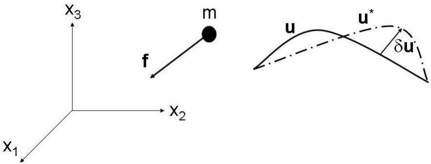
AA242B: MECHANICAL VIBRATIONS

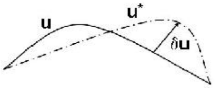
Analytical Dynamics of Discrete Systems

These slides are based on the recommended textbook: M. Géradin and D. Rixen, "Mechanical Vibrations: Theory and Applications to Structural Dynamics," Second Edition, Wiley, John & Sons, Incorporated, ISBN-13:9780471975465

Outline



- Particle mass m
- Particle force
  - force vector $\mathbf{f} = [f_1 \quad f_2 \quad f_3]^T$
  - force component f_i, $i = 1, \cdots, 3$
- Particle displacement
  - displacement vector $\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T$
  - displacement component u_i, $i = 1, \cdots, 3$
  - motion trajectory $\mathbf{u}(t)$ where t denotes time



- Particle virtual displacement
  - arbitrary displacement $\mathbf{u}^\star$ (can be zero)
  - virtual displacement $\delta\mathbf{u} = \mathbf{u}^\star - \mathbf{u} \quad \Rightarrow$ arbitrary by definition
  - family of arbitrary virtual displacements defined in a time-interval $[t_1, t_2]$ and satisfying the variational constraints

$$\boxed{\delta\mathbf{u}(t_1) = \delta\mathbf{u}(t_2) = \mathbf{0}}$$

- Important property

$$\frac{d}{dt}(\delta u_i) = \frac{d}{dt}(u_i^\star - u_i) = \frac{du_i^\star}{dt} - \frac{du_i}{dt} = \dot{u}_i{}^\star - \dot{u}_i = \delta\dot{u}_i$$

$$\Longrightarrow \boxed{\frac{d}{dt}(\delta) = \delta(\frac{d}{dt}) \quad \text{(commutativity)}}$$

- Equilibrium
  - strong form

$$m\ddot{\mathbf{u}} - \mathbf{f} = \mathbf{0} \Rightarrow m\ddot{u}_i - f_i = 0, \quad i = 1, \cdots, 3$$

  - weak form

$$\forall \delta\mathbf{u}, \quad (\delta\mathbf{u}^T)(m\ddot{\mathbf{u}} - \mathbf{f}) = 0 \Rightarrow \sum_{i=1}^{3}(m\ddot{u}_i - f_i)\delta u_i = 0$$

$$\Longrightarrow (m\ddot{u}_i - f_i)\delta u_i = 0, \quad i = 1, \cdots, 3$$

- $\delta\mathbf{u}^T(m\ddot{\mathbf{u}} - \mathbf{f}) = (m\ddot{\mathbf{u}} - \mathbf{f})^T\delta\mathbf{u}$ is homogeneous to a work $\Longrightarrow$ virtual work (δW)
- Virtual work principle

The virtual work produced by the effective forces acting on a particle during a virtual displacement is equal to zero

- N particles: $k = 1, \cdots, N$
- Equilibrium

$$m\ddot{\mathbf{u}}_k - \mathbf{f}_k = \mathbf{0}, \quad k = 1, \cdots, N$$

- Family of virtual displacements $\delta\mathbf{u}_k = \mathbf{u}_k^\star - \mathbf{u}_k$ satisfying the <u>variational constraints</u>

$$\delta\mathbf{u}_k(t_1) = \delta\mathbf{u}_k(t_2) = \mathbf{0} \tag{1}$$

- Virtual work

$$m\ddot{\mathbf{u}}_k - \mathbf{f}_k = \mathbf{0} \Rightarrow \sum_{k=1}^{N} \delta\mathbf{u}_k^T(m\ddot{\mathbf{u}}_k - \mathbf{f}_k) = \sum_{k=1}^{N} (m\ddot{\mathbf{u}}_k - \mathbf{f}_k)^T \delta\mathbf{u}_k = 0$$

- Conversely, $\forall \delta \mathbf{u}_k$ compatible with the variational constraints (1)

$$\sum_{k=1}^{N} \delta \mathbf{u}_k^T (m\ddot{\mathbf{u}}_k - \mathbf{f}_k) = 0 \Rightarrow \sum_{k=1}^{N} \sum_{i=1}^{3} (m_k \ddot{u}_{i_k} - f_{i_k}) \delta u_{i_k} = 0 \qquad (2)$$

- If (2) is true $\forall \delta \mathbf{u}_k$ compatible with (1) $\Rightarrow$ (2) is true for $\delta \mathbf{u}_j = [1 \quad 0 \quad 0]^T$ ($\delta \mathbf{u}_k = [0 \quad 0 \quad 0]^T$ for $k \neq j$), $\delta \mathbf{u}_j = [0 \quad 1 \quad 0]^T$ ($\delta \mathbf{u}_k = [0 \quad 0 \quad 0]^T$ for $k \neq j$), and $\delta \mathbf{u}_j = [0 \quad 0 \quad 1]^T$ ($\delta \mathbf{u}_k = [0 \quad 0 \quad 0]^T$ for $k \neq j$), $t \in]t_1, \ t_2[$

$$\Longrightarrow m_j \ddot{u}_{i_j} - f_{i_j} = 0, \quad i = 1, \cdots, 3, \quad j = 1, \cdots, N$$

$$\left(\Longrightarrow \sum_{k=1}^{N} m_k \ddot{u}_{i_k} - f_{i_k} = 0, \quad i = 1, \cdots, 3 \right)$$

If the virtual work equation is satisfied for any displacement compatible with the variational constraints, the system (of particles) is in dynamic equilibrium

- Major result

$$\boxed{\text{dynamic equilibrium} \Leftrightarrow \text{virtual work principle}}$$

- In the absence of (kinematic) constraints, the state of a system of N particles can be defined by $3N$ displacement components

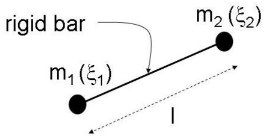
$$u_{i_k}, \quad i = 1, \cdots, 3, \quad k = 1, \cdots N$$

- Instantaneous configuration

$$\xi_{i_k} = x_{i_k} + u_{i_k}(\mathbf{x}, t) \Rightarrow 3N \text{ dofs}$$

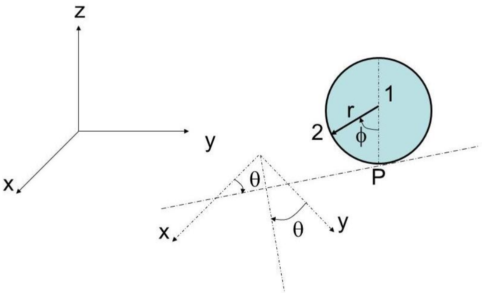
- However, most mechanical systems incorporate some sort of constraints
  - holonomic constraints
  - non-holonomic constraints

- Holonomic constraints
  - two types
    - rheonomic constraints: defined by $c(\xi_{i_k}, t) = 0$ (no explicit dependence on any velocity)
    - scleronomic constraints: defined by $c(\xi_{i_k}) = 0$ (no explicit dependence on any velocity or time)
  - a holonomic constraint reduces by 1 the number of dofs of a mechanical system
  - example



- rigidity $\Rightarrow$ conservation of length $\Rightarrow \sum_{i=1}^{3}(\xi_{i_2} - \xi_{i_1})^2 = l^2$

- Non-holonomic constraints
  - defined by $c(\dot{\xi}_{i_k}, \xi_{i_k}, t) = 0$
  - example



  - no slip $\Rightarrow$ speed of point $P = 0$

$$\Longrightarrow \begin{cases} \dot{x}_1 = 0 - r\dot{\phi}\cos\theta & \Rightarrow \quad \dot{x}_1 + r\dot{\phi}\cos\theta = 0 \\ \dot{y}_1 = 0 + r\dot{\phi}\sin\theta & \Rightarrow \quad \dot{y}_1 - r\dot{\phi}\sin\theta = 0 \end{cases}$$

  - in addition

$$\begin{cases} x_2 - x_1 = r\sin\phi\cos\theta \quad y_2 - y_1 = -r\sin\phi\sin\theta \quad z_2 - z_1 = -r\cos\phi \\ z_1 = r \end{cases}$$

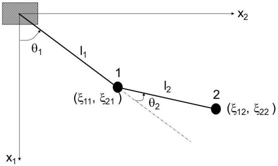
- example (continue)
  - hence, this system has
    - 8 variables: $x_1, y_1, z_1, x_2, y_2, z_2, \theta, \phi$
    - 4 holonomic constraints
    - 2 non-holonomic constraints
- in general, $c(\dot{\xi}_{i_k}, \xi_{i_k}, t) = 0$ is not integrable and therefore non-holonomic constraints do not reduce the number of dofs of a mechanical system
- therefore, the mechanical system in the above example (wheel) has 8 - 4 = 4 dofs
  - 2 translations in the rolling plane
  - 2 rotations

- Let n denote the number of dofs of a mechanical system: for example, for a system with N material points and R holonomic constraints, $n = 3N - R$
- The generalized coordinates of this system are defined as the n configuration parameters $(q_1, q_2, \cdots, q_n)$ in terms of which the displacements can be expressed as

$$u_{i_k}(\mathbf{x}, t) = U_{i_k}(q_1, q_2, \cdots, q_n, t)$$

- If the system is not constrained by any non-holonomic constraint, then the generalized coordinates $(q_1, q_2, \cdots, q_n)$ are independent: they can vary arbitrarily without violating the kinematic constraints

- Example



- holonomic constraint HC1: $\xi_{11}^2 + \xi_{21}^2 = l_1^2$
- holonomic constraint HC2: $(\xi_{12} - \xi_{11})^2 + (\xi_{22} - \xi_{21})^2 = l_2^2$

$$\Longrightarrow 4 - 2 = 2\text{dofs}$$

- one possible choice of (q_1, q_2) is (θ_1, θ_2)

$$\Longrightarrow \begin{cases} \xi_{11} = l_1 \cos\theta_1 & \xi_{21} = l_1 \sin\theta_1 \\ \xi_{12} = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & \xi_{22} = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{cases}$$

- Virtual displacements

$$u_{i_k}(\mathbf{x}, t) = U_{i_k}(q_1, q_2, \cdots, q_n, t) \Rightarrow \delta u_{i_k} = \sum_{s=1}^{n} \frac{\partial U_{i_k}}{\partial q_s} \delta q_s$$

- Virtual work equation

$$\sum_{s=1}^{n} \left[\sum_{k=1}^{N} \sum_{i=1}^{3} (m_k \ddot{u}_{i\,k} - f_{i_k}) \frac{\partial U_{i_k}}{\partial q_s} \right] \delta q_s = 0$$

- Second term in above equation can be written as $\sum_{s=1}^{n} Q_s \delta q_s$ where

$$Q_s = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{i_k} \frac{\partial U_{i_k}}{\partial q_s}$$

is the <u>generalized force conjugate to q_s</u>

- Sir William Rowan Hamilton (4 August 1805 - 2 September 1865)

  - Irish physicist, astronomer, and mathematician
  - contributions: classical mechanics, optics, and algebra (inventor of quaternions), and most importantly, reformulation of Newtonian mechanics (now called Hamiltonian mechanics)
  - impact: modern study of electromagnetism, development of quantum mechanics

- Hamilton's principle: $-\int_{t_1}^{t_2}$ virtual work principle $= 0$!

$$-\int_{t_1}^{t_2} \left[\sum_{k=1}^{N} \sum_{i=1}^{3} (m_k \ddot{u}_{i\,k} - f_{i_k}) \delta u_{i_k} \right] dt = 0$$

where δu_{i_k} are arbitrary but compatible with eventual constraints and verify the end conditions (previously referred to as variational constraints)

- First, assume that $\mathbf{f}_k$ derives from a potential $\mathcal{V}(\xi_i)$ — that is, $\mathbf{f}_k$ is a conservative force

$$\Rightarrow \exists\ \mathcal{V}(\xi_{1_k},\ \xi_{2_k},\ \xi_{3_k})\ /\ \mathbf{f}_k = -\nabla\mathcal{V}(\xi_{1_k},\ \xi_{2_k},\ \xi_{3_k})$$

- Virtual work of $\mathbf{f}_k$

$$\delta W = \sum_{k=1}^{N}\sum_{i=1}^{3} f_{i_k}\delta u_{i_k} = -\sum_{k=1}^{N}\sum_{i=1}^{3}\frac{\partial\mathcal{V}}{\partial\xi_{i_k}}\delta u_{i_k} = -\sum_{k=1}^{N}\sum_{i=1}^{3}\frac{\partial\mathcal{V}}{\partial\xi_{i_k}}\delta\xi_{i_k} = -\delta\mathcal{V}$$

- $\delta W = -\delta\mathcal{V}$ and $\delta W = \sum_{s=1}^{n} Q_s\delta q_s$

$$\Longrightarrow \boxed{Q_s = -\frac{\partial\mathcal{V}}{\partial q_s}}$$

- What about the virtual work of the inertia forces?

- Note that

$$\begin{aligned} \frac{d}{dt}(m_k \dot{u}_{i\,k} \delta u_{i_k}) &= m_k \ddot{u}_{i\,k} \delta u_{i_k} + m_k \dot{u}_{i\,k} \delta \dot{u}_{i\,k} \\ &= m_k \ddot{u}_{i\,k} \delta u_{i_k} + \delta(\frac{1}{2} m_k \dot{u}_{i\,k}^2) \\ \Longrightarrow \delta W = m_k \ddot{u}_{i\,k} \delta u_{i_k} &= \frac{d}{dt}(m_k \dot{u}_{i\,k} \delta u_{i_k}) - \delta(\frac{1}{2} m_k \dot{u}_{i\,k}^2) \end{aligned}$$

- The kinetic energy of a system of N particles can be defined as

$$\boxed{\mathcal{T} = \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \dot{u}_{i\,k}^2}$$

- Hence, Hamilton's principle for a conservative system can be written as

$$
\begin{aligned}
& -\int_{t_1}^{t_2}\left[\sum_{k=1}^{N}\sum_{i=1}^{3}(m_k \ddot{u}_{i\,k} - f_{i_k})\delta u_{i_k}\right] dt = 0 \\
\Longrightarrow \quad & -\int_{t_1}^{t_2}\left[\sum_{k=1}^{N}\sum_{i=1}^{3}\frac{d}{dt}(m_k \dot{u}_{i\,k}\delta u_{i_k}) - \delta(\frac{1}{2}m_k \dot{u}_{i\,k}^{2})\right] dt + \int_{t_1}^{t_2}(-\delta\mathcal{V})dt = 0 \\
\Longrightarrow \quad & -\sum_{k=1}^{N}\sum_{i=1}^{3} m_k \dot{u}_{i\,k}\delta u_{i_k}\,\Big|_{t_1}^{t_2} + \delta\int_{t_1}^{t_2}(\mathcal{T}-\mathcal{V})dt = 0
\end{aligned}
$$

- Recall the generalized displacements

$$
\begin{aligned}
\{q_s\} \quad & \Rightarrow u_{i_k}(\mathbf{x}, t) = U_{i_k}(q_s, t) \Rightarrow \dot{u}_{i\,k} = h(q_s, \dot{q}_s, t) \\
& \Rightarrow \xi_{i_k} = x_{i_k} + u_{i_k}(\mathbf{x}, t) = x_{i_k} + U_{i_k}(q_s, t) = g(q_s, t)
\end{aligned}
$$

$$
\Longrightarrow \boxed{\mathcal{T} = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}, t) \qquad \text{and} \qquad \mathcal{V} = \mathcal{V}(\mathbf{q}, t)}
$$

- Note that $u_{i_k}(\mathbf{x}, t) = U_{i_k}(q_s, t) \Rightarrow \delta u_{i_k} = \sum_{s=1}^{n} \frac{\partial U_{i_k}}{\partial q_s} \delta q_s$
- Now, recall the end conditions

$$\delta u_{i_k}(t_1) = 0 \Rightarrow \delta q_s(t_1) = 0 \qquad \text{and} \qquad \delta u_{i_k}(t_2) = 0 \Rightarrow \delta q_s(t_2) = 0$$

- Therefore, Hamilton's principle (HP) can be written as

$$\boxed{\delta \int_{t_1}^{t_2} \left[\mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}, t) - \mathcal{V}(\mathbf{q}, t)\right] dt = 0 \qquad \forall\ \delta\mathbf{q}\ /\ \delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = 0}$$

where

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & \cdots & q_s & \cdots & q_n \end{bmatrix}^T$$

- Equations of motion
  - $\delta \mathcal{T} = \sum_{s=1}^{n} \left(\frac{\partial \mathcal{T}}{\partial q_s} \delta q_s + \frac{\partial \mathcal{T}}{\partial \dot{q}_s} \delta \dot{q}_s \right)$
  - $\delta \mathcal{V} = -\sum_{s=1}^{n} Q_s \delta q_s$
  - HP $\rightarrow \int_{t_1}^{t_2} \left[\sum_{s=1}^{n} \frac{\partial \mathcal{T}}{\partial \dot{q}_s} \delta \dot{q}_s + \left(\frac{\partial \mathcal{T}}{\partial q_s} + Q_s \right) \delta q_s \right] dt = 0$
  - integrate by parts and apply the end conditions

$$\sum_{s=1}^{n} \frac{\partial \mathcal{T}}{\partial \dot{q}_s} \delta q_s \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{s=1}^{n} \frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right) \delta q_s dt + \int_{t_1}^{t_2} \sum_{s=1}^{n} \left(\frac{\partial \mathcal{T}}{\partial q_s} + Q_s \right) \delta q_s dt = 0$$

$$\Longrightarrow \int_{t_1}^{t_2} \sum_{s=1}^{n} \left[-\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s} \right) + \frac{\partial \mathcal{T}}{\partial q_s} + Q_s \right] \delta q_s dt = 0$$

- The Lagrange equations of motion

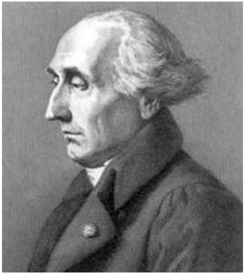
$$\int_{t_1}^{t_2} \sum_{s=1}^{n} [-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s} \quad + \quad Q_s] \, \delta q_s dt = 0$$

$$\forall \ \delta \mathbf{q} \ / \ \delta \mathbf{q}(t_1) = \delta \mathbf{q}(t_2) = 0$$

$$\Longrightarrow \boxed{-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s} + Q_s = 0}$$

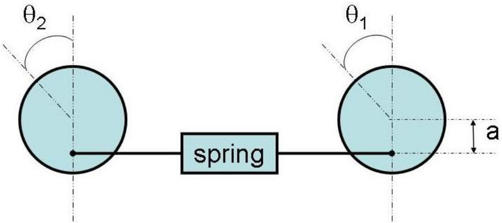
  - $-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s}$: generalized inertia forces
  - Q_s: generalized internal and external forces

- Joseph-Louis (Giuseppe Lodovico), comte de Lagrange (25 January 1736 - 10 April 1813)



- Italian-born mathematician and astronomer
- contributions: analysis, number theory, and classical and celestial mechanics
- succeeded Euler in 1766 as the director of mathematics at the Prussian Academy of Sciences in Berlin
- impact: his treatise on analytical mechanics written in Berlin and first published in 1788 formed a basis for the development of mathematical physics in the 19^{th} century
- moved to France in 1787 and became a member of the French Academy
- survived the French Revolution and became the first professor of analysis at the École Polytechnique upon its opening in 1794

- Equations of equilibrium $\sum_{k=1}^{N} m_k \ddot{u}_{i\,k} - f_{i_k} = 0, \quad i = 1, \cdots, 3$: simple, but can be difficult to formulate analytically for complex systems
- Lagrange's equations of motion $-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s} + Q_s = 0$: more complex, but can simplify the analytical solution of complex dynamic problems



- Recall $u_{i_k} = U_{i_k}(\mathbf{q}, t) \Rightarrow \dot{u}_{i\,k} = \sum_{s=1}^{n} \frac{\partial U_{i_k}}{\partial q_s} \dot{q}_s + \frac{\partial U_{i_k}}{\partial t}$
- Substitute in the expression of the kinetic energy $\Rightarrow \mathcal{T} = \mathcal{T}_0(\mathbf{q}, t) + \mathcal{T}_1(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathcal{T}_2(\mathbf{q}, \dot{\mathbf{q}}, t)$ where $\mathcal{T}_0$, $\mathcal{T}_1$, and $\mathcal{T}_2$ are homogeneous forms [1] of degree 0, 1, and 2 in $\dot{q}_s$, respectively, and are given by

$$\mathcal{T}_0(\mathbf{q}, t) = \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \left(\frac{\partial U_{i_k}}{\partial t} \right)^2 \quad \text{(transport kinetic energy)}$$

$$\mathcal{T}_1(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{s=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} \frac{\partial U_{i_k}}{\partial t} m_k \frac{\partial U_{i_k}}{\partial q_s} \dot{q}_s \quad \text{(mutual kinetic energy)}$$

$$\mathcal{T}_2(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \frac{\partial U_{i_k}}{\partial q_s} \frac{\partial U_{i_k}}{\partial q_r} \dot{q}_s \dot{q}_r \quad \text{(relative kinetic energy)}$$

[1] $\mathcal{G}(q_1, \cdots, q_n)$ is said to be a homogeneous function of degree p, where p is an integer, if $\forall \alpha \neq 0$, $\mathcal{G}(\alpha q_1, \cdots, \alpha q_n) = \alpha^p \mathcal{G}(q_1, \cdots, q_n)$

- From Euler's theorem on homogeneous functions of degree p $\left(\sum_{s=1}^{n} q_i \dfrac{\partial \mathcal{G}(\mathbf{q})}{\partial q_i} = p\mathcal{G}(\mathbf{q})\right)$ it follows that

$$\mathcal{T}_1 = \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{T}_1}{\partial \dot{q}_s} \qquad \text{and} \qquad \mathcal{T}_2 = \frac{1}{2} \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}$$

- Interpretation of the first two terms of the Lagrange equations

$$\begin{aligned}
-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s} &= -\frac{d}{dt}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s} + \frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) + \frac{\partial}{\partial q_s}(\mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2) \\
&= -\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right) - \sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r - \frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) + \frac{\partial}{\partial q_s}(\mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2) \\
&= \left(\underbrace{-\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}_0}{\partial q_s}}_{\text{transport}}\right) + \left(\underbrace{-\frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}_2}{\partial q_s}}_{\text{relative}}\right) \\
&+ \left(\underbrace{-\sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \frac{\partial \mathcal{T}_1}{\partial q_s}}_{\text{complementary}}\right)
\end{aligned}$$

- The transport inertia forces are those obtained by setting $\dot{q}_r = 0$

$$\Longrightarrow -\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}_0}{\partial q_s}$$

- The relative inertia forces are those obtained by assuming $\dfrac{\partial U_{i_k}}{\partial t} = 0$

$$\Longrightarrow -\frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}_2}{\partial q_s}$$

- The complementary inertia forces are given by the remainder

$$\Longrightarrow F_s = -\sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r}\dot{q}_r + \frac{\partial \mathcal{T}_1}{\partial q_s}$$

- Note that since $\mathcal{T}_1 = \sum_{r=1}^{n} \dot{q}_r \frac{\partial \mathcal{T}_1}{\partial \dot{q}_r}$

$$\begin{aligned} F_s &= -\sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \frac{\partial \mathcal{T}_1}{\partial q_s} = -\sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial q_s \partial \dot{q}_r} \dot{q}_r \\ &= \sum_{r=1}^{n} \left(\frac{\partial^2 \mathcal{T}_1}{\partial q_s \partial \dot{q}_r} - \frac{\partial^2 \mathcal{T}_1}{\partial q_r \partial \dot{q}_s} \right) \dot{q}_r = \sum_{r=1}^{n} g_{rs} \dot{q}_r \end{aligned}$$

  where the coefficients $g_{rs} = -g_{sr}$ do not depend on the velocities $\dot{q}_s$, but only on the generalized displacements and time

- The complementary inertia forces have the nature of *Coriolis* or *gyroscopic forces*: because of the skew-symmetry of the coefficients g_{rs}, it follows that

$$\sum_{s=1}^{n} F_s \dot{q}_s = \sum_{s=1}^{n} \sum_{r=1}^{n} g_{rs} \dot{q}_r \dot{q}_s = 0$$

- Leonhard Euler (15 April 1707 - 18 September 1783)

  - Swiss mathematician and physicist
  - contributions: infinitesimal calculus and graph theory
  - impact: mechanics, fluid dynamics, optics, and astronomy
  - the asteroid 2002 Euler was named in his honor

- Assume here, **until otherwise stated**, that the generalized displacements U_{i_k} are independent explicitly of time $\Rightarrow \mathcal{T}_0 = \mathcal{T}_1 = 0$

$$\mathcal{T} = \mathcal{T}_2 = \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{k=1}^{N}\sum_{i=1}^{3} m_k \frac{\partial U_{i_k}}{\partial q_s}\frac{\partial U_{i_k}}{\partial q_r}\dot{q}_s\dot{q}_r = \frac{1}{2}\sum_{s=1}^{n}\dot{q}_s\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}$$

- Differentiation with respect to time of the above expression leads to

$$2\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n}\ddot{q}_s\frac{\partial \mathcal{T}}{\partial \dot{q}_s} + \sum_{s=1}^{n}\dot{q}_s\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) \quad (3)$$

- From $\mathcal{T} = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}})$ it follows that

$$\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n}\ddot{q}_s\frac{\partial \mathcal{T}}{\partial \dot{q}_s} + \sum_{s=1}^{n}\dot{q}_s\frac{\partial \mathcal{T}}{\partial q_s} \quad (4)$$

- (3) - (4) $\Rightarrow \dfrac{d\mathcal{T}}{dt} = \sum_{s=1}^{n}\dot{q}_s\left[\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) - \dfrac{\partial \mathcal{T}}{\partial q_s}\right]$

$$\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n} \dot{q}_s \left[\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) - \frac{\partial \mathcal{T}}{\partial q_s} \right]$$

- From the Lagrange equations of motion $(-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right) + \frac{\partial \mathcal{T}}{\partial q_s} + Q_s = 0)$, it follow that

$$\boxed{\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n} Q_s \dot{q}_s}$$

- For a system with scleronomic constraints, $\mathcal{V} = \mathcal{V}(\mathbf{q})$
- Since for conservative forces $Q_s = -\frac{\partial \mathcal{V}}{\partial q_s}$, it follows that

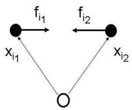
$$\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n} Q_s \dot{q}_s = -\sum_{s=1}^{n} \frac{\partial \mathcal{V}}{\partial q_s} \dot{q}_s = -\frac{d\mathcal{V}}{dt}$$

and therefore

$$\boxed{\frac{d}{dt}(\mathcal{T} + \mathcal{V}) = 0 \Leftrightarrow \mathcal{E} = \mathcal{T} + \mathcal{V} = cst}$$

- Internal forces
  - linking forces
  - elastic forces
  - dissipation forces (may have external origins too)
- External forces
  - conservative forces
  - non-conservative forces
- Both types of forces are said to be conservative if the associated virtual work is recoverable

- Linking forces



$$f_{i_1} + f_{i_2} = 0, \quad i = 1, \cdots, 3$$

- virtual work: $\delta W = \sum_{i=1}^{3} f_{i_1} \delta u_{i_1} + f_{i_2} \delta u_{i_2} = \sum_{i=1}^{3} f_{i_1} (\delta u_{i_1} - \delta u_{i_2})$
- for admissible virtual displacements – that is, virtual displacements that are compatible with the real displacements – $\delta u_{i_1} = \delta u_{i_2} \Rightarrow \delta W = 0$

Linking forces do NOT contribute to the generalized forces acting on the global system

- the above result is a nice aspect of Lagrangian mechanics

- Elastic forces
  - produced work is storable in a recoverable form
  - internal energy $\mathcal{V}_{int}(u_{i_k})$
  - $\delta\mathcal{T} = -\delta\mathcal{V}_{int} = -\displaystyle\sum_{k=1}^{N}\sum_{i=1}^{3}\frac{\partial\mathcal{V}_{int}}{\partial u_{i_k}}\delta u_{i_k}$
  - $\mathcal{V}_{int} = \mathcal{V}_{int}(\mathbf{q}) \Rightarrow \delta\mathcal{T} = -\delta\mathcal{V}_{int} = -\displaystyle\sum_{s=1}^{n}\frac{\partial\mathcal{V}_{int}}{\partial q_s}\delta q_s = \sum_{s=1}^{n} Q_s\delta q_s$

$$\Longrightarrow \boxed{Q_s = -\frac{\partial\mathcal{V}_{int}}{\partial q_s}}$$

- Dissipative forces
  - remain parallel in opposite direction to the velocity vector $\mathbf{v}$ and are function of its modulus $v_k = \sqrt{\sum_{i=1}^{3} \dot{u}_{i_k}^2}$

$$\mathbf{f}_k = -C_k f_k(v_k) \frac{\mathbf{v}_k}{v_k}$$

  - virtual work

$$\mathbf{u} = \mathbf{u}(\mathbf{q}, t) \Rightarrow \delta W = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{i_k} \delta u_{i_k} = -\sum_{s=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} C_k f_k(v_k) \frac{v_{i_k}}{v_k} \frac{\partial u_{ik}}{\partial q_s} \delta q_s = \sum_{s=1}^{n} Q_s \delta q_s$$

$$\Longrightarrow Q_s = -\sum_{k=1}^{N} \sum_{i=1}^{3} C_k f_k(v_k) \frac{v_{i_k}}{v_k} \frac{\partial u_{ik}}{\partial q_s}$$

  - since $v_{i_k} = \dfrac{du_{i_k}}{dt} = \dfrac{\partial u_{i_k}}{\partial t} + \sum_{s=1}^{n} \dfrac{\partial u_{i_k}}{\partial q_s} \dot{q}_s$ and therefore $\dfrac{\partial v_{i_k}}{\partial \dot{q}_r} = \dfrac{\partial u_{i_k}}{\partial q_r}$, it follows that

$$Q_s = -\sum_{k=1}^{N} \sum_{i=1}^{3} C_k f_k(v_k) \frac{v_{i_k}}{v_k} \frac{\partial v_{ik}}{\partial \dot{q}_s} = -\sum_{k=1}^{N} C_k \frac{f_k(v_k)}{v_k} \frac{\partial}{\partial \dot{q}_s} \left(\frac{1}{2} \sum_{i=1}^{3} v_{i_k}^2 \right)$$

$$\Longrightarrow \boxed{Q_s = -\sum_{k=1}^{N} C_k f_k(v_k) \frac{\partial v_k}{\partial \dot{q}_s}}$$

$$Q_s = -\sum_{k=1}^{N} C_k f_k(v_k) \frac{\partial v_k}{\partial \dot{q}_s}$$

- Dissipation function: $\boxed{\mathcal{D} = \sum_{k=1}^{N} \int_0^{v_k(\dot{\mathbf{q}})} C_k f_k(\gamma) d\gamma}$
- From Leibniz's integral rule

$$\frac{d}{dt}\left(\int_{a(t)}^{(b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt}$$

it follows that

$$\frac{\partial \mathcal{D}}{\partial \dot{q}_s} = \sum_{k=1}^{N} C_k f_k(v_k) \frac{\partial v_k}{\partial \dot{q}_s} \Rightarrow \boxed{Q_s = -\frac{\partial \mathcal{D}}{\partial \dot{q}_s}}$$

- Dissipated power: $P = \sum_{s=1}^{n} Q_s \dot{q}_s = -\sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{D}}{\partial \dot{q}_s}$

- If $\mathcal{D}$ is assumed to be a homogeneous function of order p in the generalized velocities, then

$$\frac{d\mathcal{T}}{dt} = \sum_{s=1}^{n} Q_s \dot{q}_s = -\frac{d\mathcal{V}}{dt} - \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{D}}{\partial \dot{q}_s} = -\frac{d\mathcal{V}}{dt} - p\mathcal{D}$$

$$\Longrightarrow \boxed{\frac{d}{dt}(\mathcal{T} + \mathcal{V}) = -p\mathcal{D}}$$

  - $p = 1$: dry friction
  - $p = 2$: viscous damping
  - $p = 3$: aerodynamic drag

- Conservative external forces: these derive from a potential

  $\Rightarrow \exists \ \mathcal{V}_{ext} \ / \ Q_s = -\dfrac{\partial \mathcal{V}_{ext}}{\partial q_s}$

  $\Rightarrow$ their virtual work during a cycle is zero

$$\delta W = \oint Q_s \delta q_s = - \oint \frac{\partial \mathcal{V}_{ext}}{\partial q_s} \delta q_s = - \oint \delta \mathcal{V}_{ext} = 0$$

- Non-conservative external forces

$$\delta W = \sum_{s=1}^{n} Q_s \delta q_s = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{i_k} \delta u_{i_k} = \sum_{i=1}^{3} \sum_{k=1}^{N} \sum_{s=1}^{n} f_{i_k} \frac{\partial u_{i_k}}{\partial q_s} \delta q_s$$

$$\Longrightarrow Q_s = \sum_{k=1}^{N} \sum_{i=1}^{3} f_{i_k} \frac{\partial u_{i_k}}{\partial q_s}$$

- Summary: taking into account the non-conservative external forces, the power balance of a system solicited by internal and external forces can be written as

$$\boxed{\frac{d}{dt}(\mathcal{T} + \mathcal{V}) = -p\mathcal{D} + \sum_{s=1}^{n} Q_s \dot{q}_s}$$

- In the general case of a non-conservative system with rheonomic constraints, the Lagrange equations of motion can be written as

$$-\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_s}\right)+\frac{\partial \mathcal{T}}{\partial q_s}-\frac{\partial \mathcal{V}}{\partial q_s}-\frac{\partial \mathcal{D}}{\partial \dot{q}_s}+Q_s(t)=0$$

where

$$\begin{aligned}\mathcal{V}=\mathcal{V}_{ext}+\mathcal{V}_{int} &= \text{total potential}\\ \mathcal{D} &= \text{dissipation function}\\ Q_s(t) &= \text{non-conservative external generalized forces}\end{aligned}$$

- They can also be written in terms of the inertia forces as follows

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}_2}{\partial \dot{q}_s}\right)-\frac{\partial \mathcal{T}_2}{\partial q_s}=Q_s(t)-\frac{\partial \mathcal{V}^{\star}}{\partial q_s}-\frac{\partial \mathcal{D}}{\partial \dot{q}_s}+F_s-\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{T}_1}{\partial \dot{q}_s}\right)$$

where

$$\begin{aligned}\mathcal{V}^{\star}=\mathcal{V}-\mathcal{T}_0 &= \text{potential modified by the tranport kinetic energy}\\ F_s=\sum_{r=1}^{n} g_{rs}\dot{q}_r &= \text{generalized gyroscopic forces}\end{aligned}$$