AA242B: MECHANICAL VIBRATIONS

Undamped Vibrations of n-DOF Systems

Outline

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Linear Vibrations

- Equilibrium configuration
  \[ q_s(t) = q_s(0), \quad \dot{q}_s(t) = 0, \quad s = 1, \ldots, n \]  
  (1)

- Recall the Lagrange equations of motion
  \[ -\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_s} \right) + \frac{\partial T}{\partial q_s} - \frac{\partial V}{\partial q_s} - \frac{\partial D}{\partial \dot{q}_s} + Q_s(t) = 0 \]
  where \( T = T_0 + T_1 + T_2 \)

- Recall the generalized gyroscopic forces
  \[ F_s = -\sum_{r=1}^{n} \frac{\partial^2 T_1}{\partial \dot{q}_s \partial q_r} \dot{q}_r + \frac{\partial T_1}{\partial q_s} = \sum_{r=1}^{n} \left( \frac{\partial^2 T_1}{\partial q_s \partial \dot{q}_r} - \frac{\partial^2 T_1}{\partial q_r \partial \dot{q}_s} \right) \dot{q}_r, \quad s = 1, \ldots, n \]

- **Definition**: the effective potential energy is defined as
  \[ V^* = V - T_0 = V^*(q, t) \]

- The Lagrange equations of motion can be re-written as
  \[ \frac{d}{dt} \left( \frac{\partial T_2}{\partial \dot{q}_s} \right) - \frac{\partial T_2}{\partial q_s} = Q_s(t) - \frac{\partial V^*}{\partial q_s} - \frac{\partial D}{\partial \dot{q}_s} + F_s - \frac{\partial}{\partial t} \left( \frac{\partial T_1}{\partial \dot{q}_s} \right) \]
Recall that
\[ T_0(q, t) = \frac{1}{2} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \left( \frac{\partial U_{ik}}{\partial t} (q, t) \right)^2 \] (transport kinetic energy)

\[ D = \sum_{k=1}^{N} \int_0^{v_k(q)} C_k f_k(\gamma) d\gamma \] (dissipation function)

From the Lagrange equations of motion
\[
\frac{d}{dt} \left( \frac{\partial T_2}{\partial \dot{q}_s} \right) - \frac{\partial T_2}{\partial q_s} = Q_s(t) - \frac{\partial V^*}{\partial q_s} (q, t) - \frac{\partial D}{\partial \dot{q}_s} + F_s - \frac{\partial}{\partial t} \left( \frac{\partial T_1}{\partial \dot{q}_s} \right)
\]
it follows that an equilibrium configuration exists if and only if
\[ 0 = Q_s(t) - \frac{\partial V^*}{\partial q_s} (q, t) ! \]

Hence, at equilibrium
\[ Q_s(t) = 0 \quad \text{and} \quad \frac{\partial V^*}{\partial q_s} = \frac{\partial (V - T_0)}{\partial q_s} = 0, \quad s = 1, \ldots, n \]
Consider *first* a system that does not undergo a transport or overall motion ⇒ $T = T_2(q)$

- The equilibrium position is then given by

\[
Q_s(t) = 0 \quad \text{and} \quad \frac{\partial V}{\partial q_s} = 0, \quad s = 1, \ldots, n
\]

Assume next that this system is conservative ⇒ $E = T + V = \text{cst}$

- Shift the origin of the generalized coordinates so that at equilibrium, $q_s = 0, \quad s = 1, \ldots, n$ (in which case the $q_s$ represent the deviation from equilibrium)

- Since the potential energy is defined only up to a constant, choose this constant so that $V(q_s = 0) = 0$

- Now, suppose that a certain energy $E(0)$ is initially given to the system in equilibrium
Linear Vibrations

Free-Vibrations About a Stable Equilibrium Position

**Definition**: the equilibrium position \((q_s = 0, s = 1, \cdots, n)\) is said to be **stable** if

\[ \exists \mathcal{E}^* \ / \ \forall \mathcal{E}(0) < \mathcal{E}^*, \mathcal{T}(t) \leq \mathcal{E}(0) \]

**Consequences**

- \(\mathcal{T} + \mathcal{V} = \mathcal{E} = \text{cst} = \mathcal{E}(0) \Rightarrow \mathcal{V}(t) = \mathcal{E}(0) - \mathcal{T}(t) \geq 0\)
- at a stable equilibrium position, the potential energy is at a relative minimum
- if \(\mathcal{E}(0)\) is small enough, \(\mathcal{V}(t)\) will be small enough \(\Rightarrow\) and therefore deviations from the equilibrium position will be small enough
"Linearization" of $T$ and $V$ around an equilibrium position

$(q_s = 0, \frac{\partial V}{\partial q_s} = 0)$

- actually, this means obtaining a quadratic form of $T$ and $V$ in $q$ and $\dot{q}$, respectively, so that the corresponding generalized forces are linear
- since $q_s(t)$ represent deviations from equilibrium, $V$ can be expanded as follows

$$V(q) = V(0) + \sum_{s=1}^{n} \frac{\partial V}{\partial q_s} \bigg|_{q=0} q_s + \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 V}{\partial q_s \partial q_r} \bigg|_{q=0} q_s q_r + O(q^3)$$

$$= V(0) + \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 V}{\partial q_s \partial q_r} \bigg|_{q=0} q_s q_r + O(q^3)$$

- since the potential energy is defined only up to a constant, if this constant is chosen so that $V(0) = 0$, then a second-order approximation of $V(q)$ is given by

$$V(q) = \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 V}{\partial q_s \partial q_r} \bigg|_{q=0} q_s q_r, \quad \text{for } q \neq 0$$
Stiffness matrix

- let $K = [k_{sr}]$ where $k_{sr} = k_{rs} = \frac{\partial^2 V}{\partial q_s \partial q_r} |_{q=0}$

$$\implies V(q) = \frac{1}{2} q^T K q > 0, \quad \text{for } q \neq 0 \implies K \text{ is symmetric positive definite}$$

- in the absence of sufficient boundary conditions – that is, in the presence of rigid body modes

$$\frac{1}{2} q^T K q \geq 0, \quad \text{for } q \neq 0 \implies K \text{ is symmetric positive semi-definite}$$
Recall that

\[ T_2 = \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} m_k \frac{\partial U_{ik}}{\partial q_s} \frac{\partial U_{ik}}{\partial q_r} \dot{q}_s \dot{q}_r \]  

(relative kinetic energy)

\[ T_2(q, \dot{q}) = T_2(0, 0) + \sum_{s=1}^{n} \frac{\partial T_2}{\partial q_s} \bigg|_{q=0, \dot{q}=0} q_s + \sum_{s=1}^{n} \frac{\partial T_2}{\partial \dot{q}_s} \bigg|_{q=0, \dot{q}=0} \dot{q}_s \]

\[ + \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 T_2}{\partial q_s \partial q_r} \bigg|_{q=0, \dot{q}=0} q_s q_r + \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 T_2}{\partial q_s \partial \dot{q}_r} \bigg|_{q=0, \dot{q}=0} \dot{q}_s \dot{q}_r \]

\[ + \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 T_2}{\partial \dot{q}_s \partial \dot{q}_r} \bigg|_{q=0, \dot{q}=0} \dot{q}_s \dot{q}_r + O(q^3, \dot{q}^3) \]

\[ = \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\partial^2 T_2}{\partial \dot{q}_s \partial \dot{q}_r} \bigg|_{q=0, \dot{q}=0} \dot{q}_s \dot{q}_r + O(q^3, \dot{q}^3) \]
Hence, a second-order approximation of $T_2(\dot{q})$ is given by

$$T_2(\dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q} > 0, \quad \text{for } \dot{q} \neq 0$$

where

$$M = \begin{bmatrix} m_{sr} = m_{rs} = \frac{\partial^2 T_2}{\partial \dot{q}_s \partial \dot{q}_r} \bigg|_{q=0} = \sum_{k=1}^{N} m_k \sum_{i=1}^{3} \frac{\partial U_{ik}}{\partial q_s} \bigg|_{q=0} \frac{\partial U_{ik}}{\partial q_r} \bigg|_{q=0} \end{bmatrix}$$

is the mass matrix and is symmetric positive definite.
Free-vibrations about a stable equilibrium position of a conservative system that does not undergo a transport or overall motion ($T_0 = T_1 = 0$)

\[
\frac{d}{dt} \left( \frac{\partial T_2}{\partial \dot{q}_s} \right) - \frac{\partial T_2}{\partial q_s} = -\frac{\partial V}{\partial q_s}
\]

\[\Rightarrow \frac{d}{dt} (M\ddot{q}) - 0 = -Kq\]

\[\Rightarrow M\ddot{q} + Kq = 0\]
Consider next the more general case of a system in steady motion (a transported system) whose equilibrium configuration defined by

$$\frac{\partial V^*}{\partial q_s} = \frac{\partial (V - T_0)}{\partial q_s} = 0, \quad s = 1, \ldots, n$$

corresponds to the balance of forces deriving from a potential \(\frac{\partial V}{\partial q_s}\) and centrifugal forces \(\frac{\partial T_0}{\partial q_s}\).

It is an equilibrium configuration in the sense that \(\dot{q}_s\) — which represent here the generalized velocities relative to a steady motion — are zero but the system is not idle.
Linear Vibrations

Free-Vibrations About an Equilibrium Configuration

- Linearizations
  - effective potential energy $\mathcal{V}^* \Rightarrow$ effective stiffness matrix $K^*$
    \[
    \mathcal{V}^*(\mathbf{q}) = \frac{1}{2} \mathbf{q}^T K^* \mathbf{q} > 0, \quad \text{for } \mathbf{q} \neq 0
    \]
    where $K^* = \left[ k_{sr}^* = k_{sr} - \left( \frac{\partial^2 \mathcal{T}_0}{\partial q_s \partial q_r} \right) \big|_{\mathbf{q}=0} \right]$

- mutual kinetic energy
  
  \[
  \mathcal{T}_1 = \sum_{s=1}^{n} \sum_{k=1}^{N} \sum_{i=1}^{3} \frac{\partial U_{ik}}{\partial t} m_k \frac{\partial U_{ik}}{\partial q_s} \dot{q}_s = \sum_{s=1}^{n} \dot{q}_s \frac{\partial \mathcal{T}_1}{\partial \dot{q}_s} (\mathbf{q})
  \]
  
  \[
  \approx \sum_{s=1}^{n} \dot{q}_s \left( \frac{\partial \mathcal{T}_1}{\partial q_s} (0) + \sum_{r=1}^{n} \frac{\partial^2 \mathcal{T}_1}{\partial q_s \partial q_r} \big|_{\mathbf{q}=0} q_r + \mathcal{O}(\mathbf{q}^2) \right) q_r \Rightarrow \mathcal{T}_1(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^T \mathbf{F} \mathbf{q}
  \]

  where $\mathbf{F} = \left[ f_{sr} = \frac{\partial^2 \mathcal{T}_1}{\partial \dot{q}_s \partial q_r} \big|_{\mathbf{q}=0} \right]$. 

Equations of free-vibration around an equilibrium configuration

- the equilibrium configuration generated by a steady motion remains stable as long as $V^* = V - T_0 \geq 0$
- this corresponds to the fact that $K^*$ remains positive definite
- in the neighborhood of such a configuration, the equations of motion (for a conservative system undergoing transport or overall motion) are

$$\frac{d}{dt} \left( \frac{\partial T_2}{\partial \dot{q}_s} \right) - \frac{\partial T_2}{\partial q_s} = Q_s(t) - \frac{\partial V^*}{\partial q_s} - \frac{\partial D}{\partial \dot{q}_s} + F_s - \frac{\partial}{\partial t} \left( \frac{\partial T_1}{\partial \dot{q}_s} \right)$$

where $F_s = \sum_{r=1}^{n} \left( \frac{\partial^2 T_1}{\partial q_s \partial \dot{q}_r} - \frac{\partial^2 T_1}{\partial q_r \partial \dot{q}_s} \right) \dot{q}_r = (F^T - F)\dot{q}$

$$\mathbf{M}\ddot{q} + \mathbf{G}\dot{q} + \mathbf{K}^*\mathbf{q} = 0$$

where $\mathbf{G} = \mathbf{F} - \mathbf{F}^T = -\mathbf{G}^T$ is the gyroscopic coupling matrix
Free-Vibrations About an Equilibrium Configuration

Example

\[ q = x \quad X = (a + x) \cos \Omega t \quad Y = (a + x) \sin \Omega t \quad v^2 = \dot{X}^2 + \dot{Y}^2 = (a + x)^2 \Omega^2 + x^2 \]

\[ \mathcal{V} = \frac{1}{2} kx^2 \quad T_0 = \frac{1}{2} \Omega^2 m(a + x)^2 \quad T_1 = 0 \quad T_2 = \frac{1}{2} mx^2 \quad \mathcal{V}^* = \frac{1}{2} kx^2 - \frac{1}{2} \Omega^2 m(a + x)^2 \]

- equilibrium configuration

\[ \frac{\partial \mathcal{V}^*}{\partial x} = 0 \implies kx - \Omega^2 m(a + x) = 0 \implies x_{eq} = \frac{\Omega^2 ma}{k - \Omega^2 m} \]

- the system becomes unstable for \( \Omega^2 = \frac{k}{m} \)

\[ k^* = \frac{\partial^2 \mathcal{V}^*}{\partial x^2} = k - \Omega^2 m \implies \text{system is unstable for } \Omega^2 \geq \frac{k}{m} \]
Natural Vibration Modes

- Free-vibration equations: \( M\ddot{q} + Kq = 0 \)
- \( q(t) = qa e^{i\omega t} \Rightarrow (K - \omega^2 M)qa = 0 \Rightarrow \det (K - \omega^2 M) = 0 \)
- If the system has \( n \) degrees of freedom (dofs), \( M \) and \( K \) are \( n \times n \) matrices \( \Rightarrow \) \( n \) eigenpairs \( (\omega_i^2, qa_i) \)
- Rigid body mode(s): \( \omega_j^2 = 0 \Rightarrow Kqa_j = 0 \)
- For a rigid body mode, \( V(qa_j) = \frac{1}{2} qa_j^T K qa_j = 0 \)
Consider two distinct eigenpairs \((\omega_i^2, q_{a_i})\) and \((\omega_j^2, q_{a_j})\)

\[
q_{a_j}^T K q_{a_i} = q_{a_j}^T \omega_i^2 M q_{a_i} \tag{2}
\]

\[
q_{a_i}^T K q_{a_j} = q_{a_i}^T \omega_j^2 M q_{a_j} \tag{3}
\]

Because \(M\) and \(K\) are symmetric

\[
(2) - (3)^T \Rightarrow 0 = (\omega_i^2 - \omega_j^2) q_{a_j}^T M q_{a_i}
\]

since \(\omega_i^2 \neq \omega_j^2 \Rightarrow q_{a_j}^T M q_{a_i} = 0\) and \(q_{a_j}^T K q_{a_i} = 0\)
Orthogonality of Natural Vibration Modes

Distinct Frequencies

Physical interpretation of the orthogonality conditions

\[ q_{aj}^T M q_{ai} = 0 \Rightarrow q_{aj}^T (\omega_i^2 M q_{ai}) = (\omega_i^2 M q_{ai})^T q_{aj} = 0 \]

which implies that the virtual work produced by the inertia forces of mode \( i \) during a virtual displacement prescribed by mode \( j \) is zero

\[ q_{aj}^T K q_{ai} = 0 \Rightarrow (K q_{ai})^T q_{aj} = 0 \]

which implies that the virtual work produced by the elastic forces of mode \( i \) during a virtual displacement prescribed by mode \( j \) is zero
Orthogonality of Natural Vibration Modes

Distinct Frequencies

- Rayleigh quotient

\[ Kq_{ai} = \omega_i^2 Mq_{ai} \Rightarrow q_{ai}^T K q_{ai} = \omega_i^2 q_{ai}^T M q_{ai} \Rightarrow \omega_i^2 = \frac{q_{ai}^T K q_{ai}}{q_{ai}^T M q_{ai}} = \frac{\gamma_i}{\mu_i} \]

- \( \gamma_i \) = generalized stiffness coefficient of mode \( i \) (measures the contribution of mode \( i \) to the elastic deformation energy)

- \( \mu_i \) = generalized mass coefficient of mode \( i \) (measures the contribution of mode \( i \) to the kinetic energy)

- Since the amplitude of \( q_{ai} \) is determined up to a factor only \( \Rightarrow \gamma_i \) and \( \mu_i \) are determined up to a constant factor only

- Mass normalization

\[ q_{aj}^T M q_{ai} = \delta_{ij} \]
\[ q_{aj}^T K q_{ai} = \omega_i^2 \delta_{ij} \]
What happens if a multiple circular frequency is encountered?

Theorem: to a multiple root $\omega_p^2$ of the system

$$Kq_a = \omega^2 Mq_a$$

*corresponds a number of linearly independent eigenvectors $\{q_{ai}\}$ equal to the root multiplicity*
Modal Superposition Analysis

- n-dof system: \( M \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^n \), and \( q \in \mathbb{R}^n \)
- Coupled system of ordinary differential equations

\[
\begin{align*}
\ddot{M}q + Kq &= 0 \\
q(0) &= q_0 \\
\dot{q}(0) &= \dot{q}_0
\end{align*}
\]
Modal Superposition Analysis

- Natural vibration modes (eigenmodes)

\[ Kq_{a_i} = \omega_i^2 Mq_{a_i}, \quad i = 1, \cdots, n \]

\[ Q = [q_{a_1} \quad q_{a_2} \quad \cdots \quad q_{a_n}] \Rightarrow \begin{cases} Q^T K Q = \Omega^2 \\ Q^T M Q = I \end{cases} \]

\[ \Omega^2 = \begin{pmatrix} \omega_1^2 \\ \vdots \\ \omega_n^2 \end{pmatrix} \]

- Truncated eigenbasis

\[ Q_r = [q_{a_1} \quad q_{a_2} \quad \cdots \quad q_{a_r}] , \quad r \ll n \Rightarrow \begin{cases} Q_r^T K Q_r = \Omega_r^2 \quad \text{(reduced stiffness matrix)} \\ Q_r^T M Q_r = I_r \quad \text{(reduced mass matrix)} \end{cases} \]

\[ \Omega_r^2 = \begin{pmatrix} \omega_1^2 \\ \vdots \\ \omega_r^2 \end{pmatrix} \]
Modal Superposition Analysis

- Modal superposition: \( \mathbf{q} = \mathbf{Q}_r \mathbf{y} = \sum_{i=1}^{r} y_i \mathbf{q}_i \) where

\[
\mathbf{y} = [y_1, y_2, \cdots, y_r]^T
\]
and \( y_i \) is called the modal displacement.

- Substitute in \( M\ddot{\mathbf{q}} + K\mathbf{q} = 0 \)

\[
\Rightarrow MQ_r \ddot{\mathbf{y}} + KQ_r \mathbf{y} = 0
\]

\[
\Rightarrow Q_r^T MQ_r \ddot{\mathbf{y}} + Q_r^T KQ_r \mathbf{y} = 0
\]

\[
\Rightarrow \mathbf{I}_r \ddot{\mathbf{y}} + \Omega_r^2 \mathbf{y} = 0
\]

- Uncoupled differential equations (modal equations)

\[
\ddot{y}_i + \omega_i^2 y_i = 0, \quad i = 1, \cdots r
\]
Modal Superposition Analysis

\[ \ddot{y}_i + \omega_i^2 y_i = 0, \quad i = 1, \ldots r \]

- **Case 1:** \( \omega_i^2 = 0 \) (rigid body mode)
  \[ y_i = a_i t + b_i \]

- **Case 2:** \( \omega_j^2 \neq 0 \)
  \[ y_j = c_j \cos \omega_j t + d_j \sin \omega_j t \]

- **General case:** \( r_b \) rigid body modes
  \[ \mathbf{q} = \sum_{i=1}^{r_b} (a_i t + b_i) \mathbf{q}_{a_i} + \sum_{j=1}^{r-r_b} (c_j \cos \omega_j t + d_j \sin \omega_j t) \mathbf{q}_{a_j} \]

- **Initial conditions**
  \[ \mathbf{q}(0) = \mathbf{q}_0 = Qy(0) \quad \text{and} \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 = Q\dot{y}(0) \]
Modal Superposition Analysis

\[
q_0 = Q_r y(0) \quad \text{and} \quad \dot{q}_0 = Q_r \dot{y}(0)
\]

\[
\implies q_{a_i}^T M q_0 = q_{a_i}^T M Q_r y(0)
\]

From the orthogonality properties of the natural mode shapes (eigenvectors) it follows that

\[
q_{a_i}^T M q_0 = q_{a_i}^T M Q_r y(0) = q_{a_i}^T M q_{a_i} y_i(0) = y_i(0) \implies y_i(0) = q_{a_i}^T M q(0)
\]

- **Case 1:** \( \omega_i^2 = 0 \) (rigid body mode) \( \implies a_i \times 0 + b_i = q_{a_i}^T M q(0) \implies b_i = q_{a_i}^T M q(0) \)

- **Case 2:** \( \omega_j^2 \neq 0 \) \( \implies c_j \times 1 + d_j \times 0 = q_{a_j}^T M q(0) \implies c_j = q_{a_j}^T M q(0) \)

- **Case 1:** \( \omega_i^2 = 0 \) (rigid body mode) \( \implies a_i = q_{a_i}^T M \dot{q}(0) \)

- **Case 2:** \( \omega_j^2 \neq 0 \) \( \implies d_j = \frac{1}{\omega_j} q_{a_j}^T M \dot{q}(0) \)

Thus, the general solution is

\[
q(t) = \sum_{i=1}^{r_b} \left( \left[ q_{a_i}^T M \dot{q}(0) \right] t + q_{a_i}^T M q(0) \right) q_{a_i} + \sum_{j=1}^{r-r_b} \left( q_{a_j}^T M q(0) \cos \omega_j t + q_{a_j}^T M \dot{q}(0) \frac{\sin \omega_j t}{\omega_j} \right) q_{a_j}
\]
Spectral Expansions

∀ \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{x} = \sum_{s=1}^{n} \alpha_s \mathbf{q}_{as} \Rightarrow \mathbf{q}_{aj}^T \mathbf{Mx} = \sum_{s=1}^{n} \alpha_s \mathbf{q}_{aj}^T \mathbf{Mq}_{as} = \alpha_j

\Rightarrow \forall \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{x} = \sum_{s=1}^{n} \left( \mathbf{q}_{as}^T \mathbf{Mx} \right) \mathbf{q}_{as} = \sum_{s=1}^{n} \mathbf{q}_{as} \left( \mathbf{q}_{as}^T \mathbf{Mx} \right) = \sum_{s=1}^{n} \left( \mathbf{q}_{as} \mathbf{q}_{as}^T \right) \mathbf{Mx} = \left( \sum_{s=1}^{n} \left( \mathbf{q}_{as} \mathbf{q}_{as}^T \right) \mathbf{M} \right) \mathbf{x}

\Rightarrow \sum_{s=1}^{n} \left( \mathbf{q}_{as} \mathbf{q}_{as}^T \right) \mathbf{M} = \mathbf{I}

- This is the same result as \( \mathbf{Q}^T \mathbf{MQ} = \mathbf{I} \)

- A given load \( \mathbf{p} \) can be expanded in terms of the inertia forces generated by the eigenmodes, \( \mathbf{Mq}_{aj} \) as follows

\[ \mathbf{p} = \sum_{j=1}^{n} \beta_j \mathbf{Mq}_{aj} \Rightarrow \mathbf{q}_{ai}^T \mathbf{p} = \sum_{j=1}^{n} \beta_j \mathbf{q}_{ai}^T \mathbf{Mq}_{aj} = \beta_i \]

\[ \Rightarrow \beta_i = \mathbf{q}_{ai}^T \mathbf{p} = \text{modal participation factor} \Rightarrow \mathbf{p} = \sum_{j=1}^{n} \left( \mathbf{q}_{aj}^T \mathbf{p} \right) \mathbf{Mq}_{aj} \]
Recall that \( \sum_{s=1}^{n} \left( q_{as} q_{as}^T \right) M = I \)

Hence, \( \forall A \in \mathbb{R}^n, \quad A = \sum_{s=1}^{n} A q_{as} q_{as}^T M \) and \( A = \sum_{s=1}^{n} q_{as} q_{as}^T M A \)

\( A = M \Rightarrow M = \sum_{s=1}^{n} M q_{as} q_{as}^T M = \sum_{s=1}^{n} M q_{as} (M q_{as})^T \) (because \( M \) is symmetric)

\( A = K \Rightarrow K = \sum_{s=1}^{n} K q_{as} q_{as}^T M = \sum_{s=1}^{n} \omega_s^2 M q_{as} q_{as}^T M = \sum_{s=1}^{n} \omega_s^2 M q_{as} (M q_{as})^T \)

\( A = M^{-1} \Rightarrow M^{-1} = \sum_{s=1}^{n} q_{as} q_{as}^T M M^{-1} \Rightarrow M^{-1} = \sum_{s=1}^{n} q_{as} q_{as}^T \)

\( A = K^{-1} \Rightarrow K^{-1} = \sum_{s=1}^{n} q_{as} q_{as}^T M K^{-1} = \sum_{s=1}^{n} q_{as} (M q_{as})^T K^{-1} \Rightarrow K^{-1} = \sum_{s=1}^{n} \frac{q_{as} q_{as}^T}{\omega_s^2} \)
Forced Harmonic Response

\[
\begin{align*}
M\ddot{q} + Kq &= s_a \cos \omega t \\
q(0) &= q_0 \\
\dot{q}(0) &= \dot{q}_0
\end{align*}
\]

Solution can be decomposed as

\[q = q_H \text{(homogeneous)} + q_P \text{(particular)}\]

\[q_p = q_a \cos \omega t \Rightarrow q_a = (K - \omega^2 M)^{-1} s_a\] where \((K - \omega^2 M)^{-1}\) is called the \textit{admittance} or \textit{dynamic influence} matrix

The forced response is the part of the response that is synchronous to the excitation — that is, \(q_p\)
Forced Harmonic Response

- Rigid body modes: \( \{u_{ai}\}_{i=1}^{rb} \)

- For all \( q_a \in \mathbb{R}^n \), \( q_a = \sum_{i=1}^{rb} \alpha_i u_{ai} + \sum_{j=1}^{n-rb} \beta_j q_{aj} \)

\[\Rightarrow s_a = (K - \omega^2 M)q_a = \sum_{i=1}^{rb} \alpha_i (K - \omega^2 M)u_{ai} + \sum_{j=1}^{n-rb} \beta_j (K - \omega^2 M)q_{aj}\]

\[\Rightarrow s_a = -\sum_{i=1}^{rb} \alpha_i \omega^2 Mu_{ai} + \sum_{j=1}^{n-rb} \beta_j (\omega_j^2 - \omega^2)Mq_{aj}\]

- Premultiply by \( u_{ai}^T \Rightarrow \alpha_j = -\frac{u_{ai}^T s_a}{\omega^2} \) and premultiply by \( q_{ai}^T \Rightarrow \beta_i = \frac{q_{ai}^T s_a}{(\omega_i^2 - \omega^2)} \)

\[\Rightarrow q_a = -\sum_{i=1}^{rb} \frac{u_{ai}^T s_a}{\omega^2} u_{ai} + \sum_{j=1}^{n-rb} \frac{q_{aj}^T s_a}{(\omega_j^2 - \omega^2)} q_{aj} = \left( -\sum_{i=1}^{rb} \frac{u_{ai}u_{ai}^T}{\omega^2} + \sum_{j=1}^{n-rb} \frac{q_{aj}q_{aj}^T}{(\omega_j^2 - \omega^2)} \right) s_a\]

- Since \( q_a = (K - \omega^2 M)^{-1}s_a \)

\[\Rightarrow (K - \omega^2 M)^{-1} = -\frac{1}{\omega^2} \sum_{i=1}^{rb} u_{ai}u_{ai}^T + \sum_{j=1}^{n-rb} \frac{q_{aj}q_{aj}^T}{\omega_j^2 - \omega^2}\]
Which excitation $s_{a m}$ will generate a harmonic response with an amplitude corresponding to $q_{a m}$?

$$q_{a m} = (K - \omega^2 M)^{-1}s_{a m} \Rightarrow s_{a m} = (K - \omega^2 M)q_{a m}$$

$$\implies s_{a m} = Kq_{a m} - \omega^2 Mq_{a m} = (\omega_m^2 - \omega^2)Mq_{a m}$$

At resonance ($\omega^2 = \omega_m^2$) $s_{a m} = 0 \Rightarrow$ no force is needed to maintain $q_{a m}$ once it is reached.
The inverse of an admittance is an impedance

\[ Z(\omega^2) = (K - \omega^2 M) \]
Forced Harmonic Response

- Application: substructuring (or domain decomposition)

- the dynamical behavior of a substructure is described by its harmonic response when forces are applied onto its interface boundaries.

- a subsystem is typically described by $\mathbf{K}$ and $\mathbf{M}$, has $n_1$ free dofs $\mathbf{q}_1$, and is connected to the rest of the system by $n_2 = n - n_1$ boundary dofs $\mathbf{q}_2$ where the reaction forces are denoted here by $\mathbf{g}_2$

\[
\begin{pmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{pmatrix}
\begin{pmatrix}
\mathbf{q}_1 \\
\mathbf{q}_2
\end{pmatrix}
= 
\begin{pmatrix}
\mathbf{0} \\
\mathbf{g}_2
\end{pmatrix}
\]

$\Rightarrow Z_{11}\mathbf{q}_1 + Z_{12}\mathbf{q}_2 = 0 \Rightarrow \mathbf{q}_1 = -Z_{11}^{-1}Z_{12}\mathbf{q}_2$

$\Rightarrow (Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})\mathbf{q}_2 = Z_{22}^*\mathbf{q}_2 = \mathbf{g}_2$

- $Z_{22}^*$ is the “reduced” impedance (reduced to the boundary)
Forced Harmonic Response

- Spectral expansion of $Z_{22}^*$
  - Look at $Z_{11}^{-1}$ and let $(\tilde{\omega}_i^2, \tilde{q}_{ai})$ denote the $n_1$ eigenpairs of the associated dynamical subsystem: from (4), it follows that
    \[
    Z_{11}^{-1} = \sum_{j=1}^{n_1} \frac{\tilde{q}_{aj} \tilde{q}_{aj}^T}{\tilde{\omega}_j^2 - \omega^2}
    \]
  - Apply twice the relation
    \[
    \frac{1}{\tilde{\omega}_j^2 - \omega^2} = \frac{1}{\tilde{\omega}_j^2} + \frac{\omega^2}{\tilde{\omega}_j^2 (\tilde{\omega}_j^2 - \omega^2)}
    \]
    \[
    \implies Z_{11}^{-1} = K_{11}^{-1} + \omega^2 \sum_{j=1}^{n_1} \frac{\tilde{q}_{aj} \tilde{q}_{aj}^T}{\tilde{\omega}_j^2 (\tilde{\omega}_j^2 - \omega^2)} = K_{11}^{-1} + \omega^2 \sum_{j=1}^{n_1} \frac{\tilde{q}_{aj} \tilde{q}_{aj}^T}{\tilde{\omega}_j^4} + \omega^4 \sum_{j=1}^{n_1} \frac{\tilde{q}_{aj} \tilde{q}_{aj}^T}{\tilde{\omega}_j^4 (\tilde{\omega}_j^2 - \omega^2)}
    \]
  - Owing to the $M_{11}$-orthonormality of the modes and the spectral expansion of $K_{11}^{-1}$, the above expression can further be written as
    \[
    Z_{11}^{-1} = K_{11}^{-1} + \omega^2 K_{11}^{-1} M_{11} K_{11}^{-1} + \omega^4 \sum_{j=1}^{n_1} \frac{\tilde{q}_{aj} \tilde{q}_{aj}^T}{\tilde{\omega}_j^4 (\tilde{\omega}_j^2 - \omega^2)}
    \]
    \[
    \implies Z_{22}^* = K_{22} - K_{21} K_{11}^{-1} K_{12} - \omega^2 [M_{22} - M_{21} K_{11}^{-1} K_{12} - K_{21} K_{11}^{-1} M_{12} + K_{21} K_{11}^{-1} M_{11} K_{11}^{-1} K_{12}] - \omega^4 \sum_{i=1}^{n_1} \frac{[\tilde{\omega}_i^2 M_{21} \tilde{q}_{ai}][\tilde{\omega}_i^2 M_{21}\tilde{q}_{ai}]}{\tilde{\omega}_i^4 (\tilde{\omega}_i^2 - \omega^2)}
    \]
Forced Harmonic Response

\[ Z_{22}^* = K_{22} - K_{21} K_{11}^{-1} K_{12} - \omega^2 [M_{22} - M_{21} K_{11}^{-1} K_{12} - K_{21} K_{11}^{-1} M_{12} + K_{21} K_{11}^{-1} M_{11} K_{11}^{-1} K_{12}] \\
-\omega^4 \sum_{i=1}^{n_1} \frac{[(K_{21} - \tilde{\omega}_i^2 M_{21}) \tilde{q}_{ai}]^T [(K_{21} - \tilde{\omega}_i^2 M_{21}) \tilde{q}_{ai}]}{\tilde{\omega}_i^4 (\tilde{\omega}_i^2 - \omega^2)} \]

- The first term \( K_{22} - K_{21} K_{11}^{-1} K_{12} \) represents the stiffness of the statically condensed system.

- The second term \( M_{22} - M_{21} K_{11}^{-1} K_{12} - K_{21} K_{11}^{-1} M_{12} + K_{21} K_{11}^{-1} M_{11} K_{11}^{-1} K_{12} \) represents the mass of the subsystem statically condensed on the boundary.

- The last term represents the contribution of the subsystem eigenmodes since it is generated by \( \tilde{q}_{ai} \tilde{q}_{ai}^T \).

- \( (K_{21} - \tilde{\omega}_i^2 M_{21}) \tilde{q}_{ai} \) is the dynamic reaction on the boundary.
Response to External Loading

\[
\begin{align*}
\{ \quad M\ddot{q} + Kq &= p(t) \\
q(0) &= q_0 \\
\dot{q}(0) &= \dot{q}_0 
\end{align*}
\]

General approach

- Consider the simpler case where there is no rigid body mode \( \Rightarrow \) eigenmodes \( (q_{a_i}, \omega_i^2), \omega_i^2 \neq 0, i = 1, \cdots, n \)
- Modal superposition: \( q = Qy = \sum_{i=1}^{n} y_i q_{a_i} \)
- Substitute in equations of dynamic equilibrium

\[
\Longrightarrow MQ\ddot{y} + KQy = p(t) \Rightarrow Q^T MQ\ddot{y} + Q^T KQy = Q^T p(t)
\]

- Modal equations

\[
\ddot{y}_i + \omega_i^2 y_i = q_{a_i}^T p(t), \quad i = 1, \cdots, n
\]

- \( y_i(t) \) depend on two constants that can be obtained from the initial conditions
- \( q(0) = Qy(0), \dot{q}(0) = Q\dot{y}(0) \) \( \Rightarrow \) orthogonality conditions

\[ y_i(0), \dot{y}_i(0) \]
Response to an impulsive force

- spring-mass system: \( m, k, \omega^2 = \frac{k}{m} \)
- impulsive force \( f(t) \): force whose amplitude could be infinitely large but which acts for a very short duration of time
- magnitude of impulse: \( I = \int_{-\epsilon}^{\epsilon} f(t) dt \)
- impulsive force \( = I\delta(t) \) where \( \delta \) is the “delta” function centered at \( t = 0 \) and satisfying \( \int_{0}^{\epsilon} \delta(t) dt = 1 \)
Response to an impulsive force (continue)

- dynamic equilibrium

\[ m \int_{\tau}^{\tau+\epsilon} \frac{d\dot{u}}{dt} \, dt + k \int_{\tau}^{\tau+\epsilon} \dot{u} \, dt = \int_{\tau}^{\tau+\epsilon} f(t) \, dt = l \]

- assume that at \( t = \tau \), the system is at rest (\( u(\tau) = 0 \) and \( \dot{u}(\tau) = 0 \))
- focus on the short (infinitesimal) interval of time \( d\tau \)

\[ \ddot{u} \approx A/\epsilon \Rightarrow \int_{\tau}^{\tau+\epsilon} \ddot{u} \, dt \approx A \text{ and } \int_{\tau}^{\tau+\epsilon} u \, dt \approx A\epsilon^2/6 \]

- hence

\[ m \int_{\tau}^{\tau+\epsilon} \frac{d\dot{u}}{dt} \, dt \approx I \Rightarrow m\Delta\dot{u} \approx I \Rightarrow \Delta\dot{u} = \dot{u}(\tau + \epsilon) - \dot{u}(\tau) \approx \frac{l}{m} \Rightarrow \dot{u}(\tau + \epsilon) \approx \frac{l}{m} \]

\[ \int_{\tau}^{\tau+\epsilon} \dot{u} \, dt \approx 0 \Rightarrow \Delta u \approx 0 \Rightarrow u(\tau + \epsilon) - u(\tau) \approx 0 \Rightarrow u(\tau + \epsilon) \approx 0 \]

- the above equations provide initial conditions for the free-vibrations that start at the end of the impulsive load

\[ \Rightarrow u(t) \approx \frac{l}{m\omega} \sin \omega t \text{ for impulses of finite duration} \]
Response to an impulsive force (continue)

for the differential time interval \( d\tau \) \((d\tau \to 0)\), the response analysis of the previous page becomes exact

\[
\Rightarrow du = \frac{f(\tau)d\tau}{m\omega} \sin \omega(t - \tau)
\]

linear system \(\Rightarrow\) superposition principle

\[
du = \frac{f(\tau)d\tau}{m\omega} \sin \omega(t - \tau) \Rightarrow u(t) = \frac{1}{m\omega} \int_0^t f(\tau) \sin \omega(t - \tau) d\tau
\]
Response to External Loading

- Time-integration of the normal equations
  - let $p_i(t) = q^T_{ai}p(t) = i$-th modal participation factor
  - modal or normal equation: $\ddot{y}_i + \omega_i^2 y_i = p_i(t)$
  - $y_i(t) = y^H_i(t) + y^P_i(t)$, $y^H_i = A_i \cos \omega_i t + B_i \sin \omega_i t$
  - the particular solution $y^P_i(t)$ depends on the form of $p_i(t)$
  - however, the general form of a particular solution that satisfies the rest initial conditions is given by the Duhamel’s integral

$$y^P_i(t) = \frac{1}{\omega_i} \int_0^t p_i(\tau) \sin \omega_i(t - \tau) d\tau$$

- complete solution

$$y_i(t) = A_i \cos \omega_i t + B_i \sin \omega_i t + \frac{1}{\omega_i} \int_0^t p_i(t) \sin \omega_i(t - \tau) d\tau$$

- $A_i = y_i(0), B_i = \frac{\dot{y}_i(0)}{\omega_i}$ and $y_i(0)$ and $\dot{y}_i(0)$ can be determined from the initial conditions $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$ and the orthogonality conditions.
Response truncation and mode displacement method

- computational efficiency \( \Rightarrow q = Q_r y = \sum_{i=1}^{r} y_i q_{ai}, \quad r \ll n \)
- what is the effect of modal truncation?
- consider the case where \( p(t) = g_{\text{static load distribution}} \times \phi(t) \)
- for a system initially at rest \( (q(0) = 0 \text{ and } \dot{q}(0) = 0) \)
  \( y_i(0) = q_{ai}^T M q(0) = 0 \text{ and } \dot{y}_i(0) = q_{ai}^T M \dot{q}(0) = 0 \Rightarrow A_i = B_i = 0 \)

\[
\Rightarrow y_i(t) = \frac{1}{\omega_i} \int_{0}^{t} p_i(\tau) \sin \omega_i(t-\tau) d\tau = \frac{q_{ai}^T g}{\omega_i} \int_{0}^{t} \phi(\tau) \sin \omega_i(t - \tau) d\tau
\]

\[
\Rightarrow q(t) = \sum_{i=1}^{r} \left[ q_{ai} q_{ai}^T g \right] \left[ \frac{1}{\omega_i} \int_{0}^{t} \phi(\tau) \sin \omega_i(t - \tau) d\tau \right]
\]
Response truncation and mode displacement method (continue)

- general solution for restrained structure initially at rest

\[ q(t) = \sum_{i=1}^{r} \left[ q_{a_i} q_{a_i}^T g \right] \left[ \frac{1}{\omega_i} \int_0^t \phi(\tau) \sin \omega_i(t - \tau) d\tau \right] \]

- truncated response is accurate if neglected terms are small, which is true if:
  - \( q_{a_i} q_{a_i}^T g \) is small for \( i = r + 1, \cdots, n \) \( \Rightarrow g \) is well approximated in the range of \( Q_r \)
  - \( \frac{1}{\omega_j} \int_0^t \phi(\tau) \sin \omega_j(t - \tau) d\tau \) is small for \( j > r \), which depends on the frequency content of \( \phi(t) \)

\[ \phi(t) = 1 \quad \Rightarrow \quad \theta_i(t) = \frac{1 - \cos \omega_i t}{\omega_i} \to 0 \text{ for large circular frequencies} \]

\[ \phi(t) = \sin \omega t \quad \Rightarrow \quad \theta_i(t) = \frac{\omega_i \sin \omega t - \omega \sin \omega_i t}{\omega_i(\omega_i^2 - \omega^2)} \]
Response to External Loading

- Mode acceleration method

\[ M\ddot{q} + Kq = p(t) \implies Kq = p(t) - M\ddot{q} \]

- apply truncated modal representation to the acceleration

\[ q(t) = Q_r y(t) \implies \dot{q}(t) = Q_r \dot{y}(t) \]

\[ \implies Kq = p(t) - \sum_{i=1}^{r} Mq_{a_i} \ddot{y}_i \implies q = K^{-1} p(t) - \sum_{i=1}^{r} \frac{q_{a_i}}{\omega_i^2} \ddot{y}_i \]

- recall that

\[ \ddot{y}_i + \omega_i^2 y_i = q_{a_i}^T p(t) \tag{5} \]

- and that for a system initially at rest

\[ y_i(t) = \frac{1}{\omega_i} \int_0^t q_{a_i}^T p(\tau) \sin \omega_i(t - \tau) d\tau \tag{6} \]

- (5) and (6) \implies \ddot{y}_i(t) = q_{a_i}^T (p(t) - \omega_i \int_0^t p(\tau) \sin \omega_i(t - \tau) d\tau)
Mode acceleration method (continue)

- substitute in $q(t) = K^{-1}p(t) - \sum_{i=1}^{r} \frac{q_{a_i}}{\omega_i^2} \ddot{y}_i$

$$\implies q(t) = \sum_{i=1}^{r} \frac{q_{a_i}}{\omega_i} q_{a_i}^T \int_0^t p(\tau) \sin \omega_i (t-\tau) d\tau + \left( K^{-1} - \sum_{i=1}^{r} \frac{q_{a_i} q_{a_i}^T}{\omega_i^2} \right) p(t)$$

- recall the spectral expansion $K^{-1} = \sum_{i=1}^{n} \frac{q_{a_i} q_{a_i}^T}{\omega_i^2}$

$$\implies q(t) = \sum_{i=1}^{r} \frac{q_{a_i} q_{a_i}^T}{\omega_i} \int_0^t p(\tau) \sin \omega_i (t-\tau) d\tau + \left( \sum_{i=r+1}^{n} \frac{q_{a_i} q_{a_i}^T}{\omega_i^2} \right) p(t)$$

- which shows that the mode acceleration method complements the truncated mode displacement solution with the missing terms using the modal expansion of the static response

- how to deal with the computational cost issue?
Response to External Loading

- Direct time-integration methods for solving

\[ \begin{align*}
    \ddot{q} + Kq &= p(t) \\
    q(0) &= q_0 \\
    \dot{q}(0) &= \dot{q}_0
\end{align*} \]

- will be covered towards the end of this course
The general case

\[
\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}
\]

where \( q_2 \) is prescribed

\[
\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ r_2(t) \end{pmatrix}
\]

The first equation gives

\[
M_{11}\ddot{q}_1 + K_{11}q_1 = -K_{12}q_2 - M_{12}\ddot{q}_2 \Rightarrow q_1(t)
\]

Substitute in second equation

\[
\Rightarrow r_2(t) = K_{21}q_1 + M_{21}\ddot{q}_1 + K_{22}q_2 + M_{22}\ddot{q}_2
\]
Quasi-static response of \( q_1 \)

\[
0 + K_{11} q_1 = -K_{12} q_2 - 0
\]

\[
\Rightarrow q_{qs}^{1} = -K^{-1}_{11} K_{12} q_2 = S q_2
\]

Decompose \( q_1 = q_{qs}^{1} + z_1 = S q_2 + z_1 \)

\[
\Rightarrow q(t) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \begin{pmatrix} z_1 \\ q_2 \end{pmatrix}
\]

where \( z_1 \) represents the sole dynamics part of the response

Substitute in the first dynamic equation and exploit above results

\[
\Rightarrow M_{11} S \ddot{q}_2 + M_{11} \ddot{z}_1 + K_{11} q_{qs}^{1} + K_{11} z_1 = -K_{12} q_2 - M_{12} \ddot{q}_2
\]

\[
\Rightarrow \begin{cases} 
M_{11} \ddot{z}_1 + K_{11} z_1 & = g_1(t) \\
- M_{11} \ddot{q}_{qs}^{1} - M_{12} \ddot{q}_2 & = -(M_{11} S + M_{12}) \ddot{q}_2
\end{cases}
\]
Consider next the system fixed to the ground and solve the corresponding EVP

\[ K_{11} x = \omega^2 M_{11} x \Rightarrow \tilde{Q} = [\cdots \tilde{q}_a, \cdots] \]

\[ \Rightarrow z_1(t) = \tilde{Q}\eta(t) \]

Solve \( M_{11}\ddot{z}_1 + K_{11}z_1 = g_1(t) \) for \( z_1(t) \) using the modal superposition technique.
Case of a global support acceleration $\ddot{q}_2(t) = u_2 \phi(t)$, where $u = [u_1 \ u_2]^T$ denotes a rigid body mode.

- Decomposition of the solution into a rigid body motion and a relative displacement $\mathbf{y}$

$$\ddot{q} = \ddot{q}^{rb} + \mathbf{y} \Rightarrow \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \phi(t) + \begin{bmatrix} \ddot{y}_1 \\ 0 \end{bmatrix}$$

- $u_1 = S u_2 \Rightarrow g_1(t) = -(M_{11} S u_2 + M_{12} u_2) \phi(t)$

$$\Rightarrow g_1(t) = -(M_{11} u_1 + M_{12} u_2) \phi(t)$$
Mechanical Systems Excited Through Support Motion

- Method of additional masses (approximate method)
  - suppose the system is subjected not to a specified $q_2(t)$ but to an imposed
    
    \[
    \begin{pmatrix}
      0 \\
      f(t)
    \end{pmatrix}
    \]
    
  - suppose that masses associated with $q_2$ are increased to $M_{22} + M_{22}^*$
  - then
    
    \[
    \begin{pmatrix}
      M_{11} & M_{12} \\
      M_{21} & M_{22} + M_{22}^*
    \end{pmatrix}
    \begin{pmatrix}
      \ddot{q}_1 \\
      \ddot{q}_2
    \end{pmatrix} + \begin{pmatrix}
      K_{11} & K_{12} \\
      K_{21} & K_{22}
    \end{pmatrix}\begin{pmatrix}
      q_1 \\
      q_2
    \end{pmatrix} = \begin{pmatrix}
      0 \\
      f(t)
    \end{pmatrix}
    \]
    
  - by elimination one obtains
    \[
    \ddot{q}_2 = (M_{22} + M_{22}^*)^{-1}(f(t) - K_{22}q_2 - K_{21}q_1 - M_{21}\ddot{q}_1)
    \]
    
    and
    
    \[
    M_{11}\ddot{q}_1 + K_{11}q_1 = -K_{12}q_2 - M_{12}(M_{22} + M_{22}^*)^{-1}(f(t) - K_{22}q_2 - K_{21}q_1 - M_{21}\ddot{q}_1)
    \]
Method of additional masses (continue)

therefore

\[
\begin{align*}
\{M_{11} - M_{12}(M_{22} + M_{22}^*)^{-1}M_{21}\} \ddot{q}_1 + \left\{K_{11} - M_{12}(M_{22} + M_{22}^*)^{-1}K_{21}\right\}q_1 = -K_{12}q_2 - M_{12}(M_{22} + M_{22}^*)^{-1}(f(t) - K_{22}q_2)
\end{align*}
\]

now if \((M_{22} + M_{22}^*)^{-1} \rightarrow 0\) and if \(f(t) = (M_{22} + M_{22}^*)\ddot{q}_2\), one obtains

\[
M_{11}\ddot{q}_1 + K_{11}q_1 = -K_{12}q_2 - M_{12}\ddot{q}_2 \implies M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + K_{11}q_1 + K_{12}q_2 = 0
\]

which is the same equation as in the general case of excitation through support motion.

⇒ for \(M_{22}^*\) very large, the response of the system with ground motion is the same as the response of the system with added lumped mass \(M_{22}^*\) and the forcing function

\[
\begin{pmatrix}
0 \\ (M_{22} + M_{22}^*)\ddot{q}_2
\end{pmatrix} \approx \begin{pmatrix}
0 \\ M_{22}^*\ddot{q}_2
\end{pmatrix}
\]

where \(\ddot{q}_2\) is given ⇒ apply same solution method as for problems of response to external loading