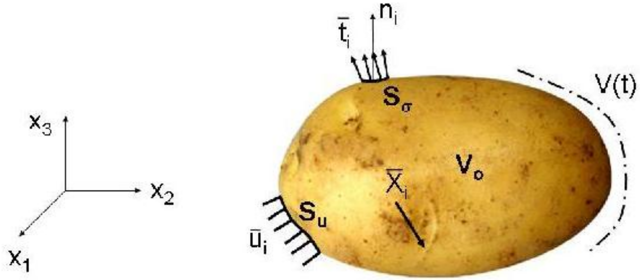
AA242B: MECHANICAL VIBRATIONS

Dynamics of Continuous Systems

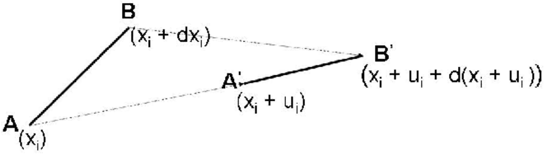
These slides are based on the recommended textbook: M. Géradin and D. Rixen, "Mechanical Vibrations: Theory and Applications to Structural Dynamics," Second Edition, Wiley, John & Sons, Incorporated, ISBN-13:9780471975465

Outline

- Elastic body



$$S = S_\sigma \,(\text{where } t_i = \sigma_{ij} n_j = \bar{t}_i) \quad \bigcup \quad S_u \,(\text{where } u_i = \bar{u}_i)$$



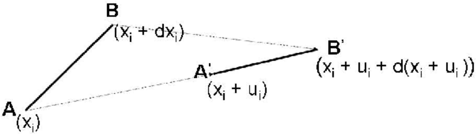
$$
\begin{array}{lcll}
ds_0^2 & = & dx_i dx_i & \text{square of the original length} \\
ds^2 & = & d(x_i + u_i) d(x_i + u_i) & \text{square of the deformed length} \\
ds^2 - ds_0^2 & = & 2\varepsilon_{ij} dx_i dx_j &
\end{array}
$$

where

$$
\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)}
$$

is the Green symmetric strain tensor

- Note that $\varepsilon_{ij} \equiv 0 \Rightarrow$ rigid body motion



- Proof
  - Einstein's notation: $dx_i dx_i = \sum_{i=1}^{3} dx_i^2$
  - $du_i = \dfrac{\partial u_i}{\partial x_j} dx_j, \; i = 1, \; 2, \; 3$

$$
\begin{aligned}
ds_0^2 &= dx_i dx_i \qquad ds^2 = d(x_i + u_i) d(x_i + u_i) \\
ds^2 - ds_0^2 &= d(x_i + u_i) d(x_i + u_i) - dx_i dx_i = dx_i du_i + du_i dx_i + du_i du_i \\
&= du_i dx_i + du_j dx_j + du_m du_m \\
&= \frac{\partial u_i}{\partial x_j} dx_j dx_i + \frac{\partial u_j}{\partial x_i} dx_i dx_j + \frac{\partial u_m}{\partial x_i} dx_i \frac{\partial u_m}{\partial x_j} dx_j \\
&= \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) dx_i dx_j \\
&= 2 \varepsilon_{ij} dx_i dx_j
\end{aligned}
$$

- Linear deformation (geometric linearity)
  - the extension strains remain infinitesimal: $\left|\frac{\partial u_i}{\partial x_i}\right| \ll 1$
  - the rotations have small amplitudes: $\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$
  - the above assumptions lead to a **linear** expression of the infinitesimal strain tensor

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

  - consider a $\vec{ds}$ parallel to $\vec{x}_1$

$$ds^2 - ds_0^2 = (ds - ds_0)(ds + ds_0) = 2\varepsilon_{11}dx_1^2 = 2\varepsilon_{11}ds_0^2$$

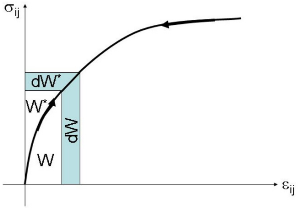
$$\Longrightarrow \varepsilon_{11} = \left(\frac{ds - ds_0}{ds_0}\right)\left(\frac{1}{2}\right)\left(1 + \frac{ds}{ds_0}\right)$$

  - for infinitesimal strains, the above result becomes

$$\boxed{\varepsilon_{11} = \frac{ds - ds_0}{ds_0} \qquad \text{(engineering or Cauchy strain)}}$$

- Hyperelastic material: the work of the mechanical stresses is stored in the form of an internal energy and thus is recoverable

$$\sigma_{ij} = f(\varepsilon_{kl})$$



- Strain energy density: to a strain increment $d\varepsilon_{ij}$ in the stress state σ_{ij} corresponds a strain energy per unit volume

$$\boxed{dW = \sigma_{ij} d\varepsilon_{ij}} \Rightarrow \boxed{\sigma_{ij} = \frac{\partial W(\varepsilon_{ij})}{\partial \varepsilon_{ij}}}$$

- σ_{ij} is energetically conjugate to the Green strain ε_{ij}. It is called the second Piola-Kirchhoff stress tensor. It does not represent the true (Cauchy) stresses inside a structure with respect to the initial reference frame. Rather, it describes the stress field in a reference frame attached to the body and therefore subjected to its deformation but is related to the elementary area of the undeformed structure. In other words, the second Piola-Kirchhoff stress tensor relates forces in the reference (undeformed) configuration to areas in the reference (undeformed) configuration.

- Complementary energy density

$$W^\star = \sigma_{ij}\varepsilon_{ij} - W \qquad \text{(Legendre transformation)}$$

$$\Longrightarrow \boxed{W^\star(\sigma_{ij}) = \int_0^{\sigma_{ij}} \varepsilon_{ij} d\sigma_{ij}}$$

$$\Longrightarrow \boxed{\varepsilon_{ij} = \frac{\partial W^*}{\partial \sigma_{ij}}}$$

- Linear material
  - linear elastic properties

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} \text{ (21 coefficients)} \Rightarrow \boxed{W = \frac{1}{2} c_{ijkl}\varepsilon_{kl}\varepsilon_{ij}}$$

- Linear material (continue)
  - Hooke's law for an isotropic linear elastic material

$$\boxed{\sigma_{ij} = \lambda \underbrace{\varepsilon_{kk}}_{\textit{volumetric strain}} \delta_{ij} + 2G \underbrace{\varepsilon_{ij}}_{\textit{shear strain}}}$$

where

$$\boxed{\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}}$$

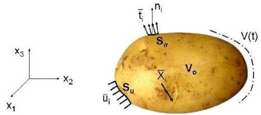
and

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

- The displacement variational principle is Hamilton's principle for a continuous system
- Recall Hamilton's principle: among all possible solutions satisfying $\delta u(t_1) = \delta u(t_2) = 0$, the true solution of the dynamic equilibrium problem is the one which is the stationary point of $\int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V}) dt$

$$\Rightarrow \delta \int_{t_1}^{t_2} \mathcal{L}[u] dt = \delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V}) dt = 0$$

- $\mathcal{T}(u) = \dfrac{1}{2}\displaystyle\int_{V_0} \rho_0 \dot{u}_i \dot{u}_i dV$
- $\mathcal{V} = \mathcal{V}_{int} + \mathcal{V}_{ext}$ where



  - $\mathcal{V}_{ext} = -\displaystyle\int_{V_0} \overline{X}_i(t) u_i dV - \int_{S_\sigma} \bar{t}_i u_i dS$, where the displacement field u_i must satisfy the **essential** Boundary Conditions (BCs) $u_i = \bar{u}_i(t)$ on S_u (recall that for particles, $\delta W = \displaystyle\sum_{s=1}^{n} Q_s \delta q_s \Rightarrow W = \sum_{s=1}^{n} Q_s q_s$)
    - the essential BCs are those which cannot be derived from Hamilton's principle
    - those which can, are called the **natural** BCs
  - $\mathcal{V}_{int} = \displaystyle\int_{V_0} W(\varepsilon_{ij}) dV = \frac{1}{2}\int_{V_0} c_{ijkl}\varepsilon_{kl}\varepsilon_{ij} dV$

$$\delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V}) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left\{ \int_{V_0} \left(\rho_0 \dot{u}_i \delta \dot{u}_i - \frac{\partial W}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} + \overline{X}_i \delta u_i \right) dV + \int_{S_\sigma} \bar{t}_i \delta u_i ds \right\} dt = 0$$

- Approach
  - consider the nonlinear Green strain tensor $\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right)$
  - integrate by parts with respect to both time and space
  - recall $\delta u_i(t_1) = \delta u_i(t_2) = 0$
  - account for the symmetry of the tensor σ_{ij}
  - account for the essential BCs $u_i = \bar{u}_i(t)$ on S_u
  - pay special attention to the evaluation of the quantity $\int_{V_0} \frac{\partial W}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} dV$

$$
\begin{aligned}
\int_{V_0} \frac{\partial W}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} dV &= \frac{1}{2} \int_{V_0} \sigma_{ij} \left(\delta \frac{\partial u_i}{\partial x_j} + \delta \frac{\partial u_j}{\partial x_i} + \delta \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} + \frac{\partial u_m}{\partial x_i} \delta \frac{\partial u_m}{\partial x_j} \right) dV \\
&= \frac{1}{2} \int_S \left[n_j \sigma_{ij} \left(\delta u_i + \delta u_m \frac{\partial u_m}{\partial x_i} \right) + n_i \sigma_{ij} \left(\delta u_j + \delta u_m \frac{\partial u_m}{\partial x_j} \right) \right] dS \\
&- \frac{1}{2} \int_{V_0} \left[\frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i + \frac{\partial \sigma_{ij}}{\partial x_i} \delta u_j + \frac{\partial}{\partial x_i} \left(\sigma_{ij} \frac{\partial u_m}{\partial x_j} \right) \delta u_m + \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_m}{\partial x_i} \right) \delta u_m \right] dv \\
&= \int_{S_\sigma} n_i \left(\sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) \delta u_j dS - \int_{V_0} \frac{\partial}{\partial x_i} \left(\sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) \delta u_j dV
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V}) dt &= \int_{t_1}^{t_2} \left\{ \int_{S_\sigma} \left(\bar{t}_j - \left(\sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) n_i \right) \delta u_j dS \right\} \\
&+ \int_{t_1}^{t_2} \left\{ \int_{V_0} \left(\frac{\partial}{\partial x_i} \left(\sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) - \rho_0 \ddot{u}_j + \overline{X}_j \right) \delta u_j dV \right\} dt \\
&= 0
\end{aligned}
$$

- Since δu_j is arbitrary inside V_0 and on S_σ, the previous equation implies

$$
\boxed{
\begin{array}{lcll}
\dfrac{\partial}{\partial x_i}\left(\sigma_{ij} + \sigma_{im}\dfrac{\partial u_j}{\partial x_m}\right) + \overline{X}_j & = & \rho_0 \ddot{u}_j & \text{in } V_0 \\
t_j = \left(\sigma_{ij} + \sigma_{im}\dfrac{\partial u_j}{\partial x_m}\right) n_i & = & \bar{t}_j & \text{on } S_\sigma \quad \text{(natural BC)}
\end{array}
}
$$

- The above equations are the equations of dynamic equilibrium of a deformable body in terms of the second Piola-Kirchhoff stresses. More specifically, they express the equilibrium of the **deformed** body and thus take into account the geometric nonlinearity.

$$\varepsilon_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij}^{(1)}\text{:linear (small displacements \& rotations)}} + \underbrace{\frac{1}{2}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}}_{\varepsilon_{ij}^{(2)}\text{:quadratic}}$$

- The pure linear case: $\varepsilon = \varepsilon_{ij}^{(1)}$
  - in this case, HP leads to

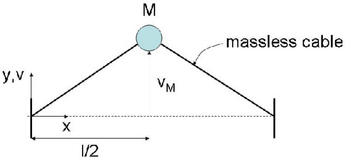
$$\boxed{\begin{array}{rclll} \dfrac{\partial \sigma_{ij}}{\partial x_i} + \overline{X}_j & = & \rho_0 \ddot{u}_j & \text{in } V_0 & \\ t_j = \sigma_{ij} n_i & = & \bar{t}_j & \text{on } S_\sigma & \text{(natural BC)} \end{array}}$$

  - these are the linearized equations of motion for an elastic body undergoing infinitesimal displacements and rotations – they express equilibrium in the undeformed state $V_0 \approx V$

- Second-order effect
  - $\varepsilon_{ij} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}$
  - strain energy density

$$W = \frac{1}{2} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} = \underbrace{\frac{1}{2} c_{ijkl} \varepsilon_{kl}^{(1)} \varepsilon_{ij}^{(1)}}_{\text{elastic forces}} + \underbrace{c_{ijkl} \varepsilon_{kl}^{(1)} \varepsilon_{ij}^{(2)} + \frac{1}{2} c_{ijjkl} \varepsilon_{kl}^{(2)} \varepsilon_{ij}^{(2)}}_{\text{second-order effect}}$$

- Example (HP approach)



$$\frac{\partial v}{\partial x} = \frac{v_M}{l/2} = \frac{2 v_M}{l} \quad 0 < x < \frac{l}{2}$$

$$\frac{\partial v}{\partial x} = -\frac{2 v_M}{l} \quad \frac{l}{2} < x < l$$

- Example (HP approach, continue)
  - if the analysis is limited to transverse motion, the axial strain can be expressed as

$$\varepsilon_{xx} = 0 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 = \frac{1}{2} \times 4 \times \frac{v_M^2}{l^2} = 2\frac{v_M^2}{l^2}$$

  - the kinetic and potential energies are given by

$$\mathcal{T} = \frac{1}{2}M\dot{v}_M^2 \qquad \mathcal{V}_{int} = \frac{1}{2}\int_0^l EA\varepsilon_{xx}^2 dx$$

  - the HP can then be expressed as

$$\begin{aligned}
\delta\int_{t_1}^{t_2}(\mathcal{T}-\mathcal{V})dt &= \delta\int_{t_1}^{t_2}\left(\frac{1}{2}M\dot{v}_M^2 - \frac{1}{2}\int_0^l EA\varepsilon_{xx}^2 dx\right)dt \\
&= \int_{t_1}^{t_2}\left(M\dot{v}_M\delta\dot{v}_M - \int_0^l EA\varepsilon_{xx}\delta\varepsilon_{xx}dx\right)dt \\
&= \int_{t_1}^{t_2}\left\{M\dot{v}_M\delta\dot{v}_M - \int_0^l \frac{EA}{2}\left(\frac{\partial v}{\partial x}\right)^2\left(\frac{\partial v}{\partial x}\right)\delta\left(\frac{\partial v}{\partial x}\right)dx\right\}dt = 0
\end{aligned}$$

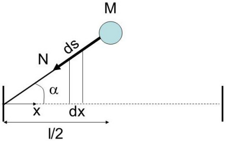
- Example (HP approach, continue)
  - approach: integrate by parts the first term and substitute all partial derivatives by their computed values

$\Longrightarrow$

$$\left[M\dot{v}_M\delta v_M\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\left\{M\ddot{v}_M\delta v_M + \int_0^{\frac{l}{2}}\frac{EA}{2}\left(\frac{2v_M}{l}\right)^3\left(\frac{2}{l}\right)\delta v_M dx\right\}dt$$
$$- \int_{t_1}^{t_2}\left\{\int_{\frac{l}{2}}^{l}\frac{EA}{2}\left(\frac{-2v_M}{l}\right)^3\left(\frac{-2}{l}\right)\delta v_M dx\right\}dt = 0$$

$$\Longrightarrow \left\{M\ddot{v}_M + 2\left(\frac{EA}{2}\right)\left(\frac{2v_M}{l}\right)^3\left(\frac{2}{l}\right)\left(\frac{l}{2}\right)\right\}\delta v_M = 0$$

$$\Longrightarrow \boxed{M\ddot{v}_M + \underbrace{EA\left(\frac{2v_M}{l}\right)^3}_{\text{restoring force is due to second-order effect}} = 0}$$

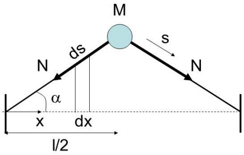


- Example (equilibrium)
  - let $N_x = A\sigma_{xx} = EA\varepsilon_{xx}$ be the axial force computed from the second Piola-Kirchhoff stress tensor and its conjugate Green strain measure. The true force N is such that its virtual work (true/Cauchy stress, engineering/Cauchy strain) is equivalent to that of N_x — that is,

    $$N\delta\left(\frac{ds - dx}{dx}\right) = N_x\delta\varepsilon_{xx} \Rightarrow N\delta\left(\frac{ds}{dx}\right) = N_x\delta\varepsilon_{xx}$$
  - recall that

    $$\varepsilon_{xx} = \frac{1}{2}\left(\frac{ds^2 - dx^2}{dx^2}\right) = \frac{1}{2}\left(\frac{ds}{dx}\right)^2 - \frac{1}{2} \Rightarrow \delta\varepsilon_{xx} = \frac{ds}{dx}\delta\left(\frac{ds}{dx}\right)$$

$$\Longrightarrow N\frac{\delta\varepsilon_{xx}}{\frac{ds}{dx}} = N_x\delta\varepsilon_{xx} \Rightarrow \Longrightarrow \underbrace{N}_{\text{true force}} = \frac{ds}{dx}\underbrace{N_x}_{\substack{\text{force from Piola-Kirchhoff stresses} \\ \text{(relative to surface of underformed cable)}}}$$



$$\cos\alpha = \frac{dx}{ds} \Rightarrow N = \frac{N_x}{\cos\alpha} = \frac{EA\varepsilon_{xx}}{\cos\alpha}$$

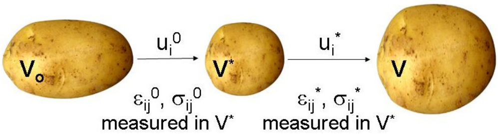
Let F denote the elastic restoring force of the massless cable

$$\Longrightarrow F = 2N\sin\alpha = 2EA\varepsilon_{xx}\tan\alpha = 2EA2\left(\frac{v_M}{l}\right)^2\frac{2v_M}{l}$$

$$\boxed{\Rightarrow F = EA\left(\frac{2v_M}{l}\right)^3}$$

which is the same as the restoring force due to second-order effect determined from the HP

- Effect of initial stress



$$u_i = u_i^0 + u_i^\star \qquad \dot{u}_i = 0 + \dot{u}_i^\star \qquad \varepsilon_{ij} = \varepsilon_{ij}^0 + \varepsilon_{ij}^\star$$

$$\delta u_i = \delta u_i^\star \qquad \delta\varepsilon_{ij} = \delta\varepsilon_{ij}^\star$$

- assume that large displacements and rotations can happen during prestress, but only small displacements and rotations occur after that

- Effect of initial stress (continue)
  - the kinetic energy is given by

$$\mathcal{T} = \frac{1}{2}\int_{V^\star} \rho^\star \dot{u}_i \dot{u}_i dt = \frac{1}{2}\int_{V^\star} \rho^\star \dot{u}_i^\star \dot{u}_i^\star dt = \mathcal{T}^\star$$

  - and the potential energy is given by

$$\mathcal{V}_{int} + \mathcal{V}_{ext}$$

where

$$\begin{aligned}
\mathcal{V}_{int} &= \frac{1}{2}\int_{V^\star} c_{ijkl}\varepsilon_{kl}\varepsilon_{ij} dV = \frac{1}{2}\int_{V^\star} c_{ijkl}(\varepsilon_{kl}^0 + \varepsilon_{kl}^\star)(\varepsilon_{ij}^0 + \varepsilon_{ij}^\star) dV \\
&= \frac{1}{2}\int_{V^\star} c_{ijkl}\varepsilon_{kl}^0\varepsilon_{ij}^0 dV + \int_{V^\star} c_{ijkl}\varepsilon_{kl}^0\varepsilon_{ij}^\star dV + \frac{1}{2}\int_{V^\star} c_{ijkl}\varepsilon_{kl}^\star\varepsilon_{ij}^\star dV \\
&= \mathcal{V}_{int}^0 + \int_{V^\star} \sigma_{ij}^0 \varepsilon_{ij}^\star dV + \mathcal{V}_{int}^\star \\
&= \underbrace{\mathcal{V}_{int}^0}_{cst} + \int_{V^\star} \sigma_{ij}^0 (\varepsilon_{ij}^{\star(1)} + \varepsilon_{ij}^{\star(2)}) dV + \underbrace{\frac{1}{2}\int_{V^\star} c_{ijkl}\varepsilon_{kl}^{\star(1)}\varepsilon_{ij}^{\star(1)} dV}_{\mathcal{V}_{int}^\star} + (HOT)
\end{aligned}$$

  - and

$$\mathcal{V}_{ext} = -\int_{V^\star} (\overline{X}_{0_i} + \overline{X}_i) u_i dV - \int_{S^\star} (\bar{t}_{0_i} + \bar{t}_i) u_i dV = \mathcal{V}_{ext}^0 + \mathcal{V}_{ext}^\star$$

- Effect of initial stress (continue)
  - Two cases:
    - case of *externally prestressed* structures in which the initial stresses result from the external dead loads $\overline{X}_{0_i}$ and $\overline{t}_{0_i}$: the equilibrium of the prestress state implies $\delta \mathcal{V}^0_{ext} + \int_{V^\star} \sigma^0_{ij} \delta \varepsilon^{\star(1)}_{ij} dV = 0$ (note the participation of only $\varepsilon^{\star(1)}_{ij}$ in this equilibrium as after prestress, only small deformations are considered here)
    - case of *internally prestressed* structures in which the initial stresses result from self-equilibrated stresses due to internal forces such as residual stresses arising from the forming or assembly process: $\int_{V^\star} \sigma^0_{ij} \delta \varepsilon^{\star(1)}_{ij} dV = 0$ and $\mathcal{V}^0_{ext} = 0$
  - the HP can then be expressed as

$$\delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V}) dt = \delta \int_{t_1}^{t_2} \left(\mathcal{T}^\star - \mathcal{V}^\star_{int} - \int_{V^\star} \sigma^0_{ij} \varepsilon^{\star(2)}_{ij} dV - \mathcal{V}^\star_{ext} \right) dt = 0$$

  - the geometric prestress potential (second-order effect) is defined as

$$\mathcal{V}_g = \int_{V^\star} \sigma^0_{ij} \varepsilon^{\star(2)}_{ij} dV$$

$$\Longrightarrow \boxed{\delta_{u_i^\star} \int_{t_1}^{t_2} (\mathcal{T}^\star - \mathcal{V}^\star_{int} - \mathcal{V}_g - \mathcal{V}^\star_{ext}) dt = 0, \quad \delta u_i^\star(t_1) = \delta u_i^\star(t_2) = 0} \qquad (1)$$

- The theory of prestressing forms the basis of structural stability analysis, which:
  - consists in computing the prestressing forces applied to a structural system which render possible the existence of a static equilibrium configuration distinct from the prestressed state $u^\star = 0$ under the geometrically linear and nonlinear elastic forces only
  - in this case, the HP is reduced to

$$\delta_{u_i^\star}(\mathcal{V}_{int}^\star + \mathcal{V}_g) = 0$$

- Equation (1) reveals that prestressing modifies the vibration eigenfrequencies, and that the limiting case of a vanishing eigenfrequency corresponds to the limit of stability ($\mathcal{T}^\star = 0$)

- Small displacements and rotations imply
  - linear expression of the infinitesimal strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

  - linear form of the equations of dynamic equilibrium

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial x_i} + \overline{X}_j &= \rho_0 \ddot{u}_j \quad \text{in } V_0 \\ t_j = \sigma_{ij} n_i &= \bar{t}_j \quad \text{on } S_\sigma \quad \text{(natural BC)} \end{aligned}$$

- Hooke's law for a linear elastic isotropic medium

$$\begin{aligned} \sigma_{ij} &= \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij} \\ &= \lambda \left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + G\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \end{aligned}$$

- Assuming a homogeneous medium (λ and G constant) leads to

$$(\lambda + G)\left(\frac{\partial}{\partial x_j}\underbrace{\frac{\partial u_i}{\partial x_i}}_{\nabla \cdot \mathbf{u}}\right) + G\underbrace{\frac{\partial^2 u_j}{\partial x_i \partial x_i}}_{\nabla^2 u_j} + \overline{X}_j = \rho_0 \ddot{u}_j, \quad j = 1, 2, 3, \quad \text{in } V_0$$

$$\Longrightarrow (\lambda + G)\frac{\partial e}{\partial x_j} + G\nabla^2 u_j + \overline{X}_j = \rho_0 \ddot{u}_j$$

  - ∇^2 is the Laplacian operator (Δ)
  - $e = \nabla \cdot \mathbf{u}$ is the divergence of the displacement field
- Propagation of free waves
  - $\overline{X}_j = 0$

$$\Longrightarrow \boxed{(\lambda + G)\frac{\partial e}{\partial x_j} + G\nabla^2 u_j = \rho_0 \ddot{u}_j, \quad j = 1, 2, 3, \quad \text{in } V_0}$$

- Solutions: Plane elastic waves, Rayleigh surface waves, and Love surface waves

- Plane waves

$$\boxed{u_i(x_j, t) = u_i(x_1 \pm ct)}$$

  - at a given time t, the displacement is identical at any point of the plane perpendicular to the direction of wave propagation (here, x_1)
  - the displacement field at the location (x_1, x_2, x_3) and time t is translated to the location $(x_1 \mp \Delta x_1, x_2, x_3)$ at time $t + \Delta t$

$$\begin{aligned} u_i(x_1, x_2, x_3, t) &= u_i(x_1 \pm ct) \\ &= u_i\left((x_1 \mp c\Delta t) \pm c(t + \Delta t)\right) \\ &= u_i(x_1 \mp \Delta x_1, x_2, x_3, t \mp \Delta t) \end{aligned}$$

    where $\Delta x_1 = c\Delta t$
  - c is the velocity of the wave propagating in the positive x_1 direction when $u_i = u_i(x_1 - ct)$ and in the negative x_1 direction when $u_i = u_i(x_1 + ct)$

- Plane elastic waves: Longitudinal waves, and transverse waves
- Longitudinal waves
  - the displacements are **parallel** to the direction of propagation
  - general form

$$\begin{aligned} u_1 &= A \sin\left(\frac{2\pi}{l}(x_1 \pm ct)\right) \\ u_2 &= 0 \\ u_3 &= 0 \end{aligned}$$

    constants A and l represent the wave amplitude and length
  - characteristic longitudinal wave speed that verifies the Navier equations

$$\boxed{c = c_L = \sqrt{\frac{\lambda + 2G}{\rho}} = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}}$$

- Transverse waves
  - the displacements are **orthogonal** to the direction of propagation
  - general form when the displacement field is parallel to x_2

$$\begin{aligned} u_1 &= 0 \\ u_2 &= A \sin\left(\frac{2\pi}{l}(x_1 \pm ct)\right) \\ u_3 &= 0 \end{aligned}$$

    constants A and l represent the wave amplitude and length
  - characteristic transverse wave speed that verifies the Navier equations

$$\boxed{c = c_T = \sqrt{\frac{G}{\rho}}}$$

  - here, (x_1, x_2) is the plane of polarization
  - the ratio of c_L and c_T depends only on the Poisson coefficient

$$\boxed{c_T = c_L \sqrt{\frac{1-2\nu}{2(1-\nu)}}}$$

- Surface waves: Rayleigh surface waves, and Love surface waves
- Rayleigh surface waves
  - two-dimensional semi-infinite medium $x_2 \geq 0$
  - no excitation on $x_2 = 0$ (stress free surface)
  - supposing that the displacement field is the **real part** of

$$\begin{aligned} u_1 &= A\, e^{-bx_2} e^{ik(x_1 - ct)} \\ u_2 &= B\, e^{-bx_2} e^{ik(x_1 - ct)} \\ u_3 &= 0 \end{aligned}$$

where $A \in \mathbb{C}$, $B \in \mathbb{C}$, wave number $k \in \mathbb{R}$, $k = \dfrac{\omega}{c}$

$$b > 0 \Rightarrow e^{-bx_2} \rightarrow 0 \;\; \text{as } x_2 \rightarrow \infty$$

  - re-writing the Navier equations as

$$\boxed{c_T^2 \nabla^2 u_j + (c_L^2 - c_T^2) \frac{\partial e}{\partial x_j} = \ddot{u}_j}$$

- Rayleigh surface waves (continue)
  - and substituting the expression of the displacement field gives

$$
\begin{aligned}
\left[c_T^2 b^2 + (c^2 - c_L^2)k^2\right] A \quad & - i(c_L^2 - c_T^2)bk\, B &= 0 \\
- i(c_L^2 - c_T^2)bk\, A \quad & + \left[c_T^2 b^2 + (c^2 - c_T^2)k^2\right] B &= 0
\end{aligned}
$$

  - $(A, B) \neq (0, 0)$ implies that the determinant vanishes
  - solving for b yields two roots

$$
b' = k\sqrt{1 - \frac{c^2}{c_L^2}} = b'(k, c), \qquad b'' = k\sqrt{1 - \frac{c^2}{c_T^2}} = b''(k, c)
$$

  - b real implies that $c < c_T < c_L$
  - corresponding amplitudes

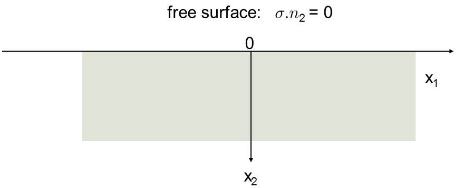
$$
\left(\frac{B}{A}\right)' = -\frac{b'}{ik} = \left(\frac{B}{A}\right)'(c), \qquad \left(\frac{B}{A}\right)'' = \frac{ik}{b''} = \left(\frac{B}{A}\right)''(c)
$$

- Rayleigh surface waves (continue)
  - the general solution becomes

$$
\begin{aligned}
u_1 &= A' e^{-b' x_2} e^{ik(x_1 - ct)} + A'' e^{-b'' x_2} e^{ik(x_1 - ct)} \\
u_2 &= -\frac{b'}{ik} A' e^{-b' x_2} e^{ik(x_1 - ct)} + \frac{ik}{b''} A'' e^{-b'' x_2} e^{ik(x_1 - ct)} \\
u_3 &= 0
\end{aligned}
$$

  - A', A'' and wave speed c are determined by the free surface conditions

$$
\sigma_{22} = \sigma_{21} = 0, \quad \text{at } x_2 = 0
$$



- Rayleigh surface waves (continue)
  - using Hooke's law and the expression of the linear strain, these conditions become

$$
\begin{array}{lcrcll}
\sigma_{21} = 0 & \Rightarrow & \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} & = & 0 & \text{at } x_2 = 0 \\
\sigma_{22} = 0 & \Rightarrow & \lambda \left(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} \right) + 2G \dfrac{\partial u_2}{\partial x_2} & = & 0 & \text{at } x_2 = 0
\end{array}
$$

- Rayleigh surface waves (continue)
  - substituting in the previous two equations the expression of the general solution, using the identities $G = \rho c_T^2$ and $\lambda = \rho(c_L^2 - 2c_T^2)$, and taking into account the expressions for b' and b'' leads to

$$
\begin{aligned}
2\sqrt{\left(1 - \frac{c^2}{c_L^2}\right)}\,A' \quad &+ \frac{\left(2 - \frac{c^2}{c_T^2}\right)}{\sqrt{\left(1 - \frac{c^2}{c_T^2}\right)}}\,A'' &= 0\\
\left(2 - \frac{c^2}{c_T^2}\right)A' \quad & \qquad\qquad -2\,A'' &= 0
\end{aligned}
$$

  - $(A', A'') \neq (0, 0)$ implies that c verifies the characteristic equation

$$
\left(2 - \frac{c^2}{c_T^2}\right)^2 = 4\sqrt{1 - \frac{c^2}{c_L^2}}\sqrt{1 - \frac{c^2}{c_T^2}}
$$

  - factorizing c^2/c_T^2 leads to the *Rayleigh equation*

$$
\boxed{\frac{c^2}{c_T^2}\left[\frac{c^6}{c_T^6} - 8\frac{c^4}{c_T^4} + c^2\left(\frac{24}{c_T^2} - \frac{16}{c_L^2}\right) - 16\left(1 - \frac{c_T^2}{c_L^2}\right)\right] = 0}
$$

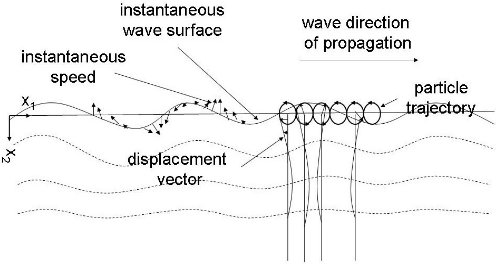
  - **k remains a free parameter**

- Rayleigh surface waves (continue)
  - Rayleigh equation

$$\boxed{\frac{c^2}{c_T^2}\left[\frac{c^6}{c_T^6} - 8\frac{c^4}{c_T^4} + c^2\left(\frac{24}{c_T^2} - \frac{16}{c_L^2}\right) - 16\left(1 - \frac{c_T^2}{c_L^2}\right)\right] = 0} \qquad (2)$$

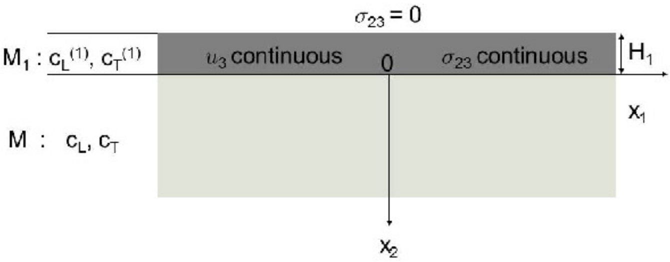
  - $c = 0 \Rightarrow A' = A'' = 0 \Rightarrow u_1 = u_2 = 0$ (trivial solution)
  - from $c_T < c_L$, it follows that the second factor of (2) is negative for $c = 0$ and positive for $c = c_T$: hence, it has a real root $0 < c < c_T$ which shows that surface waves with a velocity lower than c_T may appear in the solution of a Navier problem

- Rayleigh surface waves (continue)



- In the propagation of a Rayleigh wave the motion is backward elliptic — in contrast to the direct elliptic motion in the propagation of a surface wave in a fluid

- Love waves
  - the displacement is perpendicular to the plane of propagation (here, (x_1, x_2))
  - homogeneous layer of material M_1 with thickness H_1 superimposed on a semi-infinite space of a different material M



  - u_3 and σ_{23} are continuous at the interface $x_2 = 0$

- Love waves (continue)
  - the displacement field

$$
\begin{aligned}
u_1 &= 0 \\
u_2 &= 0 \\
u_3 &= Ae^{-kx_2\sqrt{1-\frac{c^2}{c_T^2}}}\, e^{ik(x_1-ct))} \text{ in } M \\
u_3 &= \left\{ Be^{-kx_2\sqrt{1-\left(\frac{c}{c_T^{(1)}}\right)^2}} + B'e^{kx_2\sqrt{1-\left(\frac{c}{c_T^{(1)}}\right)^2}} \right\} e^{ik(x_1-ct)} \text{ in } M_1
\end{aligned}
$$

satisfies the Navier equations and the condition $u_3 \rightarrow 0$ when $x_2 \rightarrow \infty$

- Love waves (continue)
  - u_3 and σ_{23} are continuous at the interface $x_2 = 0$
  - $\sigma_{23} = 0$ at $x_2 = -H_1$

$$\Longrightarrow \begin{cases} A & = & B + B' \\ GA\sqrt{1 - \dfrac{c^2}{c_T^2}} & = & G_1(B - B')\sqrt{1 - \left(\dfrac{c}{c_T^{(1)}}\right)^2} \\ Be^{kH_1\sqrt{1-\left(\frac{c}{c_T^{(1)}}\right)^2}} & = & B'e^{-kH_1\sqrt{1-\left(\frac{c}{c_T^{(1)}}\right)^2}} \end{cases}$$

  - eliminating A, B and B' leads to the equation governing the propagation velocity c of a surface wave with motion perpendicular to the propagation direction

$$\boxed{G\sqrt{1 - \frac{c^2}{c_T^2}} - G_1\left(\sqrt{\left(\frac{c}{c_T^{(1)}}\right)^2 - 1}\right)\tan\left[kH_1\sqrt{\left(\frac{c}{c_T^{(1)}}\right)^2 - 1}\right] = 0}$$

  - for $c_T^{(1)} < c_T$, the above equation has a real root $c_T^{(1)} < c < c_T \Rightarrow$ Love waves 
  - k **remains a free parameter**

- Harmonic motion of a linear system not subjected to external force
  - displacement

$$u_i(x_j, t) = u_{a_i}(x_j) \cos \omega t$$

  - time interval $[t_1, t_2]$ chosen such that $\delta u_i(t_1) = \delta u_i(t_2) = 0$, here for instance

$$[t_1, t_2] = \left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right]$$

  - linearity assumption $\Rightarrow$ kinetic and internal energy are quadratic in the displacement

$$\Longrightarrow \mathcal{T} = \mathcal{T}_{max} \sin^2 \omega t, \quad \mathcal{V} = \mathcal{V}_{max} \cos^2 \omega t$$

where

$$\mathcal{T}_{max} = \frac{1}{2}\omega^2 \int_{V_0} \rho_0 u_{a_i} u_{a_i} dV, \quad \mathcal{V}_{max} = \frac{1}{2} \int_{V_0} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} dV$$

- Hamilton's principle
  - eliminate the time variables by accounting for

$$\int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} \cos^2 \omega t \; dt = \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} \sin^2 \omega t \; dt = \frac{\pi}{2\omega}$$

$$\Longrightarrow \delta \mathcal{L}[u] = \delta \left[\frac{\omega^2}{2} \int_{V_0} \rho_0 u_{a_i} u_{a_i} dV - \frac{1}{2} \int_{V_0} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} dV \right] = 0$$

- Definitions
  - displacement vector $\mathbf{u} = \begin{bmatrix} u_{a_1} & u_{a_2} & u_{a_3} \end{bmatrix}^T$
  - stress vector $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{13} \end{bmatrix}^T$
  - strain vector $\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \gamma_{12} & \gamma_{23} & \gamma_{13} \end{bmatrix}^T$, where

$$\boxed{\gamma_{ij} = 2\varepsilon_{ij}}$$

  - matrix **H** of Hooke's law elastic coefficients

$$\boxed{\sigma = \mathbf{H}\varepsilon}$$

for example in 2D (plane stress – that is, $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$)

$$\mathbf{H} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

- Definitions (continue)
  - spatial differentiation operator

$$\mathbf{D}^T = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}$$

  - associated matrix of the direction cosines of the outward normal at S_σ

$$\mathbf{N}^T = \begin{bmatrix} n_1 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & n_2 & 0 & n_1 & n_3 & 0 \\ 0 & 0 & n_3 & 0 & n_2 & n_1 \end{bmatrix}$$

- Linear kinematics

$$\boxed{\varepsilon = \mathbf{Du}} \Rightarrow \boxed{\sigma = \mathbf{HDu}}$$

- Local dynamic equilibrium

$$\begin{array}{lcll} \dfrac{\partial \sigma_{ij}}{\partial x_i} & = & \rho_0 \ddot{u}_j & \text{in } V_0 \\ \sigma_{ij} n_i & = & \mathbf{0} & \text{on } S_\sigma \end{array} \Longrightarrow \left\{ \begin{array}{lcll} \mathbf{D}^T \sigma + \omega^2 \rho_0 \mathbf{u} & = & \mathbf{0} & \text{in } V_0 \\ \mathbf{N}^T \sigma & = & \mathbf{0} & \text{on } S_\sigma \end{array} \right. \qquad (3)$$

- Variational form of Hamilton's principle

$$\delta \left\{ \omega^2 \int_{V_0} \frac{1}{2} \rho_0 \mathbf{u}^T \mathbf{u} dV - \int_{V_0} \frac{1}{2} (\mathbf{Du})^T \mathbf{H} (\mathbf{Du}) \, dV \right\} = 0$$

- Using the matrix notation, the equations of local dynamic equilibrium (3) can be re-written as

$$\left\{ \begin{array}{lcll} \mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u} + \omega^2\rho_0\mathbf{u} & = & \mathbf{0} & \text{in } V_0 \\ \mathbf{N}^T\mathbf{H}\mathbf{D}\mathbf{u} & = & \mathbf{0} & \text{on } S_\sigma \end{array} \right.$$

- The homogeneous system of equations defining the local dynamic equilibrium, together with its associated variational form, defines an eigenvalue problem of the Sturm-Liouville type

$$\left\{ \begin{array}{rcll} \mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u}_{(i)} + \omega_i^2\rho_0\mathbf{u}_{(i)} & = & \mathbf{0} & \text{in } V_0 \quad \Rightarrow \mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u}_{(i)} = -\omega_i^2\rho_0\mathbf{u}_{(i)} \\ \mathbf{N}^T\mathbf{H}\mathbf{D}\mathbf{u}_{(i)} & = & \mathbf{0} & \text{on } S_\sigma \quad i = 1, \cdots, \infty \end{array} \right.$$

where

$$\mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \mathbf{u}_{(3)}, \cdots$$

are the eigenvectors[1]

[1] In this chapter, the subscript (i) is used to denote the i-th mode instead of the subscript i to avoid confusion with the i-th direction of a vector

- Orthogonality of the eigenvectors
  - equilibrium equations verified by the eigenmodes

$$\mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u}_{(i)} + \omega_i^2\rho_0\mathbf{u}_{(i)} = \mathbf{0}$$

  - multiply by $\mathbf{u}_{(j)}^T$ and integrate over the reference volume V_0
  - integrate the first term by parts

$$\int_{V_0} \underbrace{\mathbf{u}_{(j)}^T}\,\underbrace{\mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u}_{(i)}}\, dV = \int_S \mathbf{u}_{(j)}^T\mathbf{N}^T\mathbf{H}\mathbf{D}\mathbf{u}_{(i)} dS - \int_{V_0} \left(\mathbf{D}\mathbf{u}_{(j)}\right)^T \mathbf{H}\left(\mathbf{D}\mathbf{u}_{(i)}\right) dV$$

  - compatibility of the displacement field and surface equilibriun condition for $\mathbf{u}_{(i)}$

$$\begin{aligned} \mathbf{u}_{(i)}^T &= \mathbf{0} \quad \text{on } S_u \\ \mathbf{N}^T\mathbf{H}\mathbf{D}\mathbf{u}_{(i)} &= \mathbf{0} \quad \text{on } S_\sigma \end{aligned}$$

$$\Longrightarrow \int_{V_0} \left[-\left(\mathbf{D}\mathbf{u}_{(j)}\right)^T \mathbf{H}\left(\mathbf{D}\mathbf{u}_{(i)}\right) + \omega_i^2\rho_0\mathbf{u}_{(j)}^T\mathbf{u}_{(i)}\right] dV = 0 \quad (E_{ji})$$

- Orthogonality of the eigenvectors (continue)
  - similarly for $\mathbf{u}_{(j)}$

$$\int_{V_0} \left[-\left(\mathbf{D}\mathbf{u}_{(i)}\right)^T \mathbf{H} \left(\mathbf{D}\mathbf{u}_{(j)}\right) + \omega_j^2 \rho_0 \mathbf{u}_{(i)}^T \mathbf{u}_{(j)} \right] dV = 0 \quad (E_{ij})$$

  - $(E_{ji}) - (E_{ij})$

$$\Longrightarrow (\omega_j^2 - \omega_i^2) \int_{V_0} \rho_0 \mathbf{u}_{(j)}^T \mathbf{u}_{(i)} dV = 0$$

  - if $\omega_j^2 \neq \omega_i^2$

$$\int_{V_0} \rho_0 \mathbf{u}_{(j)}^T \mathbf{u}_{(i)} dV = 0$$

  - if $\omega_j^2 = \omega_i^2$ and $i \neq j$ (multiple eigenfrequency), the eigenmodes can also be orthogonalized as

$$\int_{V_0} \rho_0 \mathbf{u}_{(j)}^T \mathbf{u}_{(i)} dV = 0$$

  - normalize the eigenvector $\mathbf{u}_{(i)}$ as follows

$$\int_{V_0} \rho_0 \mathbf{u}_{(i)}^T \mathbf{u}_{(i)} dV = 1$$

- Orthogonality of the eigenvectors (continue)
  - recall (E_{ij})

$$\int_{V_0} \left(\mathbf{D}\mathbf{u}_{(j)}\right)^T \mathbf{H} \left(\mathbf{D}\mathbf{u}_{(i)}\right) dV = \omega_i^2 \int_{V_0} \rho_0 \mathbf{u}_{(j)}^T \mathbf{u}_{(i)} dV$$

$$\Longrightarrow \boxed{\begin{array}{lcl} \displaystyle\int_{V_0} \rho_0 \mathbf{u}_{(j)}^T \mathbf{u}_{(i)} dV & = & \delta_{ij} \\ \displaystyle\int_{V_0} \left(\mathbf{D}\mathbf{u}_{(j)}\right)^T \mathbf{H} \left(\mathbf{D}\mathbf{u}_{(i)}\right) dV & = & \omega_i^2 \delta_{ij} \end{array}}$$

- Response of a system with homogeneous BCs
  - eigenmodes form a complete set of solutions of the problem with homogeneous BCs

$$\mathbf{u}(x_j, t) = \sum_{s=1}^{\infty} \eta_s(t)\mathbf{u}_{(s)}(x_j)$$

    where $\eta_s(t)$ are the normal coordinates associated with each mode $\mathbf{u}_{(s)}$
  - the general solution $\mathbf{u}$ satisfies the linear equilibrium equation

$$\mathbf{D}^{\mathbf{T}}\mathbf{H}\mathbf{D}\mathbf{u} + \bar{\mathbf{X}} - \rho_0\ddot{\mathbf{u}} = \mathbf{0} \text{ in } V_0$$

  - and the homogeneous BCs

$$\begin{aligned} \mathbf{N}^{\mathbf{T}}\mathbf{H}\mathbf{D}\mathbf{u} &= \bar{\mathbf{t}} = \mathbf{0} \text{ on } S_\sigma \\ \mathbf{u} &= \bar{\mathbf{u}} = \mathbf{0} \text{ on } S_u \end{aligned}$$

- Response of a system with homogeneous BCs (continue)
  - linear equilibrium equation using the eigenmodes

$$\sum_{s=1}^{\infty} \eta_s \mathbf{D}^{\mathsf{T}}\mathbf{H}\mathbf{D}\mathbf{u}_{(s)} + \bar{\mathbf{X}} - \sum_{s=1}^{\infty} \rho_0 \ddot{\eta}_s \mathbf{u}_{(s)} = \mathbf{0} \quad \text{in } V_0$$

  - premultiply by $\mathbf{u}_{(r)}^T$
  - integrate over V_0
  - integrate by parts in space
  - use the normalization of the modal masses and the orthogonality of the eigenmodes
  - apply the BCs

$$\Longrightarrow \boxed{\ddot{\eta}_r + \omega_r^2 \eta_r = \phi_r, \quad r = 1, \cdots, \infty}$$

  - ϕ_r is the participation factor of the eigenmode $\mathbf{u}_{(r)}$ to the external excitation $\bar{\mathbf{X}}$

$$\phi_r = \int_{V_0} \mathbf{u}_{(r)}^T \bar{\mathbf{X}} dV$$

- Response of a system with homogeneous BCs (continue)
  - $\ddot{\eta}_r + \omega_r^2 \eta_r = \phi_r$ can be integrated in time as

$$\eta_r(t) = \eta_r(0)\cos\omega_r t + \dot{\eta}_r(0)\frac{\sin\omega_r t}{\omega_r} + \frac{1}{\omega_r}\int_0^t \phi_r(\tau)\sin\left(\omega_r(t-\tau)\right)d\tau$$

  where

$$\eta_r(0) = \int_{V_0} \rho_0 \mathbf{u}_{(r)}^T \mathbf{u}(0) dV, \quad \dot{\eta}_r(0) = \int_{V_0} \rho_0 \mathbf{u}_{(r)}^T \dot{\mathbf{u}}(0) dV$$

  - therefore, the general solution obtained by modal superposition is

$$\begin{aligned} \mathbf{u}(x_j, t) &= \sum_{s=1}^{\infty} \mathbf{u}_{(s)} \cos\omega_s t \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \mathbf{u}(0) dV \\ &+ \sum_{s=1}^{\infty} \mathbf{u}_{(s)} \frac{\sin\omega_s t}{\omega_s} \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \dot{\mathbf{u}}(0) dV \\ &+ \sum_{s=1}^{\infty} \frac{\mathbf{u}_{(s)}}{\omega_s} \int_0^t \phi_s(\tau)\sin\left(\omega_s(t-\tau)\right)d\tau \end{aligned}$$

- Response of a system with non-homogeneous spatial BCs
  - consider the following problem

$$\mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u} + \bar{\mathbf{X}} - \rho_0\ddot{\mathbf{u}} = \mathbf{0} \text{ in } V_0$$

    with the initial conditions $\mathbf{u}(0)$ and $\dot{\mathbf{u}}(0)$, and the non-homogeneous BCs

$$\begin{aligned} \mathbf{N}^T\mathbf{H}\mathbf{D}\mathbf{u} &= \bar{\mathbf{t}} \text{ on } S_\sigma \\ \mathbf{u} &= \bar{\mathbf{u}} \text{ on } S_u \end{aligned}$$

  - the external forces $\bar{\mathbf{t}}$ applied on the surface S_σ and the displacement $\bar{\mathbf{u}}$ specified on S_u *can be function of time*
  - solution approach: exploit linearity to split the problem into a quasi-static problem with non-homogeneous BCs and a dynamic problem with a source term and homogeneous BCs (which we already know how to solve)

- Response of a system with non-homogeneous spatial BCs (continue)
  - quasi-static displacement field $\mathbf{u}_{qs}(x_j, t)$ resulting from the application of the non-homogeneous BCs

$$\left\{ \begin{array}{rcll} \mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u}_{qs} & = & \mathbf{0} & \text{in } V_0 \\ \mathbf{N}^T\mathbf{H}\mathbf{D}\mathbf{u}_{qs} & = & \bar{\mathbf{t}} & \text{on } S_\sigma \\ \mathbf{u}_{qs} & = & \bar{\mathbf{u}} & \text{on } S_u \end{array} \right. \Longrightarrow \boxed{\text{characterizes } \mathbf{u}_{qs}}$$

  - modal superposition for the rest of the response leads to

$$\mathbf{u}(x_j, t) = \mathbf{u}_{qs}(x_j, t) + \sum_{s=1}^{\infty} \eta_s(t)\mathbf{u}_{(s)}(x_j)$$

    - equilibrium equation

$$\sum_{s=1}^{\infty} \eta_s \mathbf{D}^\mathbf{T}\mathbf{H}\mathbf{D}\mathbf{u}_{(s)} + \bar{\mathbf{X}} - \sum_{s=1}^{\infty} \rho_0 \ddot{\eta}_s \mathbf{u}_{(s)} = \rho_0 \ddot{\mathbf{u}}_{qs} \quad \text{in } V_0$$

    - BCs

$$\left\{ \begin{array}{rcll} \mathbf{N}^T\mathbf{H}\mathbf{D}\left(\displaystyle\sum_{s=1}^{\infty} \eta_s \mathbf{u}_{(s)}\right) & = & \mathbf{0} & \text{on } S_\sigma \\ \displaystyle\sum_{s=1}^{\infty} \eta_s \mathbf{u}_{(s)} & = & \mathbf{0} & \text{on } S_u \end{array} \right.$$

- Response of a system with non-homogeneous spatial BCs (continue)
  - pre-multiply by $\mathbf{u}_{(r)}^T$
  - integrate by parts over V_0
  - account for the orthogonality of the eigenmodes
  - account for the BCs satisfied by the eigenmodes

$$\Longrightarrow \boxed{\ddot{\eta}_r + \omega_r^2 \eta_r = \phi_r - \int_{V_0} \rho_0 \mathbf{u}_{(r)}^T \ddot{\mathbf{u}}_{qs} dV, \quad r = 1, \cdots, \infty}$$

  - the solution is

$$\begin{aligned} \eta_r(t) &= \eta_r(0) \cos \omega_r t + \dot{\eta}_r(0) \frac{\sin \omega_r t}{\omega_r} \\ &+ \frac{1}{\omega_r} \int_0^t \left[\phi_r(\tau) - \int_{V_0} \rho_0 \mathbf{u}_{(r)}^T \ddot{\mathbf{u}}_{qs}(\tau) \right] \sin\left(\omega_r (t - \tau)\right) d\tau \end{aligned}$$

where

$$\eta_r(0) = \int_{V_0} \rho_0 \mathbf{u}_{(r)}^T \left(\mathbf{u}(0) - \mathbf{u}_{qs}(0)\right) dV, \quad \dot{\eta}_r(0) = \int_{V_0} \rho_0 \mathbf{u}_{(r)}^T \left(\dot{\mathbf{u}}(0) - \dot{\mathbf{u}}_{qs}(0)\right) dV$$

- Response of a system with non-homogeneous spatial BCs (continue)
  - general solution

$$
\begin{aligned}
\mathbf{u}(x_j, t) \quad &= \quad \mathbf{u}_{qs}(x_j, t) - \sum_{s=1}^{\infty} \frac{\mathbf{u}_{(s)}}{\omega_s} \int_0^t \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \ddot{\mathbf{u}}_{qs}(\tau) \sin\left(\omega_s (t-\tau)\right) dV d\tau \\
&+ \quad \sum_{s=1}^{\infty} \mathbf{u}_{(s)} \cos \omega_s t \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \left(\mathbf{u}(0) - \mathbf{u}_{qs}(0)\right) dV \\
&+ \quad \sum_{s=1}^{\infty} \mathbf{u}_{(s)} \frac{\sin \omega_s t}{\omega_s} \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \left(\dot{\mathbf{u}}(0) - \dot{\mathbf{u}}_{qs}(0)\right) dV \\
&+ \quad \sum_{s=1}^{\infty} \frac{\mathbf{u}_{(s)}}{\omega_s} \int_0^t \phi_s(\tau) \sin\left(\omega_s (t-\tau)\right) d\tau
\end{aligned}
$$

  - differs from the homogeneous case by the contribution of the quasi-static displacement field (and its time-derivatives)

- Response of a system with non-homogeneous spatial BCs (continue)
  - integrate by parts *twice* the terms involving $\ddot{\mathbf{u}}_{qs}$

$$\frac{1}{\omega_s}\int_0^t \underbrace{\ddot{\mathbf{u}}_{qs}(\tau)}\,\underbrace{\sin(\omega_s(t-\tau))}\,d\tau = -\frac{\sin\omega_s t}{\omega_s}\dot{\mathbf{u}}_{qs}(0) + \mathbf{u}_{qs}(t) - \cos\omega_s t\;\mathbf{u}_{qs}(0)$$

$$-\omega_s\int_0^t \mathbf{u}_{qs}(\tau)\sin(\omega_s(t-\tau))d\tau$$

$$\begin{aligned}
\Longrightarrow \quad & -\sum_{s=1}^{\infty}\frac{\mathbf{u}_{(s)}}{\omega_s}\int_0^t\int_{V_0}\rho_0\mathbf{u}_{(s)}^T\ddot{\mathbf{u}}_{qs}(\tau)\sin\left(\omega_s(t-\tau)\right)dVd\tau \\
= \quad & \sum_{s=1}^{\infty}\mathbf{u}_{(s)}\cos\omega_s t\int_{V_0}\rho_0\mathbf{u}_{(s)}^T\mathbf{u}_{qs}(0)dV \\
+ \quad & \sum_{s=1}^{\infty}\mathbf{u}_{(s)}\frac{\sin\omega_s t}{\omega_s}\int_{V_0}\rho_0\mathbf{u}_{(s)}^T\dot{\mathbf{u}}_{qs}(0)dV \\
+ \quad & \sum_{s=1}^{\infty}\frac{\mathbf{u}_{(s)}}{\omega_s}\int_0^t\left(\omega_s^2\int_{V_0}\rho_0\mathbf{u}_{(s)}^T(\tau)\mathbf{u}_{qs}(\tau)dV\right)\sin\left(\omega_s(t-\tau)\right)d\tau \\
- \quad & \sum_{s=1}^{\infty}\mathbf{u}_{(s)}\int_{V_0}\rho_0\mathbf{u}_{(s)}^T\mathbf{u}_{qs}dV
\end{aligned}$$

- Response of a system with non-homogeneous spatial BCs (continue)
  - express $\mathbf{u}_{qs}$ in the basis of the eigenmodes

$$\mathbf{u}_{qs} = \sum_{s=1}^{\infty} \mathbf{u}_{(s)} \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \mathbf{u}_{qs} dV$$

  - substitute in previous expression of $\mathbf{u}(x_j, t)$ to keep dependence on $\mathbf{u}_{qs}$ only

$$\begin{aligned} \Longrightarrow \mathbf{u}(x_j, t) \quad &= \quad \sum_{s=1}^{\infty} \mathbf{u}_{(s)} \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \left(\mathbf{u}(0) \cos \omega_s t + \dot{\mathbf{u}}(0) \frac{\sin \omega_s t}{\omega_s} \right) dV \\ &+ \quad \sum_{s=1}^{\infty} \frac{\mathbf{u}_{(s)}}{\omega_s} \int_0^t \left(\phi_s(\tau) + \omega_s^2 \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T(\tau) \mathbf{u}_{qs}(\tau) dV \right) \sin\left(\omega_s (t - \tau)\right) d\tau \end{aligned}$$

- Response of a system with non-homogeneous spatial BCs (continue)
  - recall equilibrium equations, multiply them by $\mathbf{u}_{qs}^T$ and integrate over V_0

$$\Longrightarrow \int_{V_0} \mathbf{u}_{qs}^T \mathbf{D}^\mathbf{T} \mathbf{H} \mathbf{D} \mathbf{u}_{(s)} dV + \omega_s^2 \int_{V_0} \rho_0 \mathbf{u}_{qs}^T \mathbf{u}_{(s)} dV = 0$$

  - integrate the first term in the above equation by parts
  - recall equations satisfied by the quasi-static displacement field $\mathbf{u}_{qs}$
  - introduce $\mathbf{r}_{(s)} = -\mathbf{N}^T \mathbf{H} \mathbf{D} \mathbf{u}_{(s)}$ and eliminate dependence on $\mathbf{u}_{qs}$

$$\begin{aligned}\Longrightarrow \mathbf{u}(x_j, t) \quad = \quad & \sum_{s=1}^{\infty} \mathbf{u}_{(s)} \left(\cos \omega_s t \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \mathbf{u}(0) dV + \frac{\sin \omega_s t}{\omega_s} \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \dot{\mathbf{u}}(0) dV \right) \\ + \quad & \sum_{s=1}^{\infty} \frac{\mathbf{u}_{(s)}}{\omega_s} \int_0^t \left\{ \phi_s(\tau) + \int_{S_\sigma} \mathbf{u}_{(s)}^T \bar{\mathbf{t}}(\tau) dS + \int_{S_u} \bar{\mathbf{u}}(\tau)^T \mathbf{r}_{(s)} dS \right\} \\ & \sin(\omega_s(t-\tau)) d\tau\end{aligned}$$

  - w/r to the homogeneous BCs case, the modal participation factor is augmented by

$$\psi_s = \int_{S_\sigma} \mathbf{u}_{(s)}^T \bar{\mathbf{t}} dS + \int_{S_u} \bar{\mathbf{u}}(\tau)^T \mathbf{r}_{(s)} dS$$

which is the work produced by the boundary tractions with the eigenmode displacement and the work produced by the eigenmode boundary reaction with $\bar{\mathbf{u}}$