Outline

1. Principle of Virtual Work for a Particle
2. Principle of Virtual Work for a System of $N$ Particles
3. Hamilton’s Principle for Conservative Systems and Lagrange Equations
4. Lagrange Equations in the General Case
Principle of Virtual Work for a Particle

- Particle mass $m$
- Particle force
  - force vector $\mathbf{f} = [f_1 \quad f_2 \quad f_3]^T$
  - force component $f_i, \ i = 1, \cdots, 3$
- Particle displacement
  - displacement vector $\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T$
  - displacement component $u_i, \ i = 1, \cdots, 3$
  - motion trajectory $\mathbf{u}(t)$ where $t$ denotes time
Principle of Virtual Work for a Particle

- Particle virtual displacement
  - arbitrary displacement $u^*$ (can be zero)
  - virtual displacement $\delta u = u^* - u \Rightarrow$ arbitrary by definition
  - family of arbitrary virtual displacements defined in a time-interval $[t_1, t_2]$ and satisfying the variational constraints

$$ \delta u(t_1) = \delta u(t_2) = 0 $$

- Important property

$$ \frac{d}{dt}(\delta u_i) = \frac{d}{dt}(u_i^* - u_i) = \frac{du_i^*}{dt} - \frac{du_i}{dt} = \dot{u}_i^* - \dot{u}_i = \delta \dot{u}_i $$

$$ \Rightarrow \frac{d}{dt}(\delta) = \delta\left(\frac{d}{dt}\right) \quad \text{(commutativity)} $$
Principle of Virtual Work for a Particle

- **Equilibrium**
  - strong form
    \[
m\ddot{u} - f = 0 \Rightarrow m\ddot{u}_i - f_i = 0, \quad i = 1, \ldots, 3
\]
  - weak form
    \[
    \forall \delta u, \quad (\delta u^T)(m\ddot{u} - f) = 0 \Rightarrow \sum_{i=1}^{3} (m\ddot{u}_i - f_i)\delta u_i = 0
    \]
    \[
    \Rightarrow (m\ddot{u}_i - f_i)\delta u_i = 0, \quad i = 1, \ldots, 3
    \]
- \(\delta u^T(m\ddot{u} - f) = (m\ddot{u} - f)^T\delta u\) is homogeneous to a work
  \[
  \Rightarrow \text{virtual work} \ (\delta W)
  \]
- **Virtual work principle**
  
  The virtual work produced by the effective forces acting on a particle during a virtual displacement is equal to zero
Principle of Virtual Work for a System of \( N \) Particles

- \( N \) particles: \( k = 1, \cdots, N \)
- Equilibrium
  \[
  m\ddot{u}_k - f_k = 0, \quad k = 1, \cdots, N
  \]
- Family of virtual displacements \( \delta u_k = u_k^* - u_k \) satisfying the variational constraints
  \[
  \delta u_k(t_1) = \delta u_k(t_2) = 0
  \quad (1)
  \]
- Virtual work
  \[
  m\ddot{u}_k - f_k = 0 \Rightarrow \sum_{k=1}^{N} \delta u_k^T (m\ddot{u}_k - f_k) = \sum_{k=1}^{N} (m\ddot{u}_k - f_k)^T \delta u_k = 0
  \]
Conversely, $\forall \delta \mathbf{u}_k$ compatible with the variational constraints (1)

$$
\sum_{k=1}^{N} \delta \mathbf{u}_k^T (m\ddot{\mathbf{u}}_k - \mathbf{f}_k) = 0 \Rightarrow \sum_{k=1}^{N} \sum_{i=1}^{3} (m_k \dddot{u}_{ik} - f_{ik}) \delta u_{ik} = 0 \quad (2)
$$

If (2) is true $\forall \delta \mathbf{u}_k$ compatible with (1) $\Rightarrow$ (2) is true for

$\delta \mathbf{u}_k = [1 \ 0 \ 0]^T$, $\delta \mathbf{u}_k = [0 \ 1 \ 0]^T$, and $\delta \mathbf{u}_k = [0 \ 0 \ 1]^T$, $t \in ]t_1, t_2[

$$
\Rightarrow \sum_{k=1}^{N} m_k \dddot{u}_{ik} - f_{ik} = 0, \quad i = 1, \ldots, 3
$$

If the virtual work equation is satisfied for any displacement compatible with the variational constraints, the system (of particles) is in dynamic equilibrium.
Major result

dynamic equilibrium $\Leftrightarrow$ virtual work principle
Principle of Virtual Work for a System of \( N \) Particles

Kinematic Constraints

- In the absence of (kinematic) constraints, the state of a system of \( N \) particles can be defined by \( 3N \) displacement components

\[
   u_{ik}, \quad i = 1, \ldots, 3, \quad k = 1, \ldots, N
\]

- Instantaneous configuration

\[
   \xi_{ik} = x_{ik} + u_{ik}(x, t) \Rightarrow 3N \ \text{dofs}
\]

- However, most mechanical systems incorporate some sort of constraints
  - holonomic constraints
  - non-holonomic constraints
Principle of Virtual Work for a System of $N$ Particles

Kinematic Constraints

- Holonomic constraints
  - two types
    - rheonomic constraints: defined by $f(\xi_{ik}, t) = 0$ (no explicit dependence on any velocity)
    - scleronomic constraints: defined by $f(\xi_{ik}) = 0$

- a holonomic constraint reduces by 1 the number of dofs of a mechanical system
- example

![Diagram of a rigid bar with masses](image)

- rigidity $\Rightarrow$ conservation of length $\Rightarrow \sum_{i=1}^{3}(\xi_{i2} - \xi_{i1})^2 = l^2$
Non-holonomic constraints

- defined by \( f(\dot{\xi}_i, \xi_i, t) = 0 \)
- example

- no slip \(\Rightarrow\) speed of point \(P = 0\)

\[
\begin{align*}
\dot{x}_1 &= 0 - r \dot{\phi} \cos \theta \quad \Rightarrow \quad \dot{x}_1 + r \dot{\phi} \cos \theta = 0 \\
\dot{y}_1 &= 0 + r \dot{\phi} \sin \theta \quad \Rightarrow \quad \dot{y}_1 - r \dot{\phi} \sin \theta = 0
\end{align*}
\]

- in addition

\[
\begin{align*}
x_2 - x_1 &= r \sin \phi \cos \theta \\
y_2 - y_1 &= -r \sin \phi \sin \theta \\
z_2 - z_1 &= -r \cos \phi \\
z_1 &= r
\end{align*}
\]
example (continue)
- hence, this system has
  - 8 variables: \( x_1, y_1, z_1, x_2, y_2, z_2, \theta, \phi \)
  - 4 holonomic constraints
  - 2 non-holonomic constraints

in general, \( f(\dot{\xi}_i, \xi_i, t) = 0 \) is not integrable and therefore non-holonomic constraints do not reduce the number of dofs of a mechanical system.

therefore, the mechanical system in the above example (wheel) has \( 8 - 4 = 4 \) dofs
- 2 translations in the rolling plane
- 2 rotations
Let $n$ denote the number of dofs of a mechanical system: for example, for a system with $N$ material points and $R$ holonomic constraints, $n = 3N - R$

The generalized coordinates of this system are defined as the $n$ configuration parameters $(q_1, q_2, \cdots, q_n)$ in terms of which the displacements can be expressed as

$$u_{ik}(x, t) = U_{ik}(q_1, q_2, \cdots, q_n, t)$$

If the system is not constrained by any non-holonomic constraint, then the generalized coordinates $(q_1, q_2, \cdots, q_n)$ are independent: they can vary arbitrarily without violating the kinematic constraints.
- Principle of Virtual Work for a System of $N$ Particles

- Generalized Displacements

**Example**

- holonomic constraint HC1: $\xi_{11}^2 + \xi_{21}^2 = l_1^2$
- holonomic constraint HC2: $(\xi_{12} - \xi_{11})^2 + (\xi_{22} - \xi_{21})^2 = l_2^2$

$$\implies 4 - 2 = 2\text{ dofs}$$

- one possible choice of $(q_1, q_2)$ is $(\theta_1, \theta_2)$

$$\implies \begin{cases} 
\xi_{11} = l_1 \cos \theta_1 \\
\xi_{12} = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\
\xi_{21} = l_1 \sin \theta_1 \\
\xi_{22} = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2)
\end{cases}$$
Principle of Virtual Work for a System of \( N \) Particles

**Generalized Displacements**

- **Virtual displacements**

\[
u_{ik}(x, t) = U_{ik}(q_1, q_2, \ldots, q_n, t) \Rightarrow \delta u_{ik} = \sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial q_s} \delta q_s
\]

- **Virtual work equation**

\[
\sum_{s=1}^{n} \left[ \sum_{k=1}^{N} \sum_{i=1}^{3} (m_k \ddot{u}_{ik} - f_{ik}) \frac{\partial U_{ik}}{\partial q_s} \right] \delta q_s = 0
\]

- **Second term in above equation can be written as**

\[
\sum_{s=1}^{n} Q_s \delta q_s \quad \text{where}
\]

\[
Q_s = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{ik} \frac{\partial U_{ik}}{\partial q_s}
\]

is the generalized force conjugate to \( q_s \)
Sir William Rowan Hamilton (4 August 1805 - 2 September 1865)

- Irish physicist, astronomer, and mathematician
- contributions: classical mechanics, optics, and algebra (inventor of quaternions), and most importantly, reformulation of Newtonian mechanics (now called Hamiltonian mechanics)
- impact: modern study of electromagnetism, development of quantum mechanics
Hamilton’s principle is a statement that the action of a system is stationary. The action is given by the integral of the virtual work principle:

$$\int_{t_1}^{t_2} \left[ \sum_{k=1}^{N} \sum_{i=1}^{3} (-m_k \ddot{u}_{ik} + f_{ik}) \delta u_{ik} \right] dt = 0$$

where $\delta u_{ik}$ are arbitrary but compatible with eventual constraints and verify the end conditions (previously referred to as variational constraints).
First, assume that $f_k$ derives from a potential $V$ — that is, $(f_k$ is a conservative force) $\Rightarrow \exists \ V / f_k = -\nabla V$

Virtual work

$$\delta W = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{ik} \delta u_{ik} = - \sum_{k=1}^{N} \sum_{i=1}^{3} \frac{\partial V}{\partial \xi_i} \delta u_{ik} = - \sum_{k=1}^{N} \sum_{i=1}^{3} \frac{\partial V}{\partial \xi_i} \delta \xi_{ik} = -\delta V$$

$\delta W = -\delta V$ and $\delta W = \sum_{s=1}^{n} Q_s \delta q_s$

$$\Rightarrow Q_s = - \frac{\partial V}{\partial q_s}$$

What about the term associated with the inertia forces?
Note that

\[
\frac{d}{dt}(m_k \delta u_{ik}) = m_k \ddot{u}_{ik} \delta u_{ik} + m_k \dot{u}_{ik} \dot{\delta u}_{ik}
\]

\[
= m_k \ddot{u}_{ik} \delta u_{ik} + \delta \left( \frac{1}{2} m_k \dot{u}_{ik}^2 \right)
\]

\[
\implies m_k \ddot{u}_{ik} \delta u_{ik} = \frac{d}{dt}(m_k \dot{u}_{ik} \delta u_{ik}) - \delta \left( \frac{1}{2} m_k \dot{u}_{ik}^2 \right)
\]

The kinetic energy of a system of \( N \) particles can be defined as

\[
\mathcal{T} = \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \dot{u}_{ik}^2
\]
Hence, Hamilton’s principle for a conservative system can be written as

\[
- \int_{t_1}^{t_2} \left[ \sum_{k=1}^{N} \sum_{i=1}^{3} (m_k \ddot{u}_{ik} - f_{ik}) \delta u_{ik} \right] dt = 0
\]

\[
\Rightarrow - \int_{t_1}^{t_2} \left[ \sum_{k=1}^{N} \sum_{i=1}^{3} \frac{d}{dt} \left( m_k \dot{u}_{ik} \delta u_{ik} \right) - \delta \left( \frac{1}{2} m_k \dot{u}_{ik}^2 \right) \right] dt + \int_{t_1}^{t_2} (-\delta \mathcal{V}) dt = 0
\]

\[
\Rightarrow - \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \dot{u}_{ik} \delta u_{ik} \bigg|_{t_1}^{t_2} + \delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V}) dt = 0
\]

Recall the generalized displacements

\[
\{q_s\} \Rightarrow u_{ik}(x, t) = U_{ik}(q_s, t) \Rightarrow \dot{u}_{ik} = h(q_s, \dot{q}_s, t)
\]

\[
\Rightarrow \mathcal{T} = \mathcal{T}(q, \dot{q}, t) \quad \text{and} \quad \mathcal{V} = \mathcal{V}(q, t)
\]
Recall the end conditions

\[ \delta u_{ik}(t_1) = 0 \Rightarrow \delta q_s(t_1) = 0 \quad \text{and} \quad \delta u_{ik}(t_2) = 0 \Rightarrow \delta q_s(t_2) = 0 \]

Therefore, Hamilton’s principle (HP) can be written as

\[
\delta \int_{t_1}^{t_2} \left[ T(q, \dot{q}, t) - V(q, t) \right] dt = 0 \quad \forall \ \delta q / \delta q(t_1) = \delta q(t_2) = 0
\]

where

\[ q = [q_1 \quad q_2 \quad \cdots \quad q_s \quad \cdots \quad q_n]^T \]
Hamilton’s Principle for Conservative Systems and Lagrange Equations

- Equations of motion

\[ \delta T = \sum_{s=1}^{n} \left( \frac{\partial T}{\partial q_s} \delta q_s + \frac{\partial T}{\partial \dot{q}_s} \delta \dot{q}_s \right) \]

\[ \delta V = -\sum_{s=1}^{n} Q_s \delta q_s \]

HP \rightarrow \int_{t_1}^{t_2} \left[ \sum_{s=1}^{n} \frac{\partial T}{\partial \dot{q}_s} \delta \dot{q}_s + \left( \frac{\partial T}{\partial q_s} + Q_s \right) \delta q_s \right] dt = 0

- Integrate by parts and apply the end conditions

\[ \sum_{s=1}^{n} \frac{\partial T}{\partial \dot{q}_s} \delta q_s \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{s=1}^{n} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) \delta q_s dt \]

\[ + \int_{t_1}^{t_2} \sum_{s=1}^{n} \left( \frac{\partial T}{\partial q_s} + Q_s \right) \delta q_s dt = 0 \]

\[ \Rightarrow \int_{t_1}^{t_2} \sum_{s=1}^{n} \left[ -\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} + Q_s \right] \delta q_s dt = 0 \]
Hamilton’s Principle for Conservative Systems and Lagrange Equations

The Lagrange equations of motion

\[ \int_{t_1}^{t_2} \sum_{s=1}^{n} \left[ -\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} + Q_s \right] \delta q_s \, dt = 0 \]

\[ \forall \delta q / \delta q(t_1) = \delta q(t_2) = 0 \]

\[ \Rightarrow -\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} + Q_s = 0 \]

- \( -\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} \): generalized inertia forces
- \( Q_s \): generalized internal and external forces
Joseph-Louis (Giuseppe Lodovico), comte de Lagrange (25 January 1736 - 10 April 1813)

- Italian-born mathematician and astronomer
- Contributions: analysis, number theory, and classical and celestial mechanics
- Succeeded Euler in 1766 as the director of mathematics at the Prussian Academy of Sciences in Berlin
- Impact: his treatise on analytical mechanics written in Berlin and first published in 1788 formed a basis for the development of mathematical physics in the 19th century
- Moved to France in 1787 and became a member of the French Academy
- Survived the French Revolution and became the first professor of analysis at the École Polytechnique upon its opening in 1794
Hamilton’s Principle for Conservative Systems and Lagrange Equations

- Equations of equilibrium $\sum_{k=1}^{N} m_k \ddot{u}_{ik} - f_{ik} = 0, \quad i = 1, \cdots, 3$: simple, but can be difficult to formulate analytically for complex systems

- Lagrange’s equations of motion $-\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} + Q_s = 0$: more complex, but can simplify the analytical solution of complex dynamic problems
Recall \( u_{ik} = U_{ik}(q, t) \) \( \Rightarrow \) \( \dot{u}_{ik} = \sum_{s=1}^{n} \frac{\partial U_{ik}}{\partial q_s} \dot{q}_s + \frac{\partial U_{ik}}{\partial t} \)

Substitute in the expression of the kinetic energy

\( \Rightarrow \mathcal{T} = \mathcal{T}_0(q, t) + \mathcal{T}_1(q, \dot{q}, t) + \mathcal{T}_2(q, \dot{q}, t) \) where \( \mathcal{T}_0, \mathcal{T}_1, \) and \( \mathcal{T}_2 \) are homogeneous forms \(^1\) of degree 0, 1, and 2 in \( \dot{q}_s \), respectively, and are given by

\[
\begin{align*}
\mathcal{T}_0(q, t) &= \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \left( \frac{\partial U_{ik}}{\partial t} \right)^2 \quad \text{(transport kinetic energy)} \\
\mathcal{T}_1(q, \dot{q}, t) &= \sum_{s=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} \frac{\partial U_{ik}}{\partial t} m_k \frac{\partial U_{ik}}{\partial q_s} \dot{q}_s \quad \text{(mutual kinetic energy)} \\
\mathcal{T}_2(q, \dot{q}, t) &= \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \frac{\partial U_{ik}}{\partial q_s} \frac{\partial U_{ik}}{\partial q_r} \dot{q}_s \dot{q}_r \quad \text{(relative kinetic energy)}
\end{align*}
\]

\(^1\) \( G(q_1, \cdots, q_n) \) is said to be a homogeneous function of degree \( p \), where \( p \) is an integer, if \( \forall \alpha \neq 0, \quad G(\alpha q_1, \cdots, \alpha q_n) = \alpha^p G(q_1, \cdots, q_n) \)
Hamilton’s Principle for Conservative Systems and Lagrange Equations

Classification of Inertia Forces

- From Euler’s theorem on homogeneous functions of degree $p$ 
  \[ \left( \sum_{i=1}^{n} q_i \frac{\partial G(q)}{\partial q_i} = pG(q) \right) \] 
  it follows that 
  \[ T_1 = \sum_{s=1}^{n} \dot{q}_s \frac{\partial T_1}{\partial \dot{q}_s} \quad \text{and} \quad T_2 = \frac{1}{2} \sum_{s=1}^{n} \dot{q}_s \frac{\partial T_2}{\partial \dot{q}_s} \]

- Interpretation of the first two terms of the Lagrange equations
  
  \[
  - \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} = - \frac{d}{dt} \left( \frac{\partial T_1}{\partial \dot{q}_s} + \frac{\partial T_2}{\partial \dot{q}_s} \right) + \frac{\partial}{\partial q_s} (T_0 + T_1 + T_2) 
  \]
  
  \[
  = - \frac{\partial}{\partial t} \left( \frac{\partial T_1}{\partial \dot{q}_s} \right) - \sum_{r=1}^{n} \frac{\partial^2 T_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r - \frac{d}{dt} \left( \frac{\partial T_2}{\partial \dot{q}_s} \right) + \frac{\partial}{\partial q_s} (T_0 + T_1 + T_2) 
  \]
  
  \[
  = \left( - \frac{\partial}{\partial t} \left( \frac{\partial T_1}{\partial \dot{q}_s} \right) + \frac{\partial T_0}{\partial q_s} \right) + \left( - \frac{d}{dt} \left( \frac{\partial T_2}{\partial \dot{q}_s} \right) + \frac{\partial T_2}{\partial q_s} \right) \text{ (transport)} + \left( - \sum_{r=1}^{n} \frac{\partial^2 T_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \frac{\partial T_1}{\partial q_s} \right) \text{ (complementary)} 
  \]
The transport inertia forces are those obtained by setting $\dot{q}_r = 0$

$$\Rightarrow - \frac{\partial}{\partial t} \left( \frac{\partial T_1}{\partial \dot{q}_s} \right) + \frac{\partial T_0}{\partial q_s}$$

The relative inertia forces are those obtained by assuming $\frac{\partial U_{ik}}{\partial t} = 0$

$$\Rightarrow - \frac{d}{dt} \left( \frac{\partial T_2}{\partial \dot{q}_s} \right) + \frac{\partial T_2}{\partial q_s}$$

The complementary inertia forces are given by the remainder

$$\Rightarrow F_s = - \sum_{r=1}^{n} \frac{\partial^2 T_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \frac{\partial T_1}{\partial q_s}$$
Hamilton’s Principle for Conservative Systems and Lagrange Equations

Classification of Inertia Forces

- Note that since $T_1 = \sum_{r=1}^{n} \dot{q}_r \frac{\partial T_1}{\partial \dot{q}_r}$

$$F_s = - \sum_{r=1}^{n} \frac{\partial^2 T_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \frac{\partial T_1}{\partial q_s} = - \sum_{r=1}^{n} \frac{\partial^2 T_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \sum_{r=1}^{n} \frac{\partial^2 T_1}{\partial q_s \partial \dot{q}_r} \dot{q}_r$$

$$= \sum_{r=1}^{n} \left( \frac{\partial^2 T_1}{\partial q_s \partial \dot{q}_r} - \frac{\partial^2 T_1}{\partial q_r \partial \dot{q}_s} \right) \dot{q}_r = \sum_{r=1}^{n} g_{rs} \dot{q}_r$$

where the coefficients $g_{rs} = -g_{sr}$ do not depend on the velocities $\dot{q}_s$, but only on the generalized displacements and time.

- The complementary inertia forces have the nature of gyroscopic forces. Because of the skew-symmetry of the coefficients $g_{rs}$, it follows that

$$\sum_{s=1}^{n} F_s \dot{q}_s = \sum_{s=1}^{n} \sum_{r=1}^{n} g_{rs} \dot{q}_r \dot{q}_s = 0$$
Leonhard Euler (15 April 1707 - 18 September 1783)

- Swiss mathematician and physicist
- contributions: infinitesimal calculus and graph theory
- impact: mechanics, fluid dynamics, optics, and astronomy
- the asteroid 2002 Euler was named in his honor
Assume here, **until otherwise stated**, that the generalized displacements $U_{ik}$ are independent of time

$$T = T_2 = \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \frac{\partial U_{ik}}{\partial q_s} \frac{\partial U_{ik}}{\partial q_r} \dot{q}_s \dot{q}_r = \frac{1}{2} \sum_{s=1}^{n} \dot{q}_s \frac{\partial T_2}{\partial \dot{q}_s}$$

Differentiation with respect to time of the above expression leads to

$$2 \frac{dT}{dt} = \sum_{s=1}^{n} \ddot{q}_s \frac{\partial T}{\partial \dot{q}_s} + \sum_{s=1}^{n} \dot{q}_s \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) \quad (3)$$

From $T = T(q, \dot{q})$ it follows that

$$\frac{dT}{dt} = \sum_{s=1}^{n} \ddot{q}_s \frac{\partial T}{\partial \dot{q}_s} + \sum_{s=1}^{n} \dot{q}_s \frac{\partial T}{\partial q_s} \quad (4)$$

$$(3) - (4) \Rightarrow \frac{dT}{dt} = \sum_{s=1}^{n} \dot{q}_s \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) - \frac{\partial T}{\partial q_s} \right]$$
Hamilton’s Principle for Conservative Systems and Lagrange Equations

Energy Conservation in a System with Scleronomic Constraints

\[
\frac{dT}{dt} = \sum_{s=1}^{n} \dot{q}_s \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) - \frac{\partial T}{\partial q_s} \right]
\]

- From the Lagrange equations of motion \((-\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} + Q_s = 0\), it follow that

\[
\frac{dT}{dt} = \sum_{s=1}^{n} Q_s \dot{q}_s
\]

- For a system with scleronomic constraints, \(V = V(q)\)

- Since for conservative forces \(Q_s = -\frac{\partial V}{\partial q_s}\), it follows that

\[
\frac{dT}{dt} = \sum_{s=1}^{n} Q_s \dot{q}_s = -\sum_{s=1}^{n} \frac{\partial V}{\partial q_s} \dot{q}_s = -\frac{dV}{dt}
\]

and therefore

\[
\frac{d}{dt}(T + V) = 0 \iff E = T + V = \text{cst}
\]
Hamilton’s Principle for Conservative Systems and Lagrange Equations

Classification of Generalized Forces

- **Internal forces**
  - linking forces
  - elastic forces
  - dissipation forces (may have external origins too)

- **External forces**
  - conservative forces
  - non-conservative forces
Linking forces

\[ f_{i_1} + f_{i_2} = 0, \quad i = 1, \ldots, 3 \]

- virtual work: \( \delta W = \sum_{i=1}^{3} f_{i_1} \delta u_{i_1} + f_{i_2} \delta u_{i_2} = \sum_{i=1}^{3} f_{i_1} (\delta u_{i_1} - \delta u_{i_2}) \)

- for admissible virtual displacements (compatible with the real displacements) \( \Leftrightarrow 2 \sum_{i=1}^{3} (\xi_{i_1} - \xi_{i_2})(\delta u_{i_1} - \delta u_{i_2}) = 0 \),

\[ \delta u_{i_1} = \delta u_{i_2} \Rightarrow \delta W = 0 \]

*Linking forces do NOT contribute to the generalized forces acting on the global system*

- the above result is a nice aspect of Lagrangian mechanics
Hamilton’s Principle for Conservative Systems and Lagrange Equations

Classification of Generalized Forces

Elastic forces

- produced work is storable in a recoverable form
- internal energy \( \mathcal{V}_{int}(\mu_{ik}) \)

\[
\delta T = -\delta \mathcal{V}_{int} = - \sum_{k=1}^{N} \sum_{i=1}^{3} \frac{\partial \mathcal{V}_{int}}{\partial u_{ik}} \delta u_{ik}
\]

\[
\mathcal{V}_{int} = \mathcal{V}_{int}(q) \Rightarrow \delta T = -\delta \mathcal{V}_{int} = - \sum_{s=1}^{n} \frac{\partial \mathcal{V}_{int}}{\partial q_{s}} \delta q_{s} = \sum_{s=1}^{n} Q_{s} \delta q_{s}
\]

\[
\implies Q_{s} = - \frac{\partial \mathcal{V}_{int}}{\partial q_{s}}
\]
Dissipative forces
- remain parallel in opposite direction to the velocity vector $\mathbf{v}$ and are function of its modulus $v_k = \sqrt{\sum_{i=1}^{3} \dot{u}_{ik}^2}$

$$f_k = -C_k f_k(v_k) \frac{v_k}{v_k}$$

virtual work

$$u = u(q, t) \Rightarrow \delta W = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{ik} \delta u_{ik} = - \sum_{s=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} C_k f_k(v_k) \frac{v_k}{v_k} \frac{\partial u_{ik}}{\partial q_s} \delta q_s = \sum_{s=1}^{n} Q_s \delta q_s$$

$$\Rightarrow Q_s = - \sum_{k=1}^{N} \sum_{i=1}^{3} C_k f_k(v_k) \frac{v_k}{v_k} \frac{\partial u_{ik}}{\partial q_s}$$

since $v_{ik} = \frac{du_{ik}}{dt} = \frac{\partial u_{ik}}{\partial t} + \sum_{s=1}^{n} \frac{\partial u_{ik}}{\partial q_s} \dot{q}_s$ and therefore $\frac{\partial v_{ik}}{\partial \dot{q}_s} = \frac{\partial u_{ik}}{\partial q_r}$, it follows that

$$Q_s = - \sum_{k=1}^{N} \sum_{i=1}^{3} C_k f_k(v_k) \frac{v_k}{v_k} \frac{\partial v_{ik}}{\partial \dot{q}_s} = - \sum_{k=1}^{N} C_k \frac{f_k(v_k)}{v_k} \frac{\partial}{\partial \dot{q}_s} \left( \frac{1}{2} \sum_{i=1}^{3} v_{ik}^2 \right)$$

$$\Rightarrow Q_s = - \sum_{k=1}^{N} C_k f_k(v_k) \frac{\partial v_k}{\partial \dot{q}_s}$$
Hamilton’s Principle for Conservative Systems and Lagrange Equations

Classification of Generalized Forces

\[ Q_s = - \sum_{k=1}^{N} C_k f_k (v_k) \frac{\partial v_k}{\partial \dot{q}_s} \]

- Dissipation function:
  \[ D = \sum_{k=1}^{N} \int_0^{v_k (\dot{q})} C_k f_k (\gamma) d\gamma \]

- From Leibniz’s integral rule
  \[ \frac{d}{dt} \left( \int_{a(t)}^{b(t)} f(x, t) dx \right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f(b(t), t) \frac{db}{dt} - f(a(t), t) \frac{da}{dt} \]

  it follows that
  \[ \frac{\partial D}{\partial \dot{q}_s} = \sum_{k=1}^{N} C_k f_k (v_k) \frac{\partial v_k}{\partial \dot{q}_s} \Rightarrow Q_s = - \frac{\partial D}{\partial \dot{q}_s} \]

- Dissipated power:
  \[ P = \sum_{s=1}^{n} Q_s \dot{q}_s = - \sum_{s=1}^{n} \dot{q}_s \frac{\partial D}{\partial \dot{q}_s} \]
If $\mathcal{D}$ is assumed to be a homogeneous function of order $p$ in the generalized velocities, then

$$\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n} Q_s \dot{q}_s = -\frac{d\mathcal{V}}{dt} - \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{D}}{\partial \dot{q}_s} = -\frac{d\mathcal{V}}{dt} - p\mathcal{D}$$

$$\Rightarrow \frac{d}{dt} (\mathcal{T} + \mathcal{V}) = -p\mathcal{D}$$

- $p = 1$: dry friction
- $p = 2$: viscous damping
- $p = 3$: aerodynamic drag
Classification of Generalized Forces

- Conservative external forces: these derive from a potential

\[ \exists \mathcal{V}_{\text{ext}} / Q_s = -\frac{\partial \mathcal{V}_{\text{ext}}}{\partial q_s} \]

\Rightarrow \text{their virtual work during a cycle is zero}

\[ \delta W = \oint Q_s \delta q_s = -\oint \frac{\partial \mathcal{V}_{\text{ext}}}{\partial q_s} \delta q_s = -\oint \delta \mathcal{V}_{\text{ext}} = 0 \]

- Non-conservative external forces

\[ \delta W = \sum_{s=1}^{n} Q_s \delta q_s = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{ik} \delta u_{ik} = \sum_{i=1}^{3} \sum_{k=1}^{N} \sum_{s=1}^{n} \frac{\partial u_{ik}}{\partial q_s} \delta q_s \]

\[ \Rightarrow Q_s = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{ik} \frac{\partial u_{ik}}{\partial q_s} \]
Summary: taking into account the non-conservative external forces, the power balance of a system solicited by internal and external forces can be written as

\[
\frac{d}{dt}(T + V) = -pD + \sum_{s=1}^{n} Q_s \dot{q}_s
\]
Lagrange Equations in the General Case

In the general case of a non-conservative system with rheonomic constraints, the Lagrange equations of motion can be written as

$$\begin{align*}
- \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} \frac{\partial V}{\partial q_s} - \frac{\partial V}{\partial \dot{q}_s} - \frac{\partial D}{\partial \dot{q}_s} + Q_s(t) &= 0
\end{align*}$$

where

$$V = V_{\text{ext}} + V_{\text{int}} = \text{total potential}$$

$$D = \text{dissipation function}$$

$$Q_s(t) = \text{non-conservative external generalized forces}$$

They can also be written in terms of the inertia forces as follows

$$\begin{align*}
\frac{d}{dt} \left( \frac{\partial T_2}{\partial \dot{q}_s} \right) - \frac{\partial T_2}{\partial q_s} &= Q_s(t) - \frac{\partial V^*}{\partial q_s} - \frac{\partial D}{\partial \dot{q}_s} + F_s - \frac{\partial}{\partial t} \left( \frac{\partial T_1}{\partial \dot{q}_s} \right)
\end{align*}$$

where

$$V^* = V - \mathcal{T}_0 = \text{potential modified by the transport kinetic energy}$$

$$F_s = \sum_{r=1}^{n} g_{rs} \dot{q}_r = \text{generalized gyroscopic forces}$$