Linearization and Characteristic Relations
Outline

1. Non Conservation Form and Jacobians
2. Linearization Around a Localized Flow Condition
3. Hyperbolic Requirement
4. Characteristic Relations
5. Application to the One-Dimensional Euler Equations
6. Boundary Conditions
Consider again an equation written in conservation law form

\[
\frac{\partial W}{\partial t} + \nabla \cdot \overrightarrow{F}(W) = S
\]

where

\[
\overrightarrow{F}(W) = (\mathcal{F}_x^T(W) \mathcal{F}_y^T(W) \mathcal{F}_z^T(W))^T
\]

In three dimensions, the above equation can be re-written as follows

\[
\frac{\partial W}{\partial t} + \frac{\partial \mathcal{F}_x(W)}{\partial W} \frac{\partial W}{\partial x} + \frac{\partial \mathcal{F}_y(W)}{\partial W} \frac{\partial W}{\partial y} + \frac{\partial \mathcal{F}_z(W)}{\partial W} \frac{\partial W}{\partial z} = S
\]

or

\[
\frac{\partial W}{\partial t} + A \frac{\partial W}{\partial x} + B \frac{\partial W}{\partial y} + C \frac{\partial W}{\partial z} = S \tag{1}
\]

where

\[
A = \frac{\partial \mathcal{F}_x(W)}{\partial W}, \quad B = \frac{\partial \mathcal{F}_y(W)}{\partial W}, \quad C = \frac{\partial \mathcal{F}_z(W)}{\partial W}
\]

are called the Jacobians of \( \mathcal{F}_x, \mathcal{F}_y, \) and \( \mathcal{F}_z \) with respect to \( W \), respectively.
For example for the Euler equations in two dimensions, each of the Jacobians is a $4 \times 4$ matrix.

In general for $m$-dimensional vectors $W = (W_1, \cdots, W_m)^T$ and $F = (F_1, \cdots, F_m)^T$,

\[
\frac{\partial F}{\partial W} = \begin{pmatrix}
\frac{\partial F_1}{\partial W_1} & \cdots & \frac{\partial F_1}{\partial W_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_m}{\partial W_1} & \cdots & \frac{\partial F_m}{\partial W_m}
\end{pmatrix}
\]
If $W = W(V)$, Eq. (1) can be transformed as follows

$$\frac{\partial V}{\partial t} + A' \frac{\partial V}{\partial x} + B' \frac{\partial V}{\partial y} + C' \frac{\partial V}{\partial z} = S'$$  \hspace{1cm} (2)

where

$$A' = T^{-1}AT, \quad B' = T^{-1}BT, \quad C' = T^{-1}CT, \quad S' = T^{-1}S$$

and

$$T = \frac{\partial W}{\partial V}$$

represents the Jacobian of $W$ with respect to $V$.

The Jacobians with respect to $W$ are then given by

$$\frac{\partial}{\partial W} = \frac{\partial}{\partial V} \frac{\partial V}{\partial W} = \frac{\partial}{\partial V} T^{-1}$$
Definition: $G(W_1, \cdots, W_m)$ is said to be a homogeneous function of degree $p$, where $p$ is an integer, if

$$\forall s > 0 \quad G(sW_1, \cdots, sW_m) = s^p G(W_1, \cdots, W_m)$$

Example: A linear function is a homogeneous function of degree 1

$$\forall s > 0, \quad G(sW_1, \cdots, sW_m) = s G(W_1, \cdots, W_m)$$

Exercise: show that for a perfect gas, the fluxes $F_x, F_y,$ and $F_z$ of the Euler equations are homogeneous functions (of $W$) of degree 1

A homogeneous function of degree $p$ has **scale invariance** — that is, it has some properties that remain constant when looking at them either at different length- or time-scales
Theorem 1 (Euler’s theorem): A differentiable function $G(W_1, \cdots, W_m)$ is a homogeneous function of degree $p$ if and only if

$$\sum_{i=1}^{m} W_i \frac{\partial G}{\partial W_i}(W_1, \cdots, W_m) = pG(W_1, \cdots, W_m)$$

Proof: ($\Rightarrow$) differentiate definition with respect to $s$ and set $s = 1$

($\Leftarrow$) define $g(s) = G(sW_1, \cdots, sW_m) - s^p G(W_1, \cdots, W_m)$, differentiate $g(s)$ to get an ordinary differential equation in $g(s)$, note that $g(1) = 0$, and conclude that $g(s) = cst = 0$
Theorem 2: If $G(W_1, \cdots, W_m)$ is differentiable and homogeneous of degree $p$, then each of its partial derivatives $\frac{\partial G}{\partial W_i}$ (for $i = 1, \cdots, m$) is a homogeneous function of degree $p - 1$

$$\forall s \neq 0, \quad \frac{\partial G}{\partial W_i}(sW_1, \cdots, sW_m) = s^{p-1} \frac{\partial G}{\partial W_i}(W_1, \cdots, W_m)$$

Proof: Straightforward (differentiate both sides of the definition with respect to $W_i$)
Linearization can be either physically relevant (small perturbations), convenient for analysis, or useful for constructing a linear model problem — in either case, it leads to a linear problem.

For the purpose of constructing a linear model version of Eq. (1), the coefficient matrices $A$, $B$, and $C$ of this equation are often simply “frozen” to their values at a local flow condition designated by the subscript $o$ and represented by the fluid state vector $W_o$, which leads to

$$\frac{\partial W}{\partial t} + A_o \frac{\partial W}{\partial x} + B_o \frac{\partial W}{\partial y} + C_o \frac{\partial W}{\partial z} = S_o$$

The above linear equation can be insightful for the construction or analysis of a CFD scheme.
On the other hand, the “genuine” linearization of Eq. (1) (with $S$ independent of $W$) about a flow equilibrium condition $W_o$ — which is physically more relevant — leads to the following equation

$$\frac{\partial W}{\partial t} + A_o \frac{\partial W}{\partial x} + B_o \frac{\partial W}{\partial y} + C_o \frac{\partial W}{\partial z} - \frac{\partial S}{\partial W} |_o W$$

$$+ \frac{\partial A}{\partial W} |_o W \frac{\partial W_o}{\partial x} + \frac{\partial B}{\partial W} |_o W \frac{\partial W_o}{\partial y} + \frac{\partial C}{\partial W} |_o W \frac{\partial W_o}{\partial z} = 0$$

(4)

Hence, the following remarks are noteworthy:

- in a genuine linearization such as in Eq. (4), $W$ is a perturbation around $W_o$ which should be denoted in principle by $\delta W$
- in a genuine linearization around a dynamic equilibrium condition, the source term does not contribute a “frozen” right hand-side
- in general, Eq. (4) and Eq. (3) are different
- however, if the linearization is performed about a uniform flow condition $W_o$ and $S = 0$, Eq. (4) and Eq. (3) become identical
Consider here the linear model equation (3)

\[ \frac{\partial W}{\partial t} + A_o \frac{\partial W}{\partial x} + B_o \frac{\partial W}{\partial y} + C_o \frac{\partial W}{\partial z} = S_o \]

Linearized equations such as the above equation have exact solutions.

Let \( W(x, y, z, t^0) \) denote an initial value for \( W \) at time \( t^0 \): this initial condition can be expanded by Fourier decomposition with wave numbers \( k_{xj}, k_{yj}, \) and \( k_{zj} \) as follows

\[ W(x, y, z, t^0) = I(x, y, z) = \sum_j c_j e^{i(k_{xj}x + k_{yj}y + k_{zj}z)} \]

In this case, the exact solution of Eq. (3) for \( t > t^0 \) is

\[ W(x, y, z, t) = \sum_j e^{-i(t-t^0)(k_{xj}A_o+k_{yj}B_o+k_{zj}C_o)} c_j e^{i(k_{xj}x+k_{yj}y+k_{zj}z)} + (t - t^0)S_o \]
Linearization Around a Localized Flow Condition

\[ W(x, y, z, t) = \sum_j e^{-i(t-t^0)(k_{xj}A_0 + k_{yj}B_0 + k_{zj}C_0)} c_j e^{i(k_{xj}x + k_{yj}y + k_{zj}z)} + (t-t^0)S_0 \]

Hence, the solution of Eq. (3) has both a linear growth term and, depending on the eigenvalues of the matrix

\[ M_j = k_{xj}A_0 + k_{yj}B_0 + k_{zj}C_0 \]

a possible exponential growth in time components
Consider the following equation

$$\frac{\partial G}{\partial x_\alpha} + \frac{\partial H}{\partial x_1} = 0 \quad (5)$$

For example for the unsteady Euler equations in one dimension

$$x_\alpha = t, \quad x_1 = x, \quad G = W = (\rho \rho v_x E)^T$$

$$H = F_x = (\rho v_x \rho v_x^2 + p (E + p)v_x)^T$$

For the steady Euler equations in two dimensions

$$x_\alpha = x, \quad x_1 = y$$

$$G = F_x \left( \rho v_x \rho v_x^2 + p \rho v_x v_y (E + p)v_x \right)^T$$

$$H = F_y = \left( \rho v_y \rho v_x v_y \rho v_y^2 + p (E + p)v_y \right)^T$$
Hyperbolic Requirement

Let

\[ A = \frac{\partial H}{\partial G} \]

and let \( \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_m) \) be the diagonal matrix of eigenvalues \( \lambda_1, \cdots, \lambda_m \) of \( A \).

Eq. (5) is hyperbolic if

1. \( \lambda_k \) is real for each \( k \)
2. \( A \) has a complete set of eigenvectors \( \Leftrightarrow A \) is diagonalizable — that is

\[ \exists Q / A = \frac{\partial H}{\partial G} = Q^{-1} \Lambda Q \]

In the general multidimensional case (see Eq. (1)), the system is hyperbolic if the matrix

\[ M = k_x A + k_y B + k_z C \]

has only real eigenvalues and a complete set of eigenvectors, for all sets of real numbers \( (k_x, k_y, k_z) \)
In mathematics, the “method” of characteristics is a technique for solving partial differential equations.

Essentially, it reduces a partial differential equation to a family of ordinary differential equations along which the solution can be integrated from some initial data given on a suitable hypersurface.

It is applicable to any hyperbolic partial differential equation, but has been developed mostly for first-order partial differential equations.

Characteristic “theory” is pertinent to the treatment of boundary conditions and CFD schemes such as flux split schemes (Steger and Warming) and flux difference vector splitting schemes (Roe).
Consider the following unsteady homogeneous hyperbolic equations written in non conservation form

\[
\frac{\partial W}{\partial t} + A \frac{\partial W}{\partial x} = 0, \quad A = \frac{\partial F}{\partial W} = A(W) \tag{6}
\]

\[A\] is diagonalizable and therefore

\[A = Q^{-1} \Lambda Q \tag{7}\]

where \(\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_m) = \Lambda(W)\) and \(Q = Q(W)\)

- Let \(r_i\) denote the \(i\)-th column of \(Q^{-1}\):
  \(AQ^{-1} = Q^{-1} \Lambda \Rightarrow Ar_i = \lambda_i r_i \Rightarrow r_i\) is \(A\)'s \(i\)-th right eigenvector

- Let \(l_i\) denote the \(i\)-th column of \(Q^T\) which is the \(i\)-th row of \(Q\):
  \(QA = \Lambda Q \Rightarrow A^T Q^T = Q^T \Lambda \Rightarrow A^T l_i = \lambda_i l_i \) (or \(l_i^T A = \lambda_i l_i^T\) \(\Rightarrow l_i\) is \(A\)'s \(i\)-th left eigenvector
Characteristics Relations

- Substituting (7) into Eq. (6) and pre-multiplying by \( Q \) leads to the so-called *characteristic form* of Eq. (6)

\[
Q \frac{\partial W}{\partial t} + \Lambda Q \frac{\partial W}{\partial x} = 0
\]

- The *characteristic variables* \( \xi = (\xi_1 \cdots \xi_m)^T \) are defined as follows (note the differential form)

\[
d\xi = QdW
\]

- Substituting the definition of the characteristic variables in the characteristic form of the governing equations leads to

\[
\frac{\partial \xi}{\partial t} + \Lambda \frac{\partial \xi}{\partial x} = 0 \tag{8}
\]

which is also called the characteristic form of the governing equations and has the advantage of decoupling the characteristic variables
Each characteristic equation within Eq. (8) can be written as

$$\nabla^* \xi_i \cdot (1 \lambda_i)^T = 0, \quad i = 1, \ldots, m$$

which shows that $$\frac{\partial \xi_i}{\partial t} + \lambda_i \frac{\partial \xi_i}{\partial x}$$ is a directional derivative in the $$x - t$$ plane in the direction $$(1 \lambda_i)^T$$

Then, in English, each characteristic equation says that there is no change in the solution $$\xi_i$$ in the direction of $$(1 \lambda_i)^T$$ in the $$x - t$$ plane.

Now, consider a curve $$x = x(t)$$ that is everywhere tangent to $$(1 \lambda_i)^T$$ in the $$x - t$$ plane: Then, the slope of the vector $$(1 \lambda_i)^T$$ is the slope of the curve $$x = x(t)$$ and is given by

$$\frac{dx}{dt} = \lambda_i$$
Then Eq. (8) is equivalent to the following:

\[ d\xi_i = 0 \quad (\text{or } \xi_i = \text{cte}) \quad \text{for} \quad \frac{dx}{dt} = \lambda_i, \quad i = 1, \ldots, m \]

This is a wave solution: the eigenvalues \( \lambda_i \) are wave speeds, and the wavefronts \( \frac{dx}{dt} = \lambda_i \) are sometimes also called *characteristic curves* (or simply *characteristics*)
Application to the One-Dimensional Euler Equations

\[ \frac{\partial W}{\partial t} + \frac{\partial F_x}{\partial x} = 0, \quad W = (\rho \, \rho v_x \, E)^T, \quad F_x = (\rho v_x \, \rho v_x^2 + p \, (E + p)v_x)^T \]

with \( p = (\gamma - 1) \left( E - \rho \frac{v_x^2}{2} \right) \), and the speed of sound \( c \) given by

\[ c^2 = \frac{\gamma p}{\rho} \]

Choose \( V = (\rho \, v_x \, p)^T \) as the fluid state vector (with primitive variables) and re-write the governing equations in non conservation form (see Eq. (1) and Eq. (2))

\[ \frac{\partial V}{\partial t} + A' \frac{\partial V}{\partial x} = 0, \quad A' = \begin{pmatrix} v_x & \rho & 0 \\ 0 & v_x & 1 \\ 0 & \rho c^2 & v_x \end{pmatrix} \]
Diagonalize the resulting hyperbolic equations

\[ A' = Q^{-1} \Lambda Q \iff QA'Q^{-1} = \Lambda \]

\[ \Lambda = \begin{pmatrix} v_x & 0 & 0 \\ 0 & v_x + c & 0 \\ 0 & 0 & v_x - c \end{pmatrix} \]

\[ Q = \begin{pmatrix} 1 & 0 & -\frac{1}{c^2} \\ 0 & 1 & \frac{1}{\rho c} \\ 0 & 1 & -\frac{1}{\rho c} \end{pmatrix} \quad Q^{-1} = \begin{pmatrix} 1 & \frac{\rho}{2c} & -\frac{\rho}{2c} \\ \frac{1}{2c} & 1 & \frac{1}{2c} \\ \frac{\rho c}{2} & \frac{2}{\rho c} & -\frac{\rho c}{2} \end{pmatrix} \]
Application to the One-Dimensional Euler Equations

- Let \( \xi = (\xi_0 \; \xi_+ \; \xi_-)^T \) denote the characteristic variables
- The three characteristic equations are
  \[
  \frac{\partial \xi_0}{\partial t} + v_x \frac{\partial \xi_0}{\partial x} = 0 \\
  \frac{\partial \xi_+}{\partial t} + (v_x + c) \frac{\partial \xi_+}{\partial x} = 0 \\
  \frac{\partial \xi_-}{\partial t} + (v_x - c) \frac{\partial \xi_-}{\partial x} = 0
  \]

  with in this case \( d\xi = QdV \)

- Therefore, these equations are equivalent to
  \[
  d\xi_0 = d\rho - \frac{dp}{c^2} = ds = 0 \quad \text{for} \quad dx = v_x dt \quad (s \text{ denotes here the entropy})
  \]
  \[
  d\xi_+ = dv_x + \frac{dp}{\rho c} = 0 \quad \text{for} \quad dx = (v_x + c) dt
  \]
  \[
  d\xi_- = dv_x - \frac{dp}{\rho c} = 0 \quad \text{for} \quad dx = (v_x - c) dt
  \]
Hence, the solution of these characteristic equations can be written as:

\[ \xi_0 = s = \text{cst} \quad \text{for} \quad dx = v_x dt \quad \text{(entropy wave)} \]

\[ \xi_+ = v_x + \int \frac{dp}{\rho c} = \text{cst} \quad \text{for} \quad dx = (v_x + c)dt \quad \text{(acoustic wave)} \]

\[ \xi_- = v_x - \int \frac{dp}{\rho c} = \text{cst} \quad \text{for} \quad dx = (v_x - c)dt \quad \text{(acoustic wave)} \]

Notice that in this case, only the first characteristic equation is fully analytically integrable (but not its corresponding \( dx = v_x dt \)).

For this and other reasons, characteristics are important conceptually, but not of too great importance quantitatively.
Application to the One-Dimensional Euler Equations
Integral curves of the characteristic family

- note that $d\xi = QdV \Leftrightarrow dV = Q^{-1}d\xi$: the $i$-th column of $Q^{-1}$ ($i = 1, 2, 3$), denoted here by $r_i$, is the $i$-th right eigenvector of the Jacobian matrix $A'$ and defines a vector field
- it follows that

$$dV = Q^{-1}d\xi = \sum_i r_i d\xi_i \quad (10)$$

- hence, one can look for a set of states $V(\eta)$ that connect to some starting state $V_0$ through integration along one of these vector fields
- these constitute integral curves of the characteristic family
- two states $V_a$ and $V_b$ belong to the same $j$-characteristic integral curve ($dV = r_j d\xi_j = dr_j$) if they are connected via the integral

$$V_b = V_a + \int_a^b dr_j \quad (11)$$
Integral curves of the characteristic family (continue)

- consider now the case of a linear hyperbolic equation with a constant advection matrix
  - the state vector $V$ can be decomposed in eigen components
  - a $j$-characteristic integral curve in state-space is a set of states for which only the component along the $r_j$ eigenvector varies, while the components along the other eigenvectors may be non zero but should be non varying
- for a nonlinear hyperbolic equation, the decomposition of $V$ is no longer a useful concept, but the integral curves are the nonlinear equivalent of this idea
Riemann invariants

- one can express integral curves not only as integrals along the eigenvectors of the Jacobian (as in Eq. (11)), but also curves on which some special scalars are constant (as in Eq. (10) with only one \( d\xi_j \neq 0 \) and two \( d\xi_i = 0 \) ⇒ see Eqs. (9))

- in the 3D parameter space of \( V = (V_1, V_2, V_3) = (\rho, v_x, p) \), each curve is defined by two of such scalars

- such scalar fields are called Riemann invariants of the characteristic family

- here, \( \xi_+ \) and \( \xi_- \) are the Riemann invariants of the 1-characteristic integral curve

- \( \xi_0 \) and \( \xi_- \) are the Riemann invariants of the 2-characteristic integral curve

- \( \xi_0 \) and \( \xi_+ \) are the Riemann invariants of the 3-characteristic integral curve

- note: the 2- and 3-characteristic integral curves represent here acoustic waves which, if they do not topple to become shocks, preserve entropy: Hence, entropy \( (\xi_0) \) is a Riemann invariant of these two families
Riemann invariants (continue)

- hence, one can regard these integral curves now as the crossing lines between the two contour curves of the two Riemann invariants

- the value of each of the two Riemann invariants now identifies each of the characteristic integral curves
Riemann invariants (continue)

- in summary, the Riemann invariants are:
  - constant along characteristic integral curves of the partial differential equation
  - mathematical transformations made on a system of first-order partial differential equations to make them more easily solvable
Simple waves

- note that if the Riemann invariants are constant along the characteristic curve \( \frac{dx}{dt} = \lambda_i \), all flow properties are constant along this characteristic curve
- definition: a wave is called a simple wave if all states along the wave lie on the same integral curve of one of the characteristic families
- hence, one can say that a simple wave is a pure wave in only one of the eigenvectors
- examples
  - a simple wave in the 1-characteristic family \((dV = r_1 d\xi_0)\) is a wave (or region of the flow) in which \(v_x = \text{cst}\) and \(p = \text{cst}\) but the entropy can vary
  - a simple wave in the 3-characteristic family \((dV = r_3 d\xi_-)\) is for example an infinitesimally weak acoustic wave in one direction
- in Chapter 5, situations will be encountered where a contact discontinuity and a rarefaction wave are simple waves
Boundary Conditions

The characteristic relations coming to or from the boundaries determine the number and nature of the required boundary conditions for solving a given hyperbolic problem.