Conservation and Integral Forms and Discontinuities
Outline

1. Conservation Law Form

2. Integral Form

3. Relations at Discontinuities
   - Stationary Discontinuities
   - Moving Discontinuities
   - Shock Waves
Definition: an equation (or set of equations) is said to be in conservation law form — or more precisely, in divergence form — if it is written as follows

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{F}(W) = S$$

If $S = 0$, the equation is said to be in *strong* conservation law form.

For example, many of the equations presented in Chapter 2 are written in strong conservation form.
Recall that the transonic small disturbance equation discussed in Chapter 2 was written as

\[
\begin{align*}
\left(1 - M^2 - (\gamma + 1) M^2 \frac{\partial \phi}{\partial x} \right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
\end{align*}
\]

This equation can be re-written in strong conservation form using

\[
\vec{F} = \left( \begin{array}{c}
(1 - M^2) \frac{\partial \phi}{\partial x} - (\gamma + 1) M^2 \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} \\
2 \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial z}
\end{array} \right)^T
\]
Integral Form

The integration over an arbitrary stationary volume $\Omega$ enclosed by the surface $\partial \Omega$ of a generic equation written in conservation form can be written as

$$
\int_{\Omega} \frac{\partial W}{\partial t} d\Omega + \int_{\Omega} \nabla \cdot \mathbf{F}(W) d\Omega = \int_{\Omega} S d\Omega
$$

Dividing by $\Omega$ and using the divergence (Gauss, or Ostrogradsky) theorem leads to

$$
\frac{\partial \overline{W}}{\partial t} + \frac{1}{\Omega} \int_{\partial \Omega} \mathbf{F} \cdot d\mathbf{\partial\Omega} = \frac{1}{\Omega} \int_{\Omega} S d\Omega \quad (1)
$$

where $\overline{W} = \frac{1}{\Omega} \int_{\Omega} W d\Omega$

The above equation represents the rate of change of the mean value of $W$ over the volume $\Omega$ caused by the net flux of $\mathbf{F}$ crossing the surface $\partial \Omega$ and the volume source $S$. 
Relations at Discontinuities

Stationary Discontinuities

Let \( f(x, y, z) = 0 \) represent a surface located at a possible discontinuity within the fluid.

Assume that the flow is continuous within each of the two subdomains shown in the figure above.

Assume also that \( \Omega \) is placed symmetrically about an arbitrary point of the surface and is allowed to shrink to zero.
Now, for the case of a steady flow, Eq. (1) becomes

$$\int_{\partial \Omega} \vec{F} \cdot d\vec{\Omega} = \int_{\Omega} S d\Omega$$

As $\Omega \to 0$, the term on the right goes to zero at a faster rate than the surface integration term ($h^3$ vs $h^2$, where $h \approx \Omega^{\frac{1}{3}} = \partial \Omega^{\frac{1}{2}}$)

It follows that for an infinitesimal $\Omega$

$$\int_{\partial \Omega} \vec{F} \cdot d\vec{\Omega} = 0$$
Relations at Discontinuities

Stationary Discontinuities

\[ 0 = \int_{\partial \Omega} \vec{F} \cdot d\partial\Omega = \sum_{i=1}^{6} \vec{F}_i \cdot \vec{n}_i \, d\partial\Omega_i \]

- Since the flow is continuous within each of subdomain \( D_1 \) and \( D_2 \)
  \[ \vec{F}_3 \cdot \vec{n}_3 \, d\partial\Omega_3 + \vec{F}_4 \cdot \vec{n}_4 \, d\partial\Omega_4 = 0 \quad \text{and} \quad \vec{F}_5 \cdot \vec{n}_5 \, d\partial\Omega_5 + \vec{F}_6 \cdot \vec{n}_6 \, d\partial\Omega_6 = 0 \]

\[ \Rightarrow \int_{\partial \Omega} \vec{F} \cdot d\partial\Omega = \vec{F}_1 \cdot \vec{n}_1 \, d\partial\Omega_1 + \vec{F}_2 \cdot \vec{n}_2 \, d\partial\Omega_2 = 0 \]

\[ \Rightarrow (\vec{F}_1 - \vec{F}_2) \cdot \vec{n}_1 = 0 \quad \text{with} \quad \vec{n}_1 = \frac{\nabla f}{\|\nabla f\|} \]

\[ \Rightarrow (\vec{F}_1 - \vec{F}_2) \cdot \nabla f = 0 \]
Relations at Discontinuities

Stationary Discontinuities

The jump across the surface \( f \) is defined as

\[
(\mathbf{F}_1 - \mathbf{F}_2) = \left[ \mathbf{F} \right]_1^2
\]

\[
\Rightarrow \left[ \mathbf{F} \right]_1^2 \cdot \nabla f = 0
\]

which can also be written as

\[
\left[ \mathbf{F}_x \right]_1^2 \frac{\partial f}{\partial x} + \left[ \mathbf{F}_y \right]_1^2 \frac{\partial f}{\partial y} + \left[ \mathbf{F}_z \right]_1^2 \frac{\partial f}{\partial z} = 0
\]

If \( \mathbf{F} \) is the flux vector of the Euler equations, the above steady jump relations at surface \( f(x,y,z) = 0 \) represent the Rankine-Hugoniot relations across a shock wave.
Consider now the surface \( f(x, y, z, t) = 0 \) representing a dynamic surface located at a possible moving discontinuity within a volume \( \Omega \) of a fluid.

Let
\[
\vec{\nabla}^* = \left( \frac{\partial}{\partial t} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right)^T
\]

and
\[
\vec{F}^*(W) = \left( W^T \mathcal{F}_x^T(W) \mathcal{F}_y^T(W) \mathcal{F}_z^T(W) \right)^T
\]

Using the above notation which includes time as a dimension, the previous discussion on stationary discontinuities can be generalized to obtain the following unsteady jump relations for moving discontinuities
\[
\left[ \mathcal{F}^* \right]_1^2 \cdot \nabla^* f = \left[ W \right]_1^2 \frac{\partial f}{\partial t} + \left[ \mathcal{F}_x \right]_1^2 \frac{\partial f}{\partial x} + \left[ \mathcal{F}_y \right]_1^2 \frac{\partial f}{\partial y} + \left[ \mathcal{F}_z \right]_1^2 \frac{\partial f}{\partial z} = 0
\]
Relations at Discontinuities

Shock Waves

Simple Wave Equation

Consider the model hyperbolic equation with constant wave speed $c$ and scalar variable $u$

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

Consider first the case of a stationary discontinuity surface of the form $f(x) = x - x_0 = 0$
Relations at Discontinuities

Shock Waves

Simple Wave Equation

Consider the model hyperbolic equation with constant wave speed $c$ and scalar variable $u$

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

Consider first the case of a stationary discontinuity surface of the form

$$f(x) = x - x_0 = 0$$

In this case, $\vec{F}^* = (u \ c u)^T$ and $\vec{n}_1 = \frac{\nabla^* f}{\|\nabla^* f\|} = (0 \ 1)^T$, and therefore the jump relation is

$$\left\langle [\vec{F}^*]_1 \right\rangle \cdot \nabla^* f = c(u_1 - u_2) = c\|u\|_1^2 = 0 \iff u_1 = u_2$$

This implies that no jump is possible, which is not surprising for a linear equation.
Relations at Discontinuities

Shock Waves

Simple Wave Equation

Consider the model hyperbolic equation with constant wave speed $c$ and scalar variable $u$ (continue)

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

Consider next the case of a discontinuity surface moving at constant speed $w$, $f(x, t) = x - x_0 - w(t - t_0) = 0$
Consider the model hyperbolic equation with constant wave speed $c$ and scalar variable $u$

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0
\]

Consider next the case of a discontinuity surface moving at constant speed $w$, $f(x, t) = x - x_0 - w(t - t^0) = 0$

\[
\hat{n}_1 = \nabla \star f = \frac{1}{\sqrt{1 + w^2}} (-w \ 1)^T, \text{ and therefore the jump relation is}
\]

\[
\left[ \mathcal{F}^\star \right]_1^2 \cdot \nabla \star f = (-w(u_1 - u_2) + c(u_1 - u_2)) = 0
\]

\[
\iff (c - w)(u_1 - u_2) = (c - w)[u]_1^2 = 0
\]

this implies that any jump is possible, as long as it moves at the speed $c$
Recall the linearized small-perturbation potential equation modeling a two-dimensional steady flow in either the subsonic or supersonic regime

\[(1 - M_\infty^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\]

For \(M_\infty > 1.2\), this equation is hyperbolic and can describe purely supersonic flows with small perturbations about a supersonic free-stream with velocity \(\vec{v} = \|\vec{v}_\infty\|\vec{e}_x\) (recall also that in this case, \(\vec{v} = (\|\vec{v}_\infty\| + \frac{\partial \phi}{\partial x})\vec{e}_x + \frac{\partial \phi}{\partial y}\vec{e}_y\)

Consider as a possible stationary discontinuity surface \(f(x, y) = a(x - x_0) - b(y - y_0)\) where \(a\) and \(b\) are constants
Relations at Discontinuities

Shock Waves

Linearized Small-Perturbation Potential Equation in the Supersonic Regime

In this case, \( \vec{F} = \left( (1 - M_\infty^2) \frac{\partial \phi}{\partial x} \ \frac{\partial \phi}{\partial y} \right)^T \) and

\[
\vec{n}_1 = \frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{a^2 + b^2}} (a - b)^T,
\]
and therefore the jump relation is

\[
\left[ F \right]_1 \cdot \nabla f = a (1 - M_\infty^2) \left( \frac{\partial \phi}{\partial x} \right)_1 - \frac{\partial \phi}{\partial x} \left( \frac{\partial \phi}{\partial y} \right)_1 = 0
\]

\[
\Rightarrow (1 - M_\infty^2) a \left[ \frac{\partial \phi}{\partial x} \right]_1^2 = b \left[ \frac{\partial \phi}{\partial y} \right]_1^2
\]

- if \( a = 0 \) or \( b = 0 \), there are no permissible jumps
- a small perturbation jump can occur across a Mach line \( f(x, y) \) with angle \( \theta \) in which case the slope of the discontinuity surface is

\[
\frac{a}{b} = \tan \theta = \frac{1}{\sqrt{M_\infty^2 - 1}} \quad \text{(recall that the Mach angle is given by)}
\]

\[
\sin \theta = \frac{1}{M_\infty}
\]

along this Mach line, the jump relation simplifies to

\[
- \sqrt{M_\infty^2 - 1} \left[ \frac{\partial \phi}{\partial x} \right]_1^2 = \left[ \frac{\partial \phi}{\partial y} \right]_1^2
\]
Relations at Discontinuities

Shock Waves

Linearized Small-Perturbation Potential Equation in the Supersonic Regime

(Continue)

along the Mach line with the slope \( \frac{a}{b} = \tan \theta = \frac{1}{\sqrt{M_{\infty}^2 - 1}} \) where

\[-\sqrt{M_{\infty}^2 - 1} \left[ \frac{\partial \phi}{\partial x} \right]_1^2 = \left[ \frac{\partial \phi}{\partial y} \right]_1^2, \]

the flow can turn through an angle \( \delta \) (small value because small perturbation) from the free-stream direction (see above figure where \( \nabla \phi |_{x=1} = 0 \)) such that

\[\frac{\partial \phi}{\partial x} |_2 = \frac{-\tan \delta}{\tan \delta + \sqrt{M_{\infty}^2 - 1}} \| \vec{v}_{\infty} \| \approx \frac{-\tan \delta}{\sqrt{M_{\infty}^2 - 1}} \| \vec{v}_{\infty} \| \]

and

\[\frac{\partial \phi}{\partial y} |_2 = \frac{\tan \delta \sqrt{M_{\infty}^2 - 1}}{\tan \delta + \sqrt{M_{\infty}^2 - 1}} \| \vec{v}_{\infty} \| \approx \tan \delta \| \vec{v}_{\infty} \| \]