\partial z}$$

$$\left| \frac{\partial \phi}{\partial x} \right| < < \|\vec{v}_\infty\|, \quad \left| \frac{\partial \phi}{\partial y} \right| < < \|\vec{v}_\infty\|, \quad \left| \frac{\partial \phi}{\partial z} \right| < < \|\vec{v}_\infty\|$$
Linearized Small-Perturbation Potential Equation

Transonic Regime

Small-Perturbation Potential Equation

- Steady flow (but the small-perturbation velocity potential approach equally applies to unsteady flows)

\[
\left(1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \|\vec{v}_\infty\| \right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
\]

- The leading term of the above equation cannot be simplified because \(0.8 \leq M_\infty \leq 1.2\) (transonic regime)

- The velocity vector is obtained from \( \vec{v} = \|\vec{v}_\infty\| \vec{e}_x + \nabla \phi \) and the pressure and density from the first-order expansion of the second and third isentropic flow conditions as in the previous case

- The temperature is obtained from \( T = T(p, \rho) \) and the total energy per unit mass from \( e = e(T) \)
Linearized Small-Perturbation Potential Equation

Transonic Regime

Some Noteworthy Facts

- In the transonic regime, the small-perturbation potential equation is also known as the “transonic small-disturbance equation”
- It is a nonlinear equation of the mixed type
  - elliptic if
    \[
    (1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \frac{\partial \phi}{\partial x} \|^\mathbf{V}_\infty\|) > 0
    \]
  - hyperbolic if
    \[
    (1 - M_\infty^2 - (\gamma + 1)M_\infty^2 \frac{\partial \phi}{\partial x} \|^\mathbf{V}_\infty\|) < 0
    \]
- Linearized Small-Perturbation Potential Equation
- Subsonic or Supersonic Regime

Additional Assumptions (revisited)

- Uniform free-stream flow near Mach one (say $0.8 \leq M_\infty \leq 1.2$)
- Thin body and small angle of attack
- Flow slightly perturbed from the uniform free-stream condition

$$\vec{v} = ||\vec{v}_\infty|| \vec{e}_x + \nabla \phi$$

where $\phi$ is referred to as the small-perturbation velocity potential

$$v_x = ||\vec{v}_\infty|| + \frac{\partial \phi}{\partial x}, \quad v_y = \frac{\partial \phi}{\partial y}, \quad v_z = \frac{\partial \phi}{\partial z}$$

$$\left| \frac{\partial \phi}{\partial x} \right| << ||\vec{v}_\infty||, \quad \left| \frac{\partial \phi}{\partial y} \right| << ||\vec{v}_\infty||, \quad \left| \frac{\partial \phi}{\partial z} \right| << ||\vec{v}_\infty||$$
Linearized Small-Perturbation Potential Equation

Subsonic or Supersonic Regime

Linearized Equation

Steady flow

\[(1 - M^2_{\infty}) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0\]
Linearized Small-Perturbation Potential Equation

Subsonic or Supersonic Regime

Linearized Equation

Steady flow (continue)

the velocity vector is obtained from \( \vec{v} = \| \vec{v}_{\infty} \| \vec{e}_x + \nabla \phi \) and the pressure and density from the first-order expansion of the second and third isentropic flow conditions as follows

\[
\begin{align*}
p &= p_{\infty} \left( 1 - \gamma M_{\infty}^2 \frac{\partial \phi}{\partial x} \right) \\
\rho &= \rho_{\infty} \left( 1 - M_{\infty}^2 \frac{\partial \phi}{\partial x} \right)
\end{align*}
\]

the temperature is obtained from \( T = T(p, \rho) \) and the total energy per unit mass from \( e = e(T) \)
The linearized small-perturbation potential equation

\[(1 - M_{\infty}^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0\]

is much easier to solve than the nonlinear transonic small-perturbation potential equation, or the nonlinear full potential equation: it can be recast into Laplace’s equation using the simple coordinate stretching in the \(\tilde{e}_x\) direction

\[\tilde{x} = \frac{x}{\sqrt{(1 - M_{\infty}^2)}} \quad \text{(subsonic regime)}\]