Outline

1 Hamilton’s Principle

2 Wave Propagation in a Homogeneous Elastic Medium

3 Free Vibrations of Continuous Systems and Response to External Excitation
Hamilton’s Principle

Definitions

- Elastic body

\[ S = S_\sigma \text{ (where } t_i = \sigma_{ij} n_j = \bar{t}_i) \bigcup S_u \text{ (where } u_i = \bar{u}_i) \]
Hamilton’s Principle

Green Strains

\[ ds^2 = (x_i + u_i)^2 \] square of the original length
\[ ds^2 = d(x_i + u_i)d(x_i + u_i) \] square of the deformed length
\[ ds^2 - ds^2_0 = 2\varepsilon_{ij}dx_idx_j \]

where
\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) \]

is the Green symmetric strain tensor

Note that \( \varepsilon_{ij} \equiv 0 \Rightarrow \) rigid body motion
Proof

- Einstein’s notation: \( dx_i dx_i = \sum_{i=1}^{3} dx_i^2 \)

- \( du_i = \frac{\partial u_i}{\partial x_j} dx_j, \quad i = 1, 2, 3 \)

\[
\begin{align*}
\frac{ds^2}{ds_0^2} &= dx_i dx_i \\
&= d(x_i + u_i)d(x_i + u_i) \\
&= \left( \frac{\partial u_i}{\partial x_j} dx_j + \frac{\partial u_j}{\partial x_i} dx_i + \frac{\partial u_m}{\partial x_i} dx_i \frac{\partial u_m}{\partial x_j} dx_j \right) dx_i dx_j \\
&= 2\varepsilon_{ij} dx_i dx_j
\end{align*}
\]
Hamilton’s Principle
Green Strains

- Linear deformation (geometric linearity)
  - the extension strains remain infinitesimal: \[ \left| \frac{\partial u_i}{\partial x_i} \right| \ll 1 \]
  - the rotations have small amplitudes: \[ \left| \frac{\partial u_i}{\partial x_j} \right| \ll 1 \]
  - the above assumptions lead to a \textbf{linear} expression of the infinitesimal strain tensor

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

- consider a \( \vec{ds} \) parallel to \( \vec{x}_1 \)

\[
ds^2 - ds_0^2 = (ds - ds_0)(ds + ds_0) = 2\varepsilon_{11}dx_1^2 = 2\varepsilon_{11}ds_0^2
\]

\[\implies \varepsilon_{11} = \left( \frac{ds - ds_0}{ds_0} \right) \left( \frac{1}{2} \right) \left( 1 + \frac{ds}{ds_0} \right)\]

- for infinitesimal strains, the above result becomes

\[
\varepsilon_{11} = \frac{ds - ds_0}{ds_0} \quad \text{(engineering or Cauchy strain)}
\]
Hyperelastic material: the work of the mechanical stresses is stored in the form of an internal energy and thus is recoverable

\[ \sigma_{ij} = f(\varepsilon_{kl}) \]

Strain energy density: to a strain increment \( d\varepsilon_{ij} \) in the stress state \( \sigma_{ij} \) corresponds a strain energy per unit volume

\[ dW = \sigma_{ij} d\varepsilon_{ij} \Rightarrow \sigma_{ij} = \frac{\partial W(\varepsilon_{ij})}{\partial \varepsilon_{ij}} \]
\( \sigma_{ij} \) is energetically conjugate to the Green strain \( \varepsilon_{ij} \). It is called the second Piola-Kirchhoff stress tensor. It does not represent the true (Cauchy) stresses inside a structure with respect to the initial reference frame. Rather, it describes the stress field in a reference frame attached to the body and therefore subjected to its deformation but is related to the elementary area of the undeformed structure. In other words, the second Piola-Kirchhoff stress tensor relates forces in the reference (undeformed) configuration to areas in the reference (undeformed) configuration.
Hamilton’s Principle

Stress-Strain Relationships

- Complementary energy density

\[ W^* = \sigma_{ij} \varepsilon_{ij} - W \quad \text{(Legendre transformation)} \]

\[ \Rightarrow W^*(\sigma_{ij}) = \int_0^{\sigma_{ij}} \varepsilon_{ij} d\sigma_{ij} \]

\[ \Rightarrow \varepsilon_{ij} = \frac{\partial W^*}{\partial \sigma_{ij}} \]

- Linear material
  - linear elastic properties

\[ \sigma_{ij} = c_{ijkl} \varepsilon_{kl} \quad (21 \text{ coefficients}) \quad \Rightarrow \quad W = \frac{1}{2} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} \]
Linear material (continue)

- Hooke’s law for an isotropic linear elastic material

\[ \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G \varepsilon_{ij} \]

where

\[ \lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} \]

and

\[ G = \frac{E}{2(1 + \nu)} \]
The displacement variational principle is Hamilton’s principle for a continuous system.

Recall Hamilton’s principle: among all possible solutions satisfying $\delta u(t_1) = \delta u(t_2) = 0$, the true solution of the dynamic equilibrium problem is the one which is the stationary point of

$$\int_{t_1}^{t_2} (T - V) dt$$

$$\Rightarrow \delta \int_{t_1}^{t_2} L[u] dt = \delta \int_{t_1}^{t_2} (T - V) dt = 0$$
Hamilton’s Principle

- Displacement Variational Principle

\[ T(u) = \frac{1}{2} \int_{V_0} \rho_0 \dot{u}_i \dot{u}_i dV \]

\[ V = V_{int} + V_{ext} \text{ where} \]

\[ V_{ext} = -\int_{V_0} \overline{X}_i(t) u_i dV - \int_{S_\sigma} \overline{t}_i u_i dS, \]

where the displacement field \( u_i \) must satisfy the **essential** Boundary Conditions (BCs) \( u_i = \bar{u}_i(t) \) on \( S_u \) (recall that for particles, \( \delta W = \sum_{s=1}^{n} Q_s \delta q_s \Rightarrow W = \sum_{s=1}^{n} Q_s q_s \) )

- the essential BCs are those which cannot be derived from Hamilton’s principle
- those which can, are called the **natural** BCs

\[ V_{int} = \int_{V_0} W(\varepsilon_{ij}) dV = \frac{1}{2} \int_{V_0} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} dV \]
Hamilton’s Principle

Equations of Motion

\[ \delta \int_{t_1}^{t_2} (T - V) dt = 0 \]

\[ \Rightarrow \int_{t_1}^{t_2} \left\{ \int_{V_0} \left( \rho_0 \dot{u}_i \delta \dot{u}_i - \frac{\partial W}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} + \bar{X}_i \delta u_i \right) dV + \int_{S_\sigma} \bar{t}_i \delta u_i ds \right\} dt = 0 \]

Approach

- consider the nonlinear Green strain tensor
  \[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) \]

- integrate by parts with respect to both time and space
- recall \( \delta u_i(t_1) = \delta u_i(t_2) = 0 \)
- account for the symmetry of the tensor \( \sigma_{ij} \)
- account for the essential BCs \( u_i = \bar{u}_i(t) \) on \( S_u \)
- pay special attention to the evaluation of the quantity \( \int_{V_0} \frac{\partial W}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} dV \)
Hamilton’s Principle

Equations of Motion

\[
\int_{V_0} \frac{\partial W}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} dV = \frac{1}{2} \int_{V_0} \sigma_{ij} \left( \delta \frac{\partial u_i}{\partial x_j} + \delta \frac{\partial u_j}{\partial x_i} + \delta \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} + \frac{\partial u_m}{\partial x_i} \delta \frac{\partial u_m}{\partial x_j} \right) dV
\]

\[
= \frac{1}{2} \int_S \left[ n_j \sigma_{ij} \left( \delta u_i + \delta u_m \frac{\partial u_m}{\partial x_i} \right) + n_i \sigma_{ij} \left( \delta u_j + \delta u_m \frac{\partial u_m}{\partial x_j} \right) \right] dS
\]

\[
- \frac{1}{2} \int_V \left[ \sigma_{ij} \delta u_i + \frac{\partial \sigma_{ij}}{\partial x_i} \delta u_j + \frac{\partial}{\partial x_i} \left( \sigma_{ij} \frac{\partial u_m}{\partial x_j} \right) \delta u_m + \frac{\partial}{\partial x_j} \left( \sigma_{ij} \frac{\partial u_m}{\partial x_i} \right) \delta u_m \right] dV
\]

\[
= \int_{S_{\sigma}} n_i \left( \sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) \delta u_j dS - \int_{V_0} \frac{\partial}{\partial x_i} \left( \sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) \delta u_j dV
\]

\[\Rightarrow \delta \int_{t_1}^{t_2} (T - V) dt = \int_{t_1}^{t_2} \left\{ \int_{S_{\sigma}} \left( \delta \tilde{t}_j - \left( \sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) n_i \right) \delta u_j dS \right\}
\]

\[+ \int_{t_1}^{t_2} \left\{ \int_{V_0} \left( \frac{\partial}{\partial x_i} \left( \sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) - \rho_0 \ddot{u}_j + \overline{X}_j \right) \delta u_j dV \right\} dt
\]

\[= 0\]
Since $\delta u_j$ is arbitrary inside $V_0$ and on $S_\sigma$, the previous equation implies

$$\frac{\partial}{\partial x_i} \left( \sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) + \bar{X}_j = \rho_0 \ddot{u}_j \quad \text{in } V_0$$

$$t_j = \left( \sigma_{ij} + \sigma_{im} \frac{\partial u_j}{\partial x_m} \right) n_i = \bar{t}_j \quad \text{on } S_\sigma \quad \text{(natural BC)}$$

The above equations are the equations of dynamic equilibrium of a deformable body in terms of the second Piola-Kirchhoff stresses. More specifically, they express the equilibrium of the deformed body and thus take into account the geometric nonlinearity.
Hamilton’s Principle

The Linear Case and 2nd-Order Effects

\[ \varepsilon_{ij} = 1 \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \]

\( \varepsilon^{(1)}_{ij} \): linear (small displacements & rotations)  
\( \varepsilon^{(2)}_{ij} \): quadratic

- The pure linear case: \( \varepsilon = \varepsilon^{(1)}_{ij} \)
  - in this case, HP leads to

\[
\begin{align*}
\frac{\partial \sigma_{ij}}{\partial x_i} + \bar{X}_j &= \rho_0 \ddot{u}_j \quad \text{in } V_0 \\
t_j &= \sigma_{ij} n_i = \bar{t}_j \quad \text{on } S_{\sigma} \quad \text{(natural BC)}
\end{align*}
\]

- these are the linearized equations of motion for an elastic body undergoing infinitesimal displacements and rotations – they express equilibrium in the undeformed state \( V_0 \approx V \)
Hamilton’s Principle

The Linear Case and 2nd-Order Effects

- **Second-order effect**
  - $\varepsilon_{ij} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}$
  - strain energy density

\[
W = \frac{1}{2} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} = \frac{1}{2} c_{ijkl} \varepsilon_{kl}^{(1)} \varepsilon_{ij}^{(1)} + c_{ijkl} \varepsilon_{kl}^{(1)} \varepsilon_{ij}^{(2)} + \frac{1}{2} c_{ijkl} \varepsilon_{kl}^{(2)} \varepsilon_{ij}^{(2)}
\]

- **Example**

\[
\frac{\partial v}{\partial x} = \frac{v_M}{l/2} = \frac{2v_M}{l} \quad 0 < x < \frac{l}{2}
\]
\[
\frac{\partial v}{\partial x} = -\frac{2v_M}{l} \quad \frac{l}{2} < x < l
\]
Example (continue)

- if the analysis is limited to transverse motion, the axial strain can be expressed as

\[ \varepsilon_{xx} = 0 + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 = \frac{1}{2} \times 4 \times \frac{v_M^2}{l^2} = 2 \frac{v_M^2}{l^2} \]

- the kinetic and potential energies are given by

\[ T = \frac{1}{2} M \ddot{v}_M^2 \quad V_{int} = \frac{1}{2} \int_0^l EA \varepsilon_{xx}^2 \, dx \]

- the HP can then be expressed as

\[
\delta \int_{t_1}^{t_2} (T - V) \, dt = \delta \int_{t_1}^{t_2} \left( \frac{1}{2} M \ddot{v}_M^2 - \frac{1}{2} \int_0^l EA \varepsilon_{xx}^2 \, dx \right) \, dt \\
= \int_{t_1}^{t_2} \left( M \dddot{v}_M \ddot{v}_M - \int_0^l EA \varepsilon_{xx} \delta \varepsilon_{xx} \, dx \right) \, dt \\
= \int_{t_1}^{t_2} \left\{ M \dddot{v}_M \ddot{v}_M - \int_0^l \frac{EA}{2} \left( \frac{\partial v}{\partial x} \right)^2 \left( \frac{\partial v}{\partial x} \right) \delta \left( \frac{\partial v}{\partial x} \right) \, dx \right\} \, dt = 0
\]
Example (continue)

- approach: integrate by parts the first term and substitute all partial derivatives by their computed values

\[
\Rightarrow [M \dot{v}_M \delta v_M]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ M \ddot{v}_M \delta v_M + \int_{0}^{l/2} \frac{EA}{2} \left( \frac{2v_M}{l} \right)^3 \left( \frac{2}{l} \right) \delta v_M dx \right\} dt \\
- \int_{t_1}^{t_2} \left\{ \int_{l/2}^{l} \frac{EA}{2} \left( -\frac{2v_M}{l} \right)^3 \left( -\frac{2}{l} \right) \delta v_M dx \right\} dt = 0
\]

\[
\Rightarrow M \ddot{v}_M + 2 \left( \frac{EA}{2} \right) \left( \frac{2v_M}{l} \right)^3 \left( \frac{2}{l} \right) \left( \frac{l}{2} \right) \delta v_M = 0
\]

\[
\Rightarrow M \ddot{v}_M + \underbrace{EA \left( \frac{2v_M}{l} \right)^3}_{\text{restoring force is due to second-order effect}} = 0
\]
Example (continue)

- let \( N_x = A \sigma_{xx} = EA \varepsilon_{xx} \) be the axial force computed from the second Piola-Kirchhoff stress tensor and its conjugate Green strain measure. The true force \( N \) is such that its virtual work (true/Cauchy stress, engineering/Cauchy strain) is equivalent to that of \( N_x \) — that is,

\[
N \delta \left( \frac{ds - dx}{dx} \right) = N_x \delta \varepsilon_{xx} \Rightarrow N \delta \left( \frac{ds}{dx} \right) = N_x \delta \varepsilon_{xx}
\]

- recall that

\[
\varepsilon_{xx} = \frac{1}{2} \left( \frac{ds^2 - dx^2}{dx^2} \right) = \frac{1}{2} \left( \frac{ds}{dx} \right)^2 - \frac{1}{2} \Rightarrow \delta \varepsilon_{xx} = \frac{ds}{dx} \delta \left( \frac{ds}{dx} \right)
\]

\[
\Rightarrow N \frac{\delta \varepsilon_{xx}}{ds} = N_x \delta \varepsilon_{xx} \Rightarrow \quad \frac{N}{ds} = \frac{dx}{dx} \quad \text{true force} \quad \frac{N_x}{ds} \quad \text{force from Piola-Kirchhoff stresses (relative to surface of underformed cable)}
\]
Hamilton’s Principle

The Linear Case and 2nd-Order Effects

$$\cos \alpha = \frac{dx}{ds} \Rightarrow N = \frac{N_x}{\cos \alpha} = \frac{EA \varepsilon_{xx}}{\cos \alpha}$$

Let $F$ denote the elastic restoring force of the massless cable

$$\Rightarrow F = 2N \sin \alpha = 2EA \varepsilon_{xx} \tan \alpha = 2EA2 \left( \frac{v_M}{l} \right)^2 \frac{2v_M}{l}$$

$$\Rightarrow F = EA \left( \frac{2v_M}{l} \right)^3$$

which is the same as the restoring force due to second-order effect determined from the HP
Effect of initial stress

\[ u_i = u_i^0 + u_i^* \quad \dot{u}_i = 0 + \dot{u}_i^* \quad \varepsilon_{ij} = \varepsilon_{ij}^0 + \varepsilon_{ij}^* \]

\[ \delta u_i = \delta u_i^* \quad \delta \varepsilon_{ij} = \delta \varepsilon_{ij}^* \]

- Assume that large displacements and rotations can happen during prestress, but only small displacements and rotations occur after that.
Hamilton’s Principle

The Linear Case and 2nd-Order Effects

- Effect of initial stress (continue)
  - the kinetic energy is given by
    \[ T = \frac{1}{2} \int_{V^*} \rho^* \dot{u}_i \dot{u}_i dt = \frac{1}{2} \int_{V^*} \rho^* \dot{u}_i^* \dot{u}_i^* dt = T^* \]
  
  - and the potential energy is given by
    \[ V_{int} + V_{ext} \]
    
    where
    \[
    V_{int} = \frac{1}{2} \int_{V^*} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} dV = \frac{1}{2} \int_{V^*} c_{ijkl} (\varepsilon_{kl}^0 + \varepsilon_{kl}^*) (\varepsilon_{ij}^0 + \varepsilon_{ij}^*) dV
    \]
    \[
    = \frac{1}{2} \int_{V^*} c_{ijkl} \varepsilon_{ij}^0 dV + \int_{V^*} c_{ijkl} \varepsilon_{ij}^0 \varepsilon_{ij}^* dV + \frac{1}{2} \int_{V^*} c_{ijkl} \varepsilon_{ij}^* \varepsilon_{ij}^* dV
    \]
    \[
    = V_{int}^0 + \int_{V^*} \sigma_{ij}^0 \varepsilon_{ij}^* dV + V_{int}^*
    \]
    \[
    \geq V_{int} + \int_{V^*} \sigma_{ij}^0 (\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}) dV + \frac{1}{2} \int_{V^*} c_{ijkl} \varepsilon_{ij}^{(1)} \varepsilon_{ij}^{(1)} dV + (HOT)
    \]
    \[
    V_{int}^*
    \]
    
    and
    \[
    V_{ext} = -\int_{V^*} (X_{0i} + X_i) u_i dV - \int_{S^*} (\bar{t}_{0i} + \bar{t}_i) u_i dV = V_{ext}^0 + V_{ext}^*
    \]
Effect of initial stress (continue)

Two cases:

- case of externally prestressed structures in which the initial stresses result from the external dead loads $\overline{X}_{0,i}$ and $\overline{f}_{0,i}$: the equilibrium of the prestress state implies $\delta V^0_{\text{ext}} + \int_{V^*} \sigma^{0}_{ij} \delta \varepsilon_{ij}^{(1)} \, dV = 0$ (note the participation of only $\varepsilon_{ij}^{(1)}$ in this equilibrium as after prestress, only small deformations are considered here)

- case of internally prestressed structures in which the initial stresses result from self-equilibrated stresses due to internal forces such as residual stresses arising from the forming or assembly process: $\int_{V^*} \sigma^{0}_{ij} \delta \varepsilon_{ij}^{(1)} \, dV = 0$ and $V^0_{\text{ext}} = 0$

the HP can then be expressed as

$$\delta \int_{t_1}^{t_2} (T - V) \, dt = \delta \int_{t_1}^{t_2} \left( T^* - V^*_{\text{int}} - \int_{V^*} \sigma^{0}_{ij} \varepsilon_{ij}^{(2)} \, dV - V^*_{\text{ext}} \right) \, dt = 0$$

- the geometric prestress potential (second-order effect) is defined as

$$V_g = \int_{V^*} \sigma^{0}_{ij} \varepsilon_{ij}^{(2)} \, dV$$

$$\Rightarrow \delta u_i^* \int_{t_1}^{t_2} (T^* - V^*_{\text{int}} - V_g - V^*_{\text{ext}}) \, dt = 0, \quad \delta u_i^*(t_1) = \delta u_i^*(t_2) = 0 \quad (1)$$
The theory of prestressing forms the basis of structural stability analysis, which:

- consists in computing the prestressing forces applied to a structural system which render possible the existence of a static equilibrium configuration distinct from the prestressed state \( u^* = 0 \) under the geometrically linear and nonlinear elastic forces only
- in this case, the HP is reduced to

\[
\delta u^*_i \left( V^*_\text{int} + V_g \right) = 0
\]

Equation (1) reveals that prestressing modifies the vibration eigenfrequencies, and that the limiting case of a vanishing eigenfrequency corresponds to the limit of stability (\( \mathcal{T}^* = 0 \))
Small displacements and rotations imply
- linear expression of the infinitesimal strain tensor

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

- linear form of the equations of dynamic equilibrium

\[ \frac{\partial \sigma_{ij}}{\partial x_i} + X_j = \rho_0 \ddot{u}_j \text{ in } V_0 \]
\[ t_j = \sigma_{ij} n_i = \bar{t}_j \text{ on } S_\sigma \text{ (natural BC)} \]

- Hooke’s law for a linear elastic isotropic medium

\[ \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G \varepsilon_{ij} \]
\[ = \lambda \left( \frac{\partial u_k}{\partial x_k} \right) \delta_{ij} + G \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]
Wave Propagation in a Homogeneous Elastic Medium

The Navier Equations in Linear Dynamic Analysis

Assuming a homogeneous medium (\( \lambda \) and \( G \) constant) leads to

\[
(\lambda + G) \left( \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_i} \right) + G \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \nabla \cdot u + \nabla^2 u_j = \rho_0 \ddot{u}_j, \quad j = 1, 2, 3, \text{ in } V_0
\]

\[
\Rightarrow (\lambda + G) \frac{\partial e}{\partial x_j} + G \nabla^2 u_j + \bar{X}_j = \rho_0 \ddot{u}_j
\]

- \( \nabla^2 \) is the Laplacian operator (\( \Delta \))
- \( e = \nabla \cdot u \) is the divergence of the displacement field

Propagation of free waves

- \( \bar{X}_j = 0 \)

\[
\Rightarrow (\lambda + G) \frac{\partial e}{\partial x_j} + G \nabla^2 u_j = \rho_0 \ddot{u}_j, \quad j = 1, 2, 3, \text{ in } V_0
\]

Solutions: Plane elastic waves, Rayleigh surface waves, and Love surface waves
Wave Propagation in a Homogeneous Elastic Medium

Plane Elastic Waves

- Plane waves

\[ u_i(x_j, t) = u_i(x_1 \pm ct) \]

- at a given time \( t \), the displacement is identical at any point of the plane perpendicular to the direction of wave propagation (here, \( x_1 \))

- the displacement field at the location \( (x_1, x_2, x_3) \) and time \( t \) is translated to the location \( (x_1 \mp \Delta x_1, x_2, x_3) \) at time \( t + \Delta t \)

\[
\begin{align*}
 u_i(x_1, x_2, x_3, t) &= u_i(x_1 \pm ct) \\
 &= u_i((x_1 \mp c\Delta t) \pm c(t + \Delta t)) \\
 &= u_i(x_1 \mp \Delta x_1, x_2, x_3, t \mp \Delta t)
\end{align*}
\]

where \( \Delta x_1 = c\Delta t \)

- \( c \) is the velocity of the wave propagating in the positive \( x_1 \) direction when \( u_i = u_i(x_1 - ct) \) and in the negative \( x_1 \) direction when \( u_i = u_i(x_1 + ct) \)
Plane elastic waves: Longitudinal waves, and transverse waves

- **Longitudinal waves**
  - the displacements are **parallel** to the direction of propagation
  - general form

\[
\begin{align*}
  u_1 &= A \sin \left( \frac{2\pi}{l} (x_1 \pm ct) \right) \\
  u_2 &= 0 \\
  u_3 &= 0
\end{align*}
\]

- constants \( A \) and \( l \) represent the wave amplitude and length
- characteristic longitudinal wave speed that verifies the Navier equations

\[
c = c_L = \sqrt{\frac{\lambda + 2G}{\rho}} = \sqrt{\frac{E(1 - \nu)}{(1+\nu)(1-2\nu)\rho}}
\]
Wave Propagation in a Homogeneous Elastic Medium

Plane Elastic Waves

- Transverse waves
  - the displacements are **orthogonal** to the direction of propagation
  - general form when the displacement field is parallel to $x_2$
    \[
    u_1 = 0 \\
    u_2 = A \sin \left( \frac{2\pi}{l} (x_1 \pm ct) \right) \\
    u_3 = 0
    \]
  - constants $A$ and $l$ represent the wave amplitude and length
  - characteristic transverse wave speed that verifies the Navier equations
    \[
    c = c_T = \sqrt{\frac{G}{\rho}}
    \]
  - here, $(x_1, x_2)$ is the plane of polarization
  - the ratio of $c_L$ and $c_T$ depends only on the Poisson coefficient
    \[
    c_T = c_L \sqrt{\frac{1 - 2\nu}{2(1 - \nu)}}
    \]
Surface waves: Rayleigh surface waves, and Love surface waves

Rayleigh surface waves
- two-dimensional semi-infinite medium $x_2 \geq 0$
- no excitation on $x_2 = 0$ (stress free surface)
- supposing the displacement field is the real part of

$$
\begin{align*}
 u_1 &= A_1 e^{b x_2} e^{i k (x_1 - c t)} \\
 u_2 &= A_2 e^{b x_2} e^{i k (x_1 - c t)} \\
 u_3 &= 0
\end{align*}
$$

with $A_1, A_2 \in \mathbb{C}$, wave number $k \in \mathbb{R}$, $k = \frac{\omega}{c}$, $l = c T$, $T = \frac{2\pi}{\omega}$

$$b > 0 \Rightarrow e^{-b x_2} \to 0 \text{ as } x_2 \to \infty$$

Navier equations

$$c_T^2 \nabla^2 u_j + (c_L^2 - c_T^2) \frac{\partial e}{\partial x_j} = \ddot{u}_j$$
Rayleigh surface waves (continue)

substituting the expression of the displacement field gives

$$
\begin{align*}
\left[ c_T^2 b^2 + (c^2 - c_L^2)k^2 \right] A_1 - i(c_L^2 - c_T^2)bkA_2 &= 0 \\
- i(c_L^2 - c_T^2)bkA_1 + \left[ c_T^2 b^2 + (c^2 - c_T^2)k^2 \right] A_2 &= 0
\end{align*}
$$

$(A_1, A_2) \neq (0, 0)$ implies that the determinant vanishes

solving for $b$ yields two roots

$$
b' = k\sqrt{1 - \frac{c_T^2}{c_L^2}}, \quad b'' = k\sqrt{1 - \frac{c_T^2}{c_T^2}}
$$

$b$ real implies that $c < c_T < c_L$

corresponding amplitudes

$$
A' = \left( \frac{A_2}{A_1} \right)' = -\frac{b'}{ik}, \quad A'' = \left( \frac{A_2}{A_1} \right)'' = \frac{ik}{b''}
$$
Rayleigh surface waves (continue)

the general solution becomes

\[ u_1 = A' e^{-b' x_2} e^{i k (x_1 - c t)} + A'' e^{-b'' x_2} e^{i k (x_1 - c t)} \]

\[ u_2 = -\frac{b'}{i k} A' e^{-b' x_2} e^{i k (x_1 - c t)} + \frac{i k}{b''} A'' e^{-b'' x_2} e^{i k (x_1 - c t)} \]

\[ u_3 = 0 \]

\( A', A'' \) and \( c \) are determined by the free surface conditions

\[ \sigma_{22} = \sigma_{21} = \sigma_{23} = 0, \text{ at } x_2 = 0 \]

free surface: \( \sigma . n_2 = 0 \)
Rayleigh surface waves (continue)

using Hooke's law and the expression of the linear strain, these conditions become

\[
\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \quad \text{at } x_2 = 0
\]

\[
\lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2G \frac{\partial u_2}{\partial x_2} = 0 \quad \text{at } x_2 = 0
\]
Rayleigh surface waves (continue)

- substitute in the previous two equations the expression of the general solution and use the identities $G = \rho c_T^2$ and $\lambda = \rho (c_L^2 - 2c_T^2)$
- take into account the expressions for $b'$ and $b''$

\[
\begin{align*}
2b'A' + \left(2 - \frac{c^2}{c_T^2}\right) k^2 A'' \frac{b''}{b''} &= 0 \\
\left(2 - \frac{c^2}{c_T^2}\right) A' + 2b'' A'' \frac{A''}{b''} &= 0
\end{align*}
\]

- $(A', A'') \neq (0, 0)$ implies that $c$ verifies the characteristic equation

\[
\left(2 - \frac{c^2}{c_T^2}\right)^2 = 4 \sqrt{1 - \frac{c^2}{c_T^2}} \sqrt{1 - \frac{c^2}{c_L^2}}
\]

- after factorization, the Rayleigh equation is obtained

\[
\frac{c^2}{c_T^2} \left[ \frac{c^6}{c_T^6} - 8 \frac{c^4}{c_T^4} + c^2 \left(\frac{24}{c_T^2} - \frac{16}{c_L^2}\right) - 16 \left(1 - \frac{c_T^2}{c_L^2}\right) \right] = 0
\]

- $k$ remains a free parameter
Rayleigh surface waves (continue)

- Rayleigh equation

\[
\frac{c^2}{c_T^2} \left[ \frac{c^6}{c_T^6} - 8 \frac{c^4}{c_T^4} + c^2 \left( \frac{24}{c_T^2} - \frac{16}{c_T^2} \right) - 16 \left( 1 - \frac{c_T^2}{c_L^2} \right) \right] = 0
\] (2)

- \( c = 0 \) \( \Rightarrow u_1 = u_2 = 0 \) (trivial solution)

- from \( c_T < c_L \), it follows that the second factor of (2) is negative for \( c = 0 \) and positive for \( c = c_T \): hence, it has a real root \( 0 < c < c_T \) which shows that surface waves with a velocity lower than \( c_T \) may appear in the solution of a Navier problem
Rayleigh surface waves (continue)

In the propagation of a Rayleigh wave the motion is backward elliptic — in contrast to the direct elliptic motion in the propagation of a surface wave in a fluid.
- Love waves
  - the displacements are perpendicular to the plane of propagation (here, \((x_1, x_2)\))
  - homogeneous layer of material \(M_1\) with thickness \(H_1\) superimposed on a semi-infinite space of a different material \(M\)

\[ \sigma_{23} = 0 \]

- \(u_3\) and \(\sigma_{23}\) are continuous at the interface \(x_2 = 0\)
Love waves (continue)

- the displacement field

\[ u_1 = 0 \]
\[ u_2 = 0 \]
\[ u_3 = Ae^{-kx_2} \sqrt{1 - \frac{c^2}{c_T^2}} e^{ik(x_1 - ct)} \text{ in } M \]
\[ u_3 = \left\{ A_1 e^{-kx_2} \sqrt{1 - \left( \frac{c}{c_T} \right)^2} + A'_1 e^{kx_2} \sqrt{1 - \left( \frac{c}{c_T} \right)^2} \right\} e^{ik(x_1 - ct)} \text{ in } M_1 \]

satisfies the Navier equations and the condition \( u_3 \to 0 \) when \( x_2 \to \infty \)
Love waves (continue)

- $u_3$ and $\sigma_{23}$ are continuous at the interface $x_2 = 0$
- $\sigma_{23} = 0$ at $x_2 = -H_1$

\[
\begin{align*}
A &= A_1 + A'_1 \\
GA \sqrt{1 - \frac{c^2}{c_T^2}} &= G_1 (A_1 - A'_1) \sqrt{1 - \left(\frac{c}{c_T^{(1)}}\right)^2} \\
kH_1 \sqrt{1 - \left(\frac{c}{c_T^{(1)}}\right)^2} &= -kH_1 \sqrt{1 - \left(\frac{c}{c_T^{(1)}}\right)^2} \\
A_1 e^{-kH_1} \sqrt{1 - \left(\frac{c}{c_T^{(1)}}\right)^2} &= A'_1 e^{-kH_1} \sqrt{1 - \left(\frac{c}{c_T^{(1)}}\right)^2}
\end{align*}
\]

- eliminating $A$, $A_1$ and $A'_1$ leads to the equation governing the propagation velocity $c$ of a surface wave perpendicular to the propagation direction

\[
G \sqrt{1 - \frac{c^2}{c_T^2}} - G_1 \left(\sqrt{\left(\frac{c}{c_T^{(1)}}\right)^2 - 1}\right) \tan \left[kH_1 \sqrt{\left(\frac{c}{c_T^{(1)}}\right)^2 - 1}\right] = 0
\]

- for $c_T^{(1)} < c_T$, the above equation has a root $c_T^{(1)} < c < c_T \Rightarrow$ Love waves
- $k$ remains a free parameter
Harmonic motion of a linear system not subjected to external force

- displacement

\[ u_i(x_j, t) = u_{ai}(x_j) \cos \omega t \]

- time interval \([t_1, t_2]\) chosen such that \(\delta u_i(t_1) = \delta u_i(t_2) = 0\), here for instance

\[ [t_1, t_2] = \left[ -\frac{\pi}{2\omega}, \frac{\pi}{2\omega} \right] \]

- linearity assumption \(\Rightarrow\) kinetic and internal energy are quadratic in the displacement

\[ \Rightarrow T = T_{\text{max}} \sin^2 \omega t, \quad \mathcal{V} = \mathcal{V}_{\text{max}} \cos^2 \omega t \]

where

\[ T_{\text{max}} = \frac{1}{2} \omega^2 \int_{V_0} \rho_0 u_{ai} u_{ai} dV, \quad \mathcal{V}_{\text{max}} = \frac{1}{2} \int_{V_0} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} dV \]
Free Vibrations of Continuous Systems and Response to External Excitation

Eigenvalue Problem

- Hamilton's principle
- eliminate the time variables by accounting for

\[
\int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} \cos^2 \omega t \, dt = \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} \sin^2 \omega t \, dt = \frac{\pi}{2\omega}
\]

\[\Rightarrow \delta L[u] = \delta \left[ \frac{\omega^2}{2} \int_{V_0} \rho_0 u_{ai} u_{ai} \, dV - \frac{1}{2} \int_{V_0} c_{ijkl} \varepsilon_{kl} \varepsilon_{ij} \, dV \right] = 0\]

Definitions

- displacement vector \( \mathbf{u} = \begin{bmatrix} u_{a1} & u_{a2} & u_{a3} \end{bmatrix}^T \)
- stress vector \( \mathbf{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{13} \end{bmatrix}^T \)
- strain vector \( \mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \gamma_{12} & \gamma_{23} & \gamma_{13} \end{bmatrix}^T \), where \( \gamma_{ij} = 2\varepsilon_{ij} \)

- matrix \( \mathbf{H} \) of Hooke's law elastic coefficients

\[ \mathbf{\sigma} = \mathbf{H}\mathbf{\varepsilon} \]

for example in 2D

\[ \mathbf{H} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \]
Free Vibrations of Continuous Systems and Response to External Excitation

Eigenvalue Problem

- Definitions (continue)
  - spatial differentiation operator

\[
D^T = \begin{bmatrix}
\frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\
0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\
0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1}
\end{bmatrix}
\]

- associated matrix of the direction cosines of the outward normal at \( S_\sigma \)

\[
N^T = \begin{bmatrix}
n_1 & 0 & 0 & n_2 & 0 & n_3 \\
0 & n_2 & 0 & n_1 & n_3 & 0 \\
0 & 0 & n_3 & 0 & n_2 & n_1
\end{bmatrix}
\]

- Linear kinematics

\[
\varepsilon = Du \Rightarrow \sigma = HDu
\]

- Local dynamic equilibrium

\[
\frac{\partial \sigma_{ij}}{\partial x_i} = \rho_0 \ddot{u}_j \quad \text{in} \quad V_0 \\
\sigma_{ij} n_i = 0 \quad \text{on} \quad S_\sigma
\]

\[
\begin{cases}
D^T \sigma + \omega^2 \rho_0 u = 0 \quad \text{in} \quad V_0 \\
N^T \sigma = 0 \quad \text{on} \quad S_\sigma
\end{cases}
\quad (3)
\]

- Variational form of Hamilton’s principle

\[
\delta \left\{ \omega^2 \int_{V_0} \frac{1}{2} \rho_0 u^T u dV - \int_{V_0} \frac{1}{2} (Du)^T H (Du) dV \right\} = 0
\]

\[
\delta \left\{ \omega^2 \int_{V_0} \frac{1}{2} \rho_0 u^T u dV - \int_{V_0} \frac{1}{2} (Du)^T H (Du) dV \right\} = 0
\]
Free Vibrations of Continuous Systems and Response to External Excitation

Eigenvalue Problem

- Using the matrix notation, the equations of local dynamic equilibrium (3) can be re-written as

\[
\begin{align*}
\mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u} + \omega^2 \rho_0 \mathbf{u} &= 0 \quad \text{in } V_0 \\
\mathbf{N}^T\mathbf{H}\mathbf{D}\mathbf{u} &= 0 \quad \text{on } S_\sigma
\end{align*}
\]

- The homogeneous system of equations defining the local dynamic equilibrium, together with its associated variational form, defines an eigenvalue problem of the Sturm-Liouville type

\[
\begin{align*}
\mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u}(i) + \omega^2_i \rho_0 \mathbf{u}(i) &= 0 \quad \text{in } V_0 \quad \Rightarrow \quad \mathbf{D}^T\mathbf{H}\mathbf{D}\mathbf{u}(i) = -\omega^2_i \rho_0 \mathbf{u}(i) \\
\mathbf{N}^T\mathbf{H}\mathbf{D}\mathbf{u}(i) &= 0 \quad \text{on } S_\sigma \quad i = 1, \ldots, \infty
\end{align*}
\]

where

\[\mathbf{u}(1), \mathbf{u}(2), \mathbf{u}(3), \ldots\]

are the eigenvectors\(^1\)

\(^1\)In this chapter, the subscript \((i)\) is used to denote the \(i\)-th mode instead of the subscript \(i\) to avoid confusion with the \(i\)-th direction of a vector.
Free Vibrations of Continuous Systems and Response to External Excitation

Orthogonality of Eigensolutions

- Orthogonality of the eigenvectors
  - equilibrium equations verified by the eigenmodes
    \[ D^T H \mathbf{u}_i + \omega_i^2 \rho_0 \mathbf{u}_i = 0 \]
  - multiply by \( \mathbf{u}^T_j \) and integrate over the reference volume \( V_0 \)
  - integrate the first term by parts
    \[
    \int_{V_0} \mathbf{u}^T_j D^T H \mathbf{u}_i \, dV = \int_S \mathbf{u}^T_j N^T H \mathbf{u}_i \, dS - \int_{V_0} (\mathbf{D}_j)^T H (\mathbf{D}_i) \, dV
    \]
- compatibility of the displacement field and surface equilibrium condition for \( \mathbf{u}_i \)
  - \( \mathbf{u}^T_i = 0 \) on \( S_u \)
  - \( N^T H \mathbf{u}_i = 0 \) on \( S_\sigma \)

\[ \Rightarrow \int_{V_0} \left[ - (\mathbf{D}_j)^T H (\mathbf{D}_i) + \omega_i^2 \rho_0 \mathbf{u}^T_j \mathbf{u}_i \right] \, dV = 0 \quad (E_{ji}) \]
Orthogonality of the eigenvectors (continue)

- similarly for $u_{(j)}$

$$\int_{V_0} \left[ -(Du_{(i)})^T H(Du_{(j)}) + \omega_j^2 \rho_0 u_{(i)}^T u_{(j)} \right] dV = 0 \quad (E_{ij})$$

- $(E_{ji}) - (E_{ij})$

$$\Rightarrow (\omega_j^2 - \omega_i^2) \int_{V_0} \rho_0 u_{(j)}^T u_{(i)} dV = 0$$

- if $\omega_j^2 \neq \omega_i^2$

$$\int_{V_0} \rho_0 u_{(j)}^T u_{(i)} dV = 0$$

- if $\omega_j^2 = \omega_i^2$ and $i \neq j$ (multiple eigenfrequency), the eigenmodes can also be orthogonalized as

$$\int_{V_0} \rho_0 u_{(j)}^T u_{(i)} dV = 0$$

- normalize the eigenvector $u_{(i)}$ as follows

$$\int_{V_0} \rho_0 u_{(i)}^T u_{(i)} dV = 1$$
Orthogonality of the eigenvectors (continue)

- recall \( (E_{ij}) \)

\[
\int_{V_0} \left( Du_{(j)} \right)^T H \left( Du_{(i)} \right) dV = \omega_i^2 \int_{V_0} \rho_0 u_{(j)}^T u_{(i)} dV
\]

\[
\Rightarrow \int_{V_0} \rho_0 u_{(j)}^T u_{(i)} dV = \delta_{ij}
\]

\[
\int_{V_0} \left( Du_{(j)} \right)^T H \left( Du_{(i)} \right) dV = \delta_{ij} \omega_i^2
\]
Response of a system with homogeneous BCs

- Eigenmodes form a complete set of solutions of the problem with homogeneous BCs

\[ u(x_j, t) = \sum_{s=1}^{\infty} \eta_s(t) u_s(x_j) \]

where \( \eta_s(t) \) are the normal coordinates associated with each mode \( u_s \)

- The general solution \( u \) satisfies the linear equilibrium equation

\[ D^T H D u + \bar{X} - \rho_0 \ddot{u} = 0 \text{ in } V_0 \]

- And the homogeneous BCs

\[ N^T H D u = \bar{t} = 0 \text{ on } S_\sigma \]
\[ u = \bar{u} = 0 \text{ on } S_u \]
Response of a system with homogeneous BCs (continue)

- linear equilibrium equation using the eigenmodes

\[ \sum_{s=1}^{\infty} \eta_s D^T HDu_s + \bar{X} - \sum_{s=1}^{\infty} \rho_0 \ddot{\eta}_s u_s = 0 \text{ in } V_0 \]

- premultiply by \( u^T (r) \)
- integrate over \( V_0 \)
- integrate by parts in space
- use the normalization of the modal masses and the orthogonality of the eigenmodes
- apply the BCs

\[ \ddot{\eta}_r + \omega_r^2 \eta_r = \phi_r, \quad r = 1, \ldots, \infty \]

- \( \phi_r \) is the participation factor of the eigenmode \( u_s(r) \) to the external excitation \( \bar{X} \)

\[ \phi_r = \int_{V_0} u^T (r) \bar{X} dV \]
Response of a system with homogeneous BCs (continue)

- $\ddot{\eta}_r + \omega_r^2 \eta_r = \phi_r$ can be integrated in time as

$$\eta_r(t) = \eta_r(0) \cos \omega_r t + \dot{\eta}_r(0) \frac{\sin \omega_r t}{\omega_r} + \frac{1}{\omega_r} \int_0^t \phi_r(\tau) \sin (\omega_r (t - \tau)) \, d\tau$$

where

$$\eta_r(0) = \int_{V_0} \rho_0 u^{T}_{(r)} u(0) \, dV, \quad \dot{\eta}_r(0) = \int_{V_0} \rho_0 u^{T}_{(r)} \dot{u}(0) \, dV$$

- therefore, the general solution obtained by modal superposition is

$$u(x_j, t) = \sum_{s=1}^{\infty} u^{(s)} \cos \omega_s t \int_{V_0} \rho_0 u^{T}_{(s)} u(0) \, dV$$

$$+ \sum_{s=1}^{\infty} u^{(s)} \frac{\sin \omega_s t}{\omega_s} \int_{V_0} \rho_0 u^{T}_{(s)} \dot{u}(0) \, dV$$

$$+ \sum_{s=1}^{\infty} \frac{u^{(s)}}{\omega_s} \int_0^t \phi_s(\tau) \sin (\omega_s (t - \tau)) \, d\tau$$
Response of a system with non-homogeneous spatial BCs

- consider the following problem

\[ D^T H D u + \bar{X} - \rho_0 \ddot{u} = 0 \text{ in } V_0 \]

with the initial conditions \( u(0) \) and \( \dot{u}(0) \), and the non-homogeneous BCs

\[
N^T H D u = \bar{t} \text{ on } S_\sigma \\
u = \bar{u} \text{ on } S_u
\]

- the external forces \( \bar{t} \) applied on the surface \( S_\sigma \) and the displacement \( \bar{u} \) specified on \( S_u \) can be function of time

- solution approach: exploit linearity to split the problem into a quasi-static problem with non-homogeneous BCs and a dynamic problem with a source term and homogeneous BCs (which we already know how to solve)
Response of a system with non-homogeneous spatial BCs (continue)

- quasi-static displacement field $u_{qs}(x_j, t)$ resulting from the application of the non-homogeneous BCs

\[
\begin{align*}
D^T H u_{qs} &= 0 \quad \text{in } V_0 \\
N^T H u_{qs} &= \tilde{t} \quad \text{on } S_{\sigma} \\
u_{qs} &= \bar{u} \quad \text{on } S_u
\end{align*}
\]

- modal superposition for the rest of the response leads to

\[
u(x_j, t) = u_{qs}(x_j, t) + \sum_{s=1}^{\infty} \eta_s(t) u(s)(x_j)
\]

- equilibrium equation

\[
\sum_{s=1}^{\infty} \eta_s D^T H u(s) + \bar{\mathbf{X}} - \sum_{s=1}^{\infty} \rho_0 \ddot{\eta}_s u(s) = \rho_0 \ddot{u}_{qs} \quad \text{in } V_0
\]

- BCs

\[
\begin{align*}
N^T H \left( \sum_{s=1}^{\infty} \eta_s u(s) \right) &= 0 \quad \text{on } S_{\sigma} \\
\sum_{s=1}^{\infty} \eta_s u(s) &= 0 \quad \text{on } S_u
\end{align*}
\]
Response of a system with non-homogeneous spatial BCs (continue)

- pre-multiply by $u^T_{(r)}$
- integrate by parts over $V_0$
- account for the orthogonality of the eigenmodes
- account for the BCs satisfied by the eigenmodes

\[
\dot{\eta}_r + \omega^2_r \eta_r = \phi_r - \int_{V_0} \rho_0 u^T_{(r)} \ddot{u}_{qs} dV, \quad r = 1, \ldots, \infty
\]

the solution is

\[
\eta_r(t) = \eta_r(0) \cos \omega_r t + \dot{\eta}_r(0) \frac{\sin \omega_r t}{\omega_r}
\]

\[
+ \frac{1}{\omega_r} \int_0^t \left[ \phi_r(\tau) - \int_{V_0} \rho_0 u^T_{(r)} \ddot{u}_{qs}(\tau) \right] \sin \left( \omega_r (t - \tau) \right) d\tau
\]

where

\[
\eta_r(0) = \int_{V_0} \rho_0 u^T_{(r)} (u(0) - u_{qs}(0)) dV, \quad \dot{\eta}_r(0) = \int_{V_0} \rho_0 u^T_{(r)} (\dot{u}(0) - \dot{u}_{qs}(0)) dV
\]
Response of a system with **non-homogeneous** spatial BCs (continue)

- **general solution**

\[
\begin{align*}
  u(x_j, t) &= u_{qs}(x_j, t) - \sum_{s=1}^{\infty} \frac{u(s)}{\omega_s} \int_0^t \int_{V_0} \rho_0 u^T(s) \ddot{u}_{qs}(\tau) \sin(\omega_s(t - \tau)) \, dV d\tau \\
  &+ \sum_{s=1}^{\infty} u(s) \cos \omega_s t \int_{V_0} \rho_0 u^T(s) (u(0) - u_{qs}(0)) \, dV \\
  &+ \sum_{s=1}^{\infty} u(s) \frac{\sin \omega_s t}{\omega_s} \int_{V_0} \rho_0 u^T(s) (\dot{u}(0) - \dot{u}_{qs}(0)) \, dV \\
  &+ \sum_{s=1}^{\infty} \frac{u(s)}{\omega_s} \int_0^t \phi_s(\tau) \sin(\omega_s(t - \tau)) \, d\tau
\end{align*}
\]

- differs from the homogeneous case by the contribution of the quasi-static displacement field (and its time-derivatives)
Response of a system with non-homogeneous spatial BCs (continue)

- integrate by parts twice the terms involving $\ddot{u}_{qs}$

\[
\frac{1}{\omega_s} \int_0^t \ddot{u}_{qs}(\tau) \sin(\omega_s(t - \tau)) d\tau = -\frac{\sin \omega_s t}{\omega_s} \dot{u}_{qs}(0) + u_{qs}(t) - \cos \omega_s t \ u_{qs}(0)
\]

\[
- \omega_s \int_0^t u_{qs}(\tau) \sin(\omega_s(t - \tau)) d\tau
\]

\[
\Rightarrow - \sum_{s=1}^\infty \frac{u(s)}{\omega_s} \int_0^t \int_{V_0} \rho_0 u^T(s) \ddot{u}_{qs}(\tau) \sin(\omega_s(t - \tau)) dV d\tau
\]

\[
= \sum_{s=1}^\infty u(s) \cos \omega_s t \int_{V_0} \rho_0 u^T(s) u_{qs}(0) dV
\]

\[
+ \sum_{s=1}^\infty \frac{u(s)}{\omega_s} \sin \omega_s t \int_{V_0} \rho_0 u^T(s) \dot{u}_{qs}(0) dV
\]

\[
+ \sum_{s=1}^\infty \frac{u(s)}{\omega_s} \int_0^t \left( \omega_s^2 \int_{V_0} \rho_0 u^T(s)(\tau) u_{qs}(\tau) dV \right) \sin(\omega_s(t - \tau)) d\tau
\]

\[
- \sum_{s=1}^\infty u(s) \int_{V_0} \rho_0 u^T(s) u_{qs} dV
\]
Response of a system with non-homogeneous spatial BCs (continue)

- express $u_{qs}$ in the basis of the eigenmodes

$$u_{qs} = \sum_{s=1}^{\infty} u(s) \int_{V_0} \rho_0 u^T(s) u_{qs} dV$$

- substitute in previous expression of $u(x_j, t)$ to keep dependence on $u_{qs}$ only

$$u(x_j, t) = \sum_{s=1}^{\infty} u(s) \int_{V_0} \rho_0 u^T(s) \left( u(0) \cos \omega_s t + \dot{u}(0) \frac{\sin \omega_s t}{\omega_s} \right) dV$$

$$+ \sum_{s=1}^{\infty} \frac{u(s)}{\omega_s} \int_{0}^{t} \left( \phi_s(\tau) + \omega_s^2 \int_{V_0} \rho_0 u^T(s)(\tau) u_{qs}(\tau) dV \right) \sin (\omega_s(t-\tau)) d\tau$$
Response of a system with non-homogeneous spatial BCs (continue)

- Recall equilibrium equations, multiply them by $\mathbf{u}_{qs}^T$ and integrate over $V_0$

$$\Rightarrow \int_{V_0} \mathbf{u}_{qs}^T D^T H \mathbf{u}(s) dV + \omega_s^2 \int_{V_0} \rho_0 \mathbf{u}_{qs}^T \mathbf{u}(s) dV = 0$$

- Integrate the first term in the above equation by parts
- Recall equations satisfied by the quasi-static displacement field $\mathbf{u}_{qs}$
- Introduce $\mathbf{r}(s) = -\mathbf{N}^T H \mathbf{u}(s)$ and eliminate dependence on $\mathbf{u}_{qs}$

$$\Rightarrow \mathbf{u}(x_j, t) = \sum_{s=1}^{\infty} \mathbf{u}(s) \left( \cos \omega_s t \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \mathbf{u}(0) dV + \frac{\sin \omega_s t}{\omega_s} \int_{V_0} \rho_0 \mathbf{u}_{(s)}^T \dot{\mathbf{u}}(0) dV \right)$$

$$+ \sum_{s=1}^{\infty} \frac{\mathbf{u}(s)}{\omega_s} \int_{0}^{t} \left\{ \phi_s(\tau) + \int_{S_\sigma} \mathbf{u}_{(s)}^T \mathbf{t}(\tau) dS + \int_{S_u} \ddot{\mathbf{u}}(\tau)^T \mathbf{r}(s) dS \right\} \sin(\omega_s(t - \tau)) d\tau$$

- With respect to the homogeneous BCs case, the modal participation factor is augmented by

$$\psi_s = \int_{S_\sigma} \mathbf{u}_{(s)}^T \mathbf{t} dS + \int_{S_u} \ddot{\mathbf{u}}(\tau)^T \mathbf{r}(s) dS$$

which is the work produced by the boundary tractions with the eigenmode displacement and the work produced by the eigenmode boundary reaction with $\ddot{\mathbf{u}}$.