AA242B: MECHANICAL VIBRATIONS

Damped Vibrations of n-DOF Systems

These slides are based on the recommended textbook: M. Géradin and D. Rixen, "Mechanical Vibrations: Theory and Applications to Structural Dynamics," Second Edition, Wiley, John & Sons, Incorporated, ISBN-13:9780471975465

Outline

- Assume that the damping mechanism can be described by a viscous, quadratic, dissipation function in the generalized velocities

$$\mathbf{D} = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{C}\dot{\mathbf{q}} \geq 0$$

where **C** is a symmetric[1] and non-negative damping matrix

- Lagrange's equations

$$\Longrightarrow \boxed{\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t)}$$

- Consider again the eigenpairs $(\omega_i^2, \mathbf{q}_{a_i}), i = 1, \cdots, n$ of the *undamped* system
- Look for a solution of the damped equations of dynamic equilibrium of the form

$$\mathbf{q} = \sum_{i=1}^{n} y_i(t)\mathbf{q}_{a_i}$$

[1] Symmetry is required to obtain a conservative system (or obtain that $\mathcal{D}$ is a homogeneous function of degree 2 in $\dot{q}$).

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t); \qquad \mathbf{q} = \sum_{i=1}^{n} y_i(t)\mathbf{q}_{a_i} = \mathbf{Q}\mathbf{y}(t)$$

$$\Longrightarrow \mathbf{Q}^T\mathbf{M}\mathbf{Q}\ddot{\mathbf{y}} + \mathbf{Q}^T\mathbf{C}\mathbf{Q}\dot{\mathbf{y}} + \mathbf{Q}^T\mathbf{K}\mathbf{Q}\mathbf{y} = \mathbf{Q}^T\mathbf{p}(t)$$

$$\Longrightarrow \mathbf{I}\ddot{\mathbf{y}} + \mathbf{Q}^T\mathbf{C}\mathbf{Q}\dot{\mathbf{y}} + \mathbf{\Omega}^2\mathbf{y} = \mathbf{Q}^T\mathbf{p}(t)$$

- In general, $\mathbf{Q}^T\mathbf{C}\mathbf{Q} = [\beta_{ij}]$ – where $\beta_{ij} = \mathbf{q}_{a_i}^T\mathbf{C}\mathbf{q}_{a_j}$ – is a full matrix
- Hence,

$$\ddot{y}_i + \sum_{j=1}^{n} \beta_{ij}\dot{y}_j + \omega_i^2 y_i = \mathbf{q}_{a_i}^T\mathbf{p}(t), \qquad i = 1, \cdots, n$$

- The above equation shows that unless some assumptions are introduced, the method of modal superposition is not that interesting for solving the damped equations of dynamic equilibrium, because the resulting modal equations are coupled

- However, if a small number of modes $m \ll n$ suffices to compute an accurate solution, the modal superposition technique can still be interesting because in this case, the size of the modal equations is much smaller than that of the original equations (reduced-order modeling)

$$\underset{n\times n}{\mathbf{C}} \longrightarrow \underbrace{\underset{m\times n}{\mathbf{Q}^T} \underset{n\times n}{\mathbf{C}} \underset{n\times m}{\mathbf{Q}}}_{m\times m}$$

$$\Longrightarrow \ddot{y}_i + \sum_{j=1}^{m\ll n} \beta_{ij}\dot{y}_j + \omega_i^2 y_i = \mathbf{q}_{a_i}^T \mathbf{p}(t), \qquad i = 1, \cdots, m$$

- If the structure is lightly damped, a diagonal matrix $\mathbf{Q}^T\mathbf{C}\mathbf{Q}$ is a consistent even though not a physical assumption
  - consider $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$
  - search for a solution of the form $\mathbf{q} = \mathbf{z}_a e^{\lambda t}$

$$\Longrightarrow (\lambda_k^2\mathbf{M} + \lambda_k\mathbf{C} + \mathbf{K})\mathbf{z}_{a_k} = \mathbf{0}$$

  - without damping, one would have $\lambda_k = \pm i\omega_k$ and $\mathbf{z}_{a_k} = \mathbf{q}_{a_k}$
  - if the system is lightly damped, it can be assumed that λ_k and $\mathbf{z}_{a_k}$ differ only slightly from ω_k and $\mathbf{q}_{a_k}$, respectively

$$\lambda_k = i\omega_k + \Delta\lambda \qquad \mathbf{z}_{a_k} = \mathbf{q}_{a_k} + \Delta\mathbf{z}$$

  - substituting in the characteristic equation and neglecting the second-order terms gives

$$(\mathbf{K} - \omega_k^2\mathbf{M})\Delta\mathbf{z} + (i2\omega_k\mathbf{M} + \mathbf{C})\mathbf{q}_{a_k}\Delta\lambda + i\omega_k\mathbf{C}(\mathbf{q}_{a_k} + \Delta\mathbf{z}) \approx \mathbf{0}$$

  - the light-damping assumption allows one to neglect the terms $\mathbf{C}\Delta\lambda$ and $\mathbf{C}\Delta\mathbf{z}$ in the above equation

$$\Longrightarrow (\mathbf{K} - \omega_k^2\mathbf{M})\Delta\mathbf{z} + i\omega_k(\mathbf{C} + 2\Delta\lambda\mathbf{M})\mathbf{q}_{a_k} \approx \mathbf{0}$$

- If the structure is lightly damped, a diagonal matrix $\mathbf{Q}^T\mathbf{C}\mathbf{Q}$ is a consistent even though not a physical assumption (continue)

$$\Longrightarrow (\mathbf{K} - \omega_k^2\mathbf{M})\Delta\mathbf{z} + i\omega_k(\mathbf{C} + 2\Delta\lambda\mathbf{M})\mathbf{q}_{a_k} \approx \mathbf{0}$$

$$\Longrightarrow \mathbf{q}_{a_k}^T(\mathbf{K} - \omega_k^2\mathbf{M})\Delta\mathbf{z} + \mathbf{q}_{a_k}^T i\omega_k(\mathbf{C} + 2\Delta\lambda\mathbf{M})\mathbf{q}_{a_k} \approx 0$$

$$\Longrightarrow \mathbf{q}_{a_k}^T(\mathbf{C} + 2\Delta\lambda\mathbf{M})\mathbf{q}_{a_k} \approx 0$$

$$\Longrightarrow \Delta\lambda \approx -\frac{1}{2}\beta_{kk}$$

$$\Longrightarrow \boxed{\lambda_k \approx -\frac{1}{2}\beta_{kk} + i\omega_k}$$

  - the first-order correction involves only the diagonal damping terms $\beta_{kk} = \mathbf{q}_{a_k}^T\mathbf{C}\mathbf{q}_{a_k}$, and thus the influence of the non-diagonal terms is only second-order

- If the structure is lightly damped, a diagonal matrix $\mathbf{Q}^T\mathbf{C}\mathbf{Q}$ is a consistent even though not a physical assumption (continue)
  - it is also possible to obtain the eigenmode correction as follows

$$\Delta \mathbf{z} = \sum_{\substack{j=1 \\ j\neq k}}^{n} \alpha_j \mathbf{q}_{a_j}$$

$$\forall l \neq k, \quad \mathbf{q}_{a_l}^T(\mathbf{K} - \omega_k^2\mathbf{M})\sum_{\substack{j=1 \\ j\neq k}}^{n} \alpha_j \mathbf{q}_{a_j} + \mathbf{q}_{a_l}^T i\omega_k(\mathbf{C} + 2\Delta\lambda\mathbf{M})\mathbf{q}_{a_k} \approx 0$$

$$\Longrightarrow \boxed{\alpha_l = \frac{i\omega_k\beta_{kl}}{\omega_k^2 - \omega_l^2}} \qquad \Longrightarrow \qquad \boxed{\mathbf{z}_{a_k} = \mathbf{q}_{a_k} + i\sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{\omega_k\beta_{kj}}{\omega_k^2 - \omega_j^2}\mathbf{q}_{a_j}}$$

  - note that the above expression for $\mathbf{z}_{a_k}$ is valid only if
    - the coefficients β_{kj} are first-order quantities
    - the undamped eigenfrequencies are well separated

- If the structure is lightly damped, a diagonal matrix $\mathbf{Q}^T\mathbf{C}\mathbf{Q}$ is a consistent even though not a physical assumption (continue)

$$\Delta \mathbf{z} = \sum_{\substack{j=1 \\ j\neq k}}^{n} \alpha_j \mathbf{q}_{a_j} = i \sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{\omega_k \beta_{kj}}{\omega_k^2 - \omega_j^2} \mathbf{q}_{a_j} \qquad (1)$$

  - note also that
    - $\Delta \mathbf{z}$ is pure imaginary $\Rightarrow$ a free-vibration of the damped system is no longer a synchronous motion of the whole system

$$\mathbf{q}_k = \mathbf{z}_{a_k} e^{\lambda_k t} = \left(\mathbf{q}_{a_k} + i\Im(\Delta \mathbf{z})\right) e^{(i\omega_k + \Delta\lambda)t} \qquad (2)$$

$\Longrightarrow$ dof q_i is no longer in phase with another dof q_j!

    - from (1), it follows that for lightly damped systems with well-separated undamped eigenfrequencies, $\mathbf{z}_{a_k} = \mathbf{q}_{a_k} + \Delta \mathbf{z} \approx \mathbf{q}_{a_k}$ and therefore (2) takes the form

$$\mathbf{q}_k \approx \mathbf{q}_{a_k} e^{(i\omega_k \Delta\lambda)t}$$

which is consistent with the approximation

$$\mathbf{q}_{a_k}^T \mathbf{C} \mathbf{q}_{a_l} = \beta_{kj}\delta_{kj}$$

- In summary, the lightly-damped assumption simplifies the modal equations to

$$\ddot{y}_i + \beta_i \dot{y}_i + \omega_i^2 y_i = \mathbf{q}_{a_i}^T \mathbf{p}(t), \qquad i = 1, \cdots, n$$

where

$$\beta_i \equiv \beta_{ii} = \mathbf{q}_{a_i}^T \mathbf{C} \mathbf{q}_{a_i}$$

in which case the modal damping coefficient is defined as

$$\boxed{\xi_i = \frac{\beta_i}{2\omega_i}}$$

$$\Longrightarrow \boxed{\ddot{y}_i + 2\xi_i \omega_i \dot{y}_i + \omega_i^2 y_i = \mathbf{q}_{a_i}^T \mathbf{p}(t), \qquad i = 1, \cdots, n}$$

- A simple method for constructing a diagonal damping matrix is the so-called Rayleigh proportional damping

$$\boxed{\mathbf{C} = a\mathbf{K} + b\mathbf{M}}$$

$$\Longrightarrow \boxed{\beta_{ij} = \mathbf{q}_{a_i}^T \mathbf{C} \mathbf{q}_{a_j} = (a\omega_i^2 + b)\delta_{ij}; \qquad \beta_i \equiv \beta_{ii}}$$

$$\Longrightarrow \boxed{\xi_i = \frac{1}{2}(a\omega_i + \frac{b}{\omega_i})}$$

- Whenever possible, the modal coefficients ξ_i should be determined from experimental vibration testing

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}_a e^{i\omega t}$$

- After a certain amount of time, the homogeneous response is damped out $\Rightarrow$ response can be limited to the forced term (particular solution) $\mathbf{q} = \mathbf{z}_a e^{i\omega t}$

$$\Longrightarrow (\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C})\mathbf{z}_a = \mathbf{f}_a$$

- Develop the solution in the form $\mathbf{z}_a = \sum_{l=1}^{n} y_l \mathbf{q}_{a_l}$

$$\Longrightarrow (\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C}) \sum_{l=1}^{n} y_l \mathbf{q}_{a_l} = \mathbf{f}_a$$

- Pre-multiply by $\mathbf{q}_{a_j}^T$

$$\Longrightarrow y_j = \frac{\mathbf{q}_{a_j}^T \mathbf{f}_a}{\omega_j^2 - \omega^2 + i2\xi_j\omega\omega_j}$$

- Note: $(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C})^{-1}$ is the dynamic influence (or admittance) matrix
- Since $\mathbf{z}_a = (\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C})^{-1}\mathbf{f}_a$ and $\mathbf{z}_a = \sum_{j=1}^{n} y_j \mathbf{q}_{a_j}$ with $y_j = \dfrac{\mathbf{q}_{a_j}^T \mathbf{f}_a}{\omega_j^2 - \omega^2 + i2\xi_j\omega\omega_j}$, it follows that the spectral expansion of this matrix in terms of the *undamped* modes is given by

$$\boxed{(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C})^{-1} = \sum_{j=1}^{n} \frac{\mathbf{q}_{a_j}\mathbf{q}_{a_j}^T}{\omega_j^2 - \omega^2 + i2\xi_j\omega\omega_j}}$$

- Observe that if $\omega \to 0$, the right hand-side converges to $\mathbf{K}^{-1}$, which is consistent with the limit when $\omega \to 0$ of the left hand-side

- Vibration test $\Rightarrow$ modal characteristics (natural modes and frequencies)
- Procedure: force the system to vibrate along successive modes by tuning the frequency and relative force amplitude of the excitation
- There is no direct method to determine the appropriate excitation of a given vibration mode: it must be obtained through successive approximations from criteria such as the *phase lag quadrature* or the *stationary nature of reactive power* to verify that the mode appropriation is effectively achieved

- Phase lag quadrature criterion
  - damped system, harmonic vibration test
  - $(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C})\mathbf{z}_a = \mathbf{f}_a$ with $\mathbf{z}_a = \mathbf{q}_{a_j}$ and $\omega = \omega_j$

$$\Longrightarrow (\mathbf{K} - \omega_j^2\mathbf{M} + i\omega_j\mathbf{C})\mathbf{q}_{a_j} = \mathbf{f}_{a_j}$$

    where $\mathbf{f}_{a_j}$ is the appropriate excitation to achieve $\mathbf{z}_a = \mathbf{q}_{a_j}$ and $\omega = \omega_j$
  - since $(\mathbf{q}_{a_j}, \omega_j^2)$ satisfies $(\mathbf{K} - \omega_j^2\mathbf{M})\mathbf{q}_{a_j} = \mathbf{0}$, it follows that

$$\boxed{\mathbf{f}_{a_j} = i\omega_j\mathbf{C}\mathbf{q}_{a_j}}$$

    which shows that the excitation force is in phase with the dissipation forces and has a 90^{deg} phase lag (phase quadrature) with respect to the response $\mathbf{z}_a = \mathbf{q}_{a_j}$
  - is the converse true?

- Suppose all the exciting forces are synchronous and the response at every point of the structure is in phase quadrature with the excitation
- The phase relationship between response and excitation may be expressed by assuming that $\mathbf{z}_a$ is a real vector and $\mathbf{f}_a$ an imaginary one

$$(\mathbf{K}-\omega^2\mathbf{M}+i\omega\mathbf{C})\mathbf{z}_a = \mathbf{f}_a \Rightarrow \underbrace{(\mathbf{K}-\omega^2\mathbf{M})\mathbf{z}_a}_{real} = \underbrace{\mathbf{0}}_{real} \text{ and } \underbrace{\mathbf{f}_a}_{imaginary} = \underbrace{i\omega\mathbf{C}\mathbf{z}_a}_{imaginary}$$

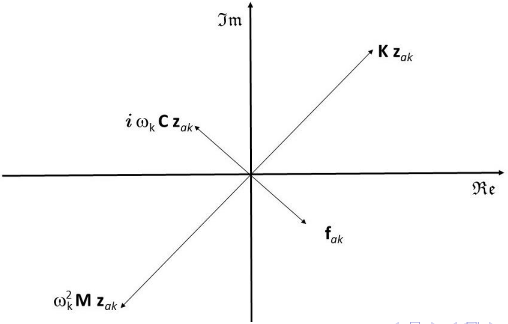
showing that the only admissible solution for ω and $\mathbf{z}_a$ are the eigensolutions of the associated undamped system

- Hence, the converse is true and the 90^{deg} phase idea is one way to measure $\mathbf{q}_{a_k}$ and ω_k^2

- Understanding the concept of phase quadrature

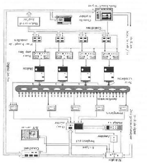
$$(\mathbf{K} - \omega_k^2 \mathbf{M} + i\omega_k \mathbf{C})\mathbf{z}_{a_k} - \mathbf{f}_{a_k} = \mathbf{0}$$

- The Phase Quadrature Criterion: *the structure vibrates according to one of the eigenmodes of the associated undamped system* **if and only if** *all degrees of freedom vibrate synchronously and have a phase lag of* $\frac{\pi}{2}$ *with respect to the excitation*

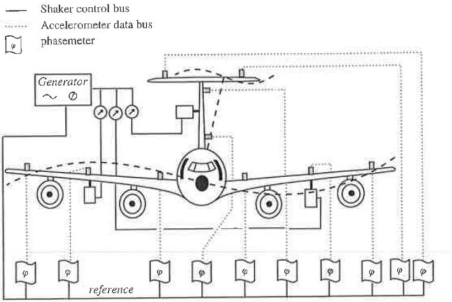


- The methods based on excitation appropriation are by far the most reliable ones to determine the modal characteristics of structures (eigenfrequencies, mode shapes, generalized masses, modal damping coefficients)

- They are however time consuming, delicate to implement (simultaneous excitation of multiple degrees of freedom $\Rightarrow$ trial and error procedures to reach the appropriate excitation conditions), and require a lot of equipment



- Their usage is therefore limited to the testing of structures for which it is necessary to have very accurate knowledge of modal properties – for example, airplanes and spacecraft



- Damped equations of motion

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t)$$

  where $\mathbf{C}$ is also symmetric

- Above equation of dynamic equilibrium can be written in first-order form as

$$\underbrace{\begin{pmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{pmatrix}}_{\dot{\mathbf{r}}} + \underbrace{\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}}_{\mathbf{r}} = \underbrace{\begin{pmatrix} \mathbf{p}(t) \\ \mathbf{0} \end{pmatrix}}_{\mathbf{s}(t)}$$

$$\Longrightarrow \boxed{\mathbf{B}\dot{\mathbf{r}} + \mathbf{A}\mathbf{r} = \mathbf{s}(t)}$$

- Above first-order form is called the *space-state* form of the equation of dynamic equilibrium

$$\mathbf{B}\dot{\mathbf{r}} + \mathbf{A}\mathbf{r} = \mathbf{0}$$

$$\mathbf{r} = \mathbf{y}_a e^{\lambda t} \Rightarrow (\lambda \mathbf{B}\mathbf{y}_a + \mathbf{A}\mathbf{y}_a) e^{\lambda t} = \mathbf{0}$$

$$\Rightarrow \lambda \mathbf{B}\mathbf{y}_a + \mathbf{A}\mathbf{y}_a = \mathbf{0}$$

$$\Rightarrow \boxed{\mathbf{A}\mathbf{y}_a = -\lambda \mathbf{B}\mathbf{y}_a \quad \text{(EVP)}}$$

- $\mathbf{A}$ and $\mathbf{B}$ are $2n \times 2n$ matrices if $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are $n \times n$ matrices $\Rightarrow$ $2n$ eigensolutions $(\lambda_k, \mathbf{y}_{a_k})$
- Orthogonality relationships
  - since $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are symmetric, then $\mathbf{A}$ and $\mathbf{B}$ are symmetric
  - recall that the eigenvectors of a symmetric pencil $(\mathbf{A}, \mathbf{B})$ are orthogonal with respect to $\mathbf{A}$ and $\mathbf{B}$

$$\Longrightarrow \begin{cases} \mathbf{y}_{a_j}^T \mathbf{B} \mathbf{y}_{a_i} = b_i \delta_{ij} \\ \mathbf{y}_{a_j}^T \mathbf{A} \mathbf{y}_{a_i} = a_i \delta_{ij} \end{cases}$$

- Conjugate eigensolutions
  - if $(\lambda_k, \mathbf{y}_{a_k})$ is an eigensolution, the complex conjugate pair $(\bar{\lambda}_k, \bar{\mathbf{y}}_{a_k})$ is also an eigensolution of $\mathbf{A}\mathbf{y}_a = -\lambda \mathbf{B}\mathbf{y}_a$
  - proof

$$\lambda_k = \mu_k + i\nu_k \qquad \mathbf{y}_{a_k} = \mathbf{u}_{a_k} + i\mathbf{v}_{a_k}$$

$$\Longrightarrow \mathbf{A}(\mathbf{u}_{a_k} + i\mathbf{v}_{a_k}) + (\mu_k + i\nu_k)\mathbf{B}(\mathbf{u}_{a_k} + i\mathbf{v}_{a_k}) = \mathbf{0}$$

$$\Longrightarrow \left\{ \begin{array}{lcll} \mathbf{A}\mathbf{u}_{a_k} + \mu_k \mathbf{B}\mathbf{u}_{a_k} - \nu_k \mathbf{B}\mathbf{v}_{a_k} & = & \mathbf{0} & \text{(real part)} \\ \mathbf{A}\mathbf{v}_{a_k} + \nu_k \mathbf{B}\mathbf{u}_{a_k} + \mu_k \mathbf{B}\mathbf{v}_{a_k} & = & \mathbf{0} & \text{(imaginary part)} \end{array} \right. \qquad (3)$$

  note that equations (3) above are invariant with respect to the *simultaneous* changes $\nu_k \rightarrow -\nu_k$ and $\mathbf{v}_k \rightarrow -\mathbf{v}_k$, which implies that $\left((\bar{\lambda}_k = \mu_k - i\nu_k), (\bar{\mathbf{y}}_{a_k} = \mathbf{u}_{a_k} - i\mathbf{v}_{a_k})\right)$ is also a solution of the eigenvalue problem $\mathbf{A}\mathbf{y}_a = -\lambda \mathbf{B}\mathbf{y}_a$
  - modal superposition delivers a real-valued solution of a real-valued problem

- Stability of the general solution
  - recall that $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \Longleftrightarrow \mathbf{B}\dot{\mathbf{r}} + \mathbf{A}\mathbf{r} = \mathbf{0}$ where $\mathbf{r} = [\mathbf{q}\ \dot{\mathbf{q}}]^T$
  - $\mathbf{r} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \mathbf{z}_a \\ \lambda \mathbf{z}_a \end{pmatrix} e^{\lambda t} = \mathbf{y}_a e^{\lambda t} \Rightarrow \mathbf{y}_a = \begin{pmatrix} \mathbf{z}_a \\ \lambda \mathbf{z}_a \end{pmatrix}$
  - hence, the $2n$ eigenmodes of $\mathbf{B}\dot{\mathbf{r}} + \mathbf{A}\mathbf{r} = \mathbf{0}$ are of the form $\mathbf{y}_{a_k} = \begin{pmatrix} \mathbf{z}_{a_k} \\ \lambda_k \mathbf{z}_{a_k} \end{pmatrix}$ where $\mathbf{z}_{a_k}$ is a complex mode of the damped system, and λ_k is a complex eigenvalue of either system above and therefore solution of

$$(\lambda_k^2 \mathbf{M} + \lambda_k \mathbf{C} + \mathbf{K})\mathbf{z}_{a_k} = \mathbf{0}, \qquad k = 1, \cdots n$$

  - pre-multiply the above equation by $\bar{\mathbf{z}}_{a_k}^T$ and assume that the system is lightly damped

$$\Longrightarrow \boxed{\lambda_k^2 m_k + \lambda_k c_k + k_k = 0}$$

    where $m_k = \bar{\mathbf{z}}_{a_k}^T \mathbf{M}\mathbf{z}_{a_k}$, $c_k = \bar{\mathbf{z}}_{a_k}^T \mathbf{C}\mathbf{z}_{a_k}$, and $k_k = \bar{\mathbf{z}}_{a_k}^T \mathbf{K}\mathbf{z}_{a_k}$
  - stability condition

$$\boxed{\mu_k = \Re(\lambda_k) < 0}$$

- Stability of the general solution (continue)

$$\boxed{\lambda_k^2 m_k + \lambda_k c_k + k_k = 0}$$

$$m_k = \bar{\mathbf{z}}_{a_k}^T \mathbf{M} \mathbf{z}_{a_k}, \;\; c_k = \bar{\mathbf{z}}_{a_k}^T \mathbf{C} \mathbf{z}_{a_k}, \;\; k_k = \bar{\mathbf{z}}_{a_k}^T \mathbf{K} \mathbf{z}_{a_k}$$

- recall that if $\lambda_k = \mu_k + i\nu_k$ is solution of the above characteristic equation, $\bar{\lambda}_k = \mu_k - i\nu_k$ is also solution of this characteristic equation
- $\lambda_{k_1} + \lambda_{k_2} = 2\mu_k = -\dfrac{c_k}{m_k} \Rightarrow \mu_k = -\dfrac{c_k}{2m_k}$
- stability: $\mu_k = \Re(\lambda_k) < 0 \Leftrightarrow -\dfrac{c_k}{2m_k} < 0 \Leftrightarrow \boxed{c_k > 0}$
- the general solution of $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$ (or for that matter $\mathbf{p}(t)$) remains stable if $\mathbf{C}$ is positive definite, which is assumed to be the case in the remainder of this chapter

- Solution of the free-vibrations equation $\mathbf{B}\dot{\mathbf{r}} + \mathbf{A}\mathbf{r} = \mathbf{0}$

$$\mathbf{r} = \sum_{i=1}^{n} \zeta_i e^{\lambda_i t} \mathbf{y}_{a_i} + \bar{\zeta}_i e^{\bar{\lambda}_i t} \bar{\mathbf{y}}_{a_i}$$

where the constants ζ_i and $\bar{\zeta}_i$ are determined from the initial conditions and orthogonality relationships of $\mathbf{y}_{a_i}$, and the eigenvalues λ_i and $\bar{\lambda}_i$ are the solutions of

$$\lambda_i^2 m_i + \lambda_i c_i + k_i = 0$$

- recall that $\xi_i = \dfrac{c_i}{2 m_i \omega_i}$ and $\omega_i^2 = \dfrac{k_i}{m_i}$

$$\Longrightarrow \lambda_i^2 + 2\xi_i \omega_i \lambda_i + \omega_i^2 = 0$$

- hence

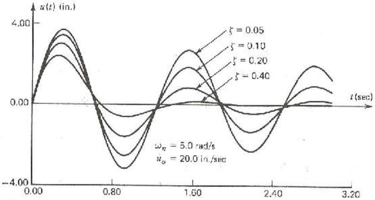
$$\boxed{\lambda_i = -\xi_i \omega_i \pm \omega_i \sqrt{\xi_i^2 - 1}}$$

$$\lambda_i = -\xi_i \omega_i \pm \omega_i \sqrt{\xi_i^2 - 1}$$

- **underdamped** (lightly damped) mode i: $\xi_i < 1$

$$\Longrightarrow \begin{cases} \lambda_i & = -\xi_i \omega_i \pm i\omega_i \sqrt{1 - \xi_i^2} = -\xi_i \omega_i \pm i\omega_i^d \\ \text{where} & \\ \omega_i^d & = \omega_i \sqrt{1 - \xi_i^2} \\ \mathbf{r} & = \sum_{i=1}^{n} e^{-\xi_i \omega_i t} \left[(A_i \cos \omega_i^d t + B_i \sin \omega_i^d t)\mathbf{u}_{a_i} + (B_i \cos \omega_i^d t - A_i \sin \omega_i^d t)\mathbf{v}_{a_i} \right] \end{cases}$$

and the constants A_i and B_i are determined from the initial conditions and orthogonality relationships of $\mathbf{y}_{a_i} = \mathbf{u}_{a_i} \pm i\mathbf{v}_{a_i}$

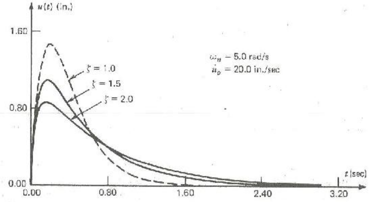


$$\lambda_i = -\xi_i \omega_i \pm \omega_i \sqrt{\xi_i^2 - 1}$$

- **critically damped** mode i: $\xi_i = 1$

$$\Longrightarrow \begin{cases} \lambda_i & = -\omega_i \\ \mathbf{r} & = \displaystyle\sum_{i=1}^{n} e^{-\omega_i t}(A_i + B_i t)\mathbf{y}_{a_i} \end{cases}$$

where the constants A_i and B_i are determined from the initial conditions and orthogonality relationships of the eigenvectors $\mathbf{y}_{a_i}$ which are in this case real-valued

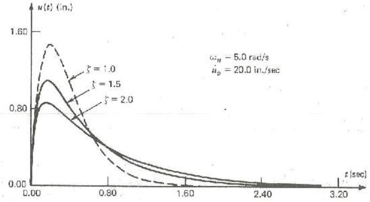


$$\lambda_i = -\xi_i \omega_i \pm \omega_i \sqrt{\xi_i^2 - 1}$$

- **overdamped** mode i: $\xi_i > 1$

$$\Longrightarrow \left\{ \begin{array}{ll} \lambda_i & = -\omega_i \left(\xi_i \mp \sqrt{\xi_i^2 - 1} \right) < 0 \\ \mathbf{r} & = \sum_{i=1}^{n} (A_i e^{-\omega_i \left(\xi_i - \sqrt{\xi_i^2 - 1} \right)} + B_i e^{-\omega_i \left(\xi_i + \sqrt{\xi_i^2 - 1} \right)}) \mathbf{y}_{a_i} \end{array} \right.$$

where the constants A_i and B_i are determined from the initial conditions and orthogonality relationships of the eigenvectors $\mathbf{y}_{a_i}$ which are in this case real-valued



- In the following section, the system is assumed to be underdamped and therefore ω_k^d is real and λ_k is complex
- Governing system of equations

$$\mathbf{B}\dot{\mathbf{r}} + \mathbf{A}\mathbf{r} = \mathbf{s}(t); \qquad \mathbf{s}(t) = \begin{pmatrix} \mathbf{p}^T(t) & \mathbf{0}^T \end{pmatrix}^T$$

- Modal superposition: $\mathbf{r} = \sum_{i=1}^{2n} \eta_i(t)\mathbf{y}_{a_i} \left(= \sum_{i=1}^{n} \eta_i(t)\mathbf{y}_{a_i} + \bar{\eta}_i(t)\bar{\mathbf{y}}_{a_i} \right)$

$$\Longrightarrow \sum_{i=1}^{2n} \mathbf{B}\mathbf{y}_{a_i}\dot{\eta}_i(t) + \sum_{i=1}^{2n} \mathbf{A}\mathbf{y}_{a_i}\eta_i = \mathbf{s}(t)$$

- Pre-multiply the above equation by $\mathbf{y}_{a_j}^T$ and recall the orthogonality conditions

$$\Longrightarrow (\mathbf{y}_{a_j}^T\mathbf{B}\mathbf{y}_{a_j})\dot{\eta}_j + (\mathbf{y}_{a_j}^T\mathbf{A}\mathbf{y}_{a_j})\eta_j = \mathbf{y}_{a_j}^T\mathbf{s}$$

- Recall that an eigenvector $\mathbf{y}_{a_j}$ satisfies

$$\mathbf{y}_{a_j}^T\mathbf{A}\mathbf{y}_{a_j} = -\lambda_j\mathbf{y}_{a_j}^T\mathbf{B}\mathbf{y}_{a_j} \Rightarrow \lambda_j = -\frac{\mathbf{y}_{a_j}^T\mathbf{A}\mathbf{y}_{a_j}}{\mathbf{y}_{a_j}^T\mathbf{B}\mathbf{y}_{a_j}}$$

$$\Longrightarrow \dot{\eta}_j - \lambda_j\eta_j = \underbrace{\frac{\mathbf{y}_{a_j}^T\mathbf{s}}{\mathbf{y}_{a_j}^T\mathbf{B}\mathbf{y}_{a_j}}}_{\Phi_j}$$

$$\boxed{\dot{\eta}_j - \lambda_j \eta_j = \underbrace{\frac{\mathbf{y}_{a_j}^T \mathbf{s}}{\mathbf{y}_{a_j}^T \mathbf{B} \mathbf{y}_{a_j}}}_{\Phi_j}}$$

- Pre-multiply by $e^{-\lambda_j t}$

$$\Longrightarrow e^{-\lambda_j t}\dot{\eta}_j - e^{-\lambda_j t}\lambda_j \eta_j = e^{-\lambda_j t}\Phi_j$$

$$\Longrightarrow \frac{d}{dt}\left(e^{-\lambda_j t}\eta_j\right) = e^{-\lambda_j t}\Phi_j$$

$$\Longrightarrow \boxed{\eta_j(t) = \underbrace{e^{\lambda_j t}\int_0^t \Phi_j(\tau) e^{-\lambda_j \tau} d\tau}_{particular \ \ solution} + \underbrace{\eta_j(0)e^{\lambda_j t}}_{homogeneous \ \ solution}}$$

$$\mathbf{s}(t) = \mathbf{s}_a e^{i\omega t} \quad \text{where} \quad \mathbf{s}_a = \begin{pmatrix} \mathbf{p}_a^T & \mathbf{0}^T \end{pmatrix}^T \Rightarrow \Phi_j(t) = \frac{\mathbf{y}_{a_j}^T \mathbf{s}_a}{\mathbf{y}_{a_j}^T \mathbf{B} \mathbf{y}_{a_j}} e^{i\omega t} = \Phi_{a_j} e^{i\omega t}$$

- After the transient response is damped out, the amplitude $\eta_j(t)$ of the forced response becomes

$$\eta_j(t) = \text{amp}\left(e^{\lambda_j t} \int_0^t \Phi_{a_j} e^{(i\omega - \lambda_j)\tau} d\tau \right) = \left(\frac{e^{i\omega t}}{i\omega - \lambda_j} \right) \Phi_{a_j}, \quad \forall t > t^\star,\ t^\star \neq 0$$

- Recalling that $\lambda_i = -\xi_i \omega_i \pm i\omega_i^d$, it follows that the forced harmonic response of an underdamped system can be written as

$$\mathbf{r} = \sum_{j=1}^{n} \left\{ \left(\frac{1}{\mathbf{y}_{a_j}^T \mathbf{B} \mathbf{y}_{a_j}} \right) \frac{\mathbf{y}_{a_j} \mathbf{y}_{a_j}^T}{\xi_j \omega_j + i(\omega + \omega_j^d)} + \left(\frac{1}{\bar{\mathbf{y}}_{a_j}^T \mathbf{B} \bar{\mathbf{y}}_{a_j}} \right) \frac{\bar{\mathbf{y}}_{a_j} \bar{\mathbf{y}}_{a_j}^T}{\xi_j \omega_j + i(\omega - \omega_j^d)} \right\} \mathbf{s}_a e^{i\omega t}$$

$$\mathbf{r} = \sum_{j=1}^{n} \left\{ \left(\frac{1}{\mathbf{y}_{a_j}^T \mathbf{B} \mathbf{y}_{a_j}} \right) \frac{\mathbf{y}_{a_j} \mathbf{y}_{a_j}^T}{\xi_j \omega_j + i(\omega + \omega_j^d)} + \left(\frac{1}{\bar{\mathbf{y}}_{a_j}^T \mathbf{B} \bar{\mathbf{y}}_{a_j}} \right) \frac{\bar{\mathbf{y}}_{a_j} \bar{\mathbf{y}}_{a_j}^T}{\xi_j \omega_j + i(\omega - \omega_j^d)} \right\} \mathbf{s}_a e^{i\omega t}$$

- Recall that (summary for harmonic case)

$$\mathbf{r} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} = \mathbf{y}_a e^{i\omega t} = \begin{pmatrix} \mathbf{z}_a \\ i\omega \mathbf{z}_a \end{pmatrix} e^{i\omega t}, \quad \mathbf{s} = \mathbf{s}_a e^{i\omega t} = \begin{pmatrix} \mathbf{p}_a \\ \mathbf{0} \end{pmatrix} e^{i\omega t}$$

- And note that

$$\begin{aligned}
\mathbf{y}_{a_j}^T \mathbf{B} \mathbf{y}_{a_j} &= \begin{pmatrix} \mathbf{z}_{a_j}^T & i\omega_j \mathbf{z}_{a_j}^T \end{pmatrix} \begin{pmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z}_{a_j} \\ i\omega_j \mathbf{z}_{a_j} \end{pmatrix} = \mathbf{z}_{a_j}^T \mathbf{C} \mathbf{z}_{a_j} + 2i\omega_j \mathbf{z}_{a_j}^T \mathbf{M} \mathbf{z}_{a_j} = \rho_j \\
\bar{\mathbf{y}}_{a_j}^T \mathbf{B} \bar{\mathbf{y}}_{a_j} &= \begin{pmatrix} \bar{\mathbf{z}}_{a_j}^T & -i\omega_j \bar{\mathbf{z}}_{a_j}^T \end{pmatrix} \begin{pmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{z}}_{a_j} \\ -i\omega_j \bar{\mathbf{z}}_{a_j} \end{pmatrix} = \bar{\mathbf{z}}_{a_j}^T \mathbf{C} \bar{\mathbf{z}}_{a_j} - 2i\omega_j \bar{\mathbf{z}}_{a_j}^T \mathbf{M} \bar{\mathbf{z}}_{a_j} = \bar{\rho}_j \\
\mathbf{y}_{a_j} \mathbf{y}_{a_j}^T \mathbf{s}_a &= \begin{pmatrix} \mathbf{z}_{a_j} \mathbf{z}_{a_j}^T \\ i\omega_j \mathbf{z}_{a_j} \mathbf{z}_{a_j}^T \end{pmatrix} \mathbf{p}_a, \qquad \bar{\mathbf{y}}_{a_j} \bar{\mathbf{y}}_{a_j}^T \mathbf{s}_a = \begin{pmatrix} \bar{\mathbf{z}}_{a_j} \bar{\mathbf{z}}_{a_j}^T \\ -i\omega_j \bar{\mathbf{z}}_{a_j} \bar{\mathbf{z}}_{a_j}^T \end{pmatrix} \mathbf{p}_a
\end{aligned}$$

and

$$\mathbf{z}_a = (\mathbf{K} - \omega^2 \mathbf{M} + i\omega \mathbf{C})^{-1} \mathbf{p}_a$$

- It follows that

$$\mathbf{z}_a = \sum_{j=1}^{n} \left\{ \left(\frac{1}{\rho_j} \right) \frac{\mathbf{z}_{a_j} \mathbf{z}_{a_j}^T}{\xi_j \omega_j + i(\omega + \omega_j^d)} + \left(\frac{1}{\bar{\rho}_j} \right) \frac{\bar{\mathbf{z}}_{a_j} \bar{\mathbf{z}}_{a_j}^T}{\xi_j \omega_j + i(\omega - \omega_j^d)} \right\} \mathbf{p}_a$$

and therefore the expansion of the admittance matrix in terms of the *damped* modes is given by

$$\boxed{(\mathbf{K} - \omega^2 \mathbf{M} + i\omega \mathbf{C})^{-1} = \sum_{k=1}^{n} \left\{ \left(\frac{1}{\xi_k \omega_k + i(\omega + \omega_k^d)} \right) \frac{\mathbf{z}_{a_k}^T \mathbf{z}_{a_k}^T}{\rho_k} + \left(\frac{1}{\xi_k \omega_k + i(\omega - \omega_k^d)} \right) \frac{\bar{\mathbf{z}}_{a_k} \bar{\mathbf{z}}_{a_k}^T}{\bar{\rho}_k} \right\}}$$