Outline

1. Eigenvector Iteration Methods
2. Subspace Construction Methods
Eigenvector Iteration Methods

Introduction

- Generalized eigenvalue problem associated with an \( n \)-degree-of-freedom system

\[
Kx = \omega^2 Mx
\]

- \( n \) eigenvalues

\[
0 \leq \omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_n^2
\]

and \( n \) associated eigenvectors

\[
x_1, x_2, \cdots, x_n
\]
### Eigenvector Iteration Methods

#### Introduction

- **Dynamic flexibility matrix**
  - Consider a system that has no rigid-body modes $\Rightarrow K$ is non-singular
  - Construct the dynamic flexibility matrix defined as
    \[
    D = K^{-1}M
    \]
  - The associated eigenvalue problem is
    \[
    Kx = \omega^2 Mx \Rightarrow Dx = \lambda x \quad \text{with} \quad \lambda = \frac{1}{\omega^2}
    \]
  - Eigenvectors are same as for the generalized eigenproblem $Kx = \omega^2 Mx$
  - Eigenvalues are given by
    \[
    \lambda_i = \frac{1}{\omega_i^2} \Rightarrow \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n
    \]

- **Drawbacks of working with $D$**
  - Building cost is $\mathcal{O}(n^3)$
  - Non-symmetric matrix and as such must be stored completely
  - These drawbacks can be dealt with (for example, see next the inverse iteration form of the power iteration algorithm)
Eigenvector Iteration Methods

The Power Algorithm

- Determination of the fundamental eigenmode

  - An iterative solution to $\mathbf{Dx} = \lambda \mathbf{x}$ can be obtained by considering the iteration

    $$\mathbf{z}_{p+1} = \mathbf{Dz}_p$$

  - Proof: expand $\mathbf{z}_0$ in the basis of the eigenmodes of the system

    $$\mathbf{z}_0 = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$$

    Then, at the $p$-th iteration,

    $$\mathbf{z}_p = \mathbf{D}^p \mathbf{z}_0 = \sum_{i=1}^{n} \alpha_i \mathbf{D}^p \mathbf{x}_i = \sum_{i=1}^{n} \alpha_i \lambda_i^p \mathbf{x}_i = \lambda_1^p \left( \alpha_1 \mathbf{x}_1 + \sum_{i=2}^{n} \alpha_i \left( \frac{\lambda_i}{\lambda_1} \right)^p \mathbf{x}_i \right)$$

    Recall that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and assume that $\lambda_1 > \lambda_2$

    $$\Rightarrow \quad \mathbf{z}_p \rightarrow \lambda_1^p \alpha_1 \mathbf{x}_1 \Rightarrow \frac{\|\mathbf{z}_{p+1}\|}{\|\mathbf{z}_p\|} \rightarrow \lambda_1 \quad \text{as} \quad p \rightarrow \infty$$

    $$\Rightarrow \quad \text{the power iteration algorithm converges to the lowest eigenvalue } \lambda_1^2$$
Eigenvector Iteration Methods

The Power Algorithm

**Convergence analysis of the power iteration algorithm**

- \( \forall p, z_p = \lambda_1^p x_1 + \lambda_2^p x_2 + \cdots \) (recall that \( x_i \) is defined up to a multiplicative constant)

\[
\frac{z_{p+1}^T e_j}{z_p^T e_j} = \frac{\lambda_1^{p+1} x_1^T e_j + \lambda_2^{p+1} x_2^T e_j + \cdots}{\lambda_1^p x_1^T e_j + \lambda_2^p x_2^T e_j + \cdots} = \frac{\lambda_1^{p+1} (x_1 + r^{p+1} x_2 + \cdots)^T e_j}{\lambda_1^p (x_1 + r^p x_2 + \cdots)^T e_j}
\]

\[
= \frac{(x_1 + r^{p+1} x_2 + \cdots)^T e_j}{(x_1 + r^p x_2 + \cdots)^T e_j}
\]

where \( r = \frac{\lambda_2}{\lambda_1} \)

- define \( a_j = \frac{x_2^T e_j}{x_1^T e_j} \) and assume that \( |\lambda_2| < |\lambda_1| \Rightarrow r < 1 \)

\[
\lambda_1 = \frac{z_{p+1}^T e_j}{z_p^T e_j} = \frac{\lambda_1 (1 + a_j r^{p+1} + \cdots)}{1 + a_j r^p + \cdots} \approx_{p \to \infty} \lambda_1 (1 + a_j r^{p+1}) (1 + a_j r^p)^{-1} 
\]

\[
\approx_{p \to \infty} \lambda_1 (1 + a_j r^p (r - 1))
\]

- number of iterations \( N \) after which \( \lambda_1^N \) is stabilized with \( t \) significant digits

\[
\left| \frac{\lambda_1^{N+1} - \lambda_1^N}{\lambda_1^N} \right| \leq 10^{-t} \Rightarrow a_j r^N \approx 10^{-t} \Rightarrow N \approx \frac{t}{\log \left( \frac{\lambda_1}{\lambda_2} \right)}
\]

\[\implies N \] — and therefore the convergence rate of the power algorithm — depends only on the ratio of the first two eigenvalues
Determinition of the higher modes by orthogonal deflation

- once $x_1$ is determined, the following orthogonal projection operator can be formed (orthogonal projection $\iff P_1^2 = P_1$ and $\text{Ker}(P_1)$ is $M$-orthogonal to span $(P_1)$)

$$P_1 = I - \frac{x_1x_1^TM}{x_1^TMx_1} \quad (\Rightarrow P_1x_1 = 0)$$

- next, the following starting vector can be constructed

$$z_0^* = P_1z_0 = \sum_{i=2}^{n} \alpha_i x_i$$

this vector does not have any component in the $x_1$ “direction” $\Rightarrow$ the power method equipped with the starting vector $z_0^*$ converges in principle to $(\lambda_2, x_2)$

- in practice, the mode $x_1$ reappears in the iterations because of roundoff errors; to remediate this issue, the following algorithm can be used instead

$$\begin{cases} 
  z_p^* = P_1z_p \\
  z_{p+1} = Dz_p^*
\end{cases}$$

- this approach can be generalized to extract higher modes
Alternatively, one can construct a power iteration algorithm based on

$$\overline{D} = M^{-1}K$$

in which case

- the associated eigenvalue problem is

$$Kx = \omega^2 Mx \Rightarrow \overline{D}x = \omega^2 x$$

- the eigenvectors and eigenvalues of this problem are the same as for the generalized eigenproblem $Kx = \omega^2 Mx$

- the eigenvalues can be sorted and labeled as follows

$$\omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_n^2$$

- the power iteration algorithm converges to $\omega_n^2$ – that is, to the highest eigenvalue

- the drawbacks of working with $\overline{D}$ are identical to those of working with $D$ and can be dealt with in a similar way
Eigenvector Iteration Methods

The Inverse Iteration Method

- Problem: it is not desirable to construct the dynamical matrix \( D = K^{-1}M \) (or \( \bar{D} = M^{-1}K \)) at the base of the power iteration algorithm.

- Solution: power iteration approach using the more computationally efficient iterate \( z_{p+1} = K^{-1}(Mz_p) \), which can be organized in two steps as follows:

\[
\begin{align*}
    y_p &= Mz_p \\
    Kz_{p+1} &= y_p
\end{align*}
\]

- Incurs the solution of an algebraic problem of the form \( Kq = g \).

- Factorization-based algorithm:
  1. Factorization of \( K \) (once)
  2. Forward and backward substitutions (repeated)

- Recall that convergence to the highest \( \lambda \) is convergence to the lowest \( \omega^2 \) (\( \lambda = \frac{1}{\omega^2} \)).
Eigenvector Iteration Methods

Inverse Iteration with Spectral Shifting

- Spectral shifting (by $\mu$) of the eigenvalue problem

$$\begin{align*}
(K - \mu M)x &= (\omega^2 - \mu)Mx \\
&\Rightarrow (K - \mu M)^{-1}Mx = \frac{1}{\omega^2 - \mu}x
\end{align*}$$

- associated inverse iteration scheme

$$\begin{align*}
\begin{cases}
y_p &= Mz_p \\
(K - \mu M)z_{p+1} &= y_p
\end{cases}
\end{align*}$$

- iterations

$$z_0 = \sum_{i=1}^{n} \alpha_i x_i \Rightarrow z_p = \sum_{i=1}^{n} \alpha_i \left((K - \mu M)^{-1}M\right)^p x_i = \sum_{i=1}^{n} \frac{\alpha_i}{(\omega_i^2 - \mu)^p} x_i$$

converge toward the eigenmode $x_r$ whose associated eigenvalue $\omega_r^2$ satisfies

$$|\omega_r^2 - \mu| = \min_j \left\{|\omega_j^2 - \mu|\right\}$$

and therefore is closest to $\mu$, and this convergence accelerates as $\mu$ approaches $\omega_r^2$. 
Subspace Construction Methods

The Subspace Iteration Method

The subspace concept

- let \( X \) denote the \( M \)-normalized solution of the generalized eigenvalue problem \( Kx = \omega^2 Mx \)

\[
X^T MX = I_n, \quad X^T KX = \Lambda = \text{diag}(\omega_1^2, \ldots, \omega_n^2)
\]

then \( \text{span}(X) = E \) of dimension \( n \)

- consider the first \( m \ll n \) eigenmodes

\[
X^* = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix} \in \mathbb{R}^{n \times m}
\]

\( \text{span}(X^*) = E^* \) of dimension \( m \); \( E^* \subset E \)

- property 1: \( \forall a \in \mathbb{R}^m \)

\[
z = X^* a \in E^*
\]

- property 2: if \( A \in \mathbb{R}^{m \times m} \) and \( \det(A) \neq 0 \)

\[
\text{span}(X^* A) = \text{span}(X^*)
\]
Interaction problem

once any set \( Z^* \) of \( m \ll n \) linearly independent vectors in \( \mathbb{E}^* \) is found (\( Z^* \in \mathbb{R}^{n \times m} \))

- the first \( m \) eigensolutions are determined by forming and solving the reduced eigenproblem

\[
K^* y = \omega^2 M^* y
\]

where \( K^* = Z^*^T K Z^* \in \mathbb{R}^{m \times m} \) and \( M^* = Z^*^T M Z^* \in \mathbb{R}^{m \times m} \)

- the matrix \( Y = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \in \mathbb{R}^{m \times m} \) is assembled

- the matrix \( X^* = Z^* Y \in \mathbb{R}^{n \times m} \), which has the same span as the matrix \( Z^* \), is constructed
Subspace Construction Methods

The Subspace Iteration Method

- Using the inverse iteration algorithm, the subspace method generates a sequence of matrices \( Z_0, Z_1, \cdots, Z_k \in \mathbb{R}^{n \times m} \) spanning the subspaces \( E^*_0, E^*_1, \cdots, E^*_k \) that converge toward \( E^*_\infty = E^* = \text{span}(X^*) \) (iterative method on the Grassmann manifold \( G(m, n) \))

- **Algorithm**
  - iteration kernel
    \[
    \begin{cases}
    V_k &= MZ_k \\
    KZ_{k+1} &= V_k
    \end{cases}
    \]
  - aim is convergence toward \( E^* \) of the subspace as a whole, and not of the individual vectors in \( Z_k \)
  - to make sure that the reduced-order bases \( Z_i \) span subspaces of dimension \( m \), orthogonality of the column vectors is ensured by one of two ways
    - Gram-Schmidt orthogonalization
    - re-orthogonalization with respect to the \( M \)-induced norm
Convergence analysis

- the subspaces spanned by the matrices $Z_k$ converge toward the subspace spanned by $X^*$, provided that $Z_0$ is not orthogonal to any of the first $m$ eigenvectors
- global convergence of the subspace iteration algorithm guarantees that subspace $E_k^* \subset E$ converges toward $E^*$
- the convergence rate toward each mode $x_j$ is of the order of $O(r_j^k)$ where

$$r_j = \frac{\omega_j^2}{\omega_{m+1}^2}$$

measures the ratio of the eigenvalue $\omega_j^2$ and that which is immediately outside the subspace $E^*$. This indicates that a higher convergence rate is obtained by using $q > m$ iteration vectors (which increases computational cost however). A practical trade-off suggests to use $q = \min(2m, m + 8)$.

- considering the convergence rate mentioned above, multiple eigenvalues do not decrease the rate of convergence provided that $\omega_{q+1}^2 > \omega_m^2$
Principle of the method

- generate a subspace that includes the fundamental eigensolutions by applying the inverse iteration algorithm to a starting vector $z_0$
- construct the following Krylov sequence

$$\left\{ z_0, K^{-1}Mz_0, \left(K^{-1}M\right)^2 z_0, \cdots \right\}$$

- orthogonalize the terms of the sequence

Properties (for the basic form of the algorithm)

- the search subspace is built column by column: its dimension is increased at each iteration
- fast convergence
- however, orthonormality rapidly deteriorates (and re-orthogonalization procedures can be computationally expensive)
- inability to detect multiple eigenvalues
Subspace Construction Methods

The Lanczos Method

Algorithm

- iteration

\[ \gamma_{p+1} x_{p+1} = K^{-1} M x_p - \alpha_p x_p - \beta_{p-1} x_{p-1} \]

where \( \beta_{p-1}, \alpha_p, \gamma_{p+1} \) are determined such that

\[
\begin{cases}
  x_{p+1}^T M x_j & = 0 \quad j < p + 1 \\
  x_{p+1}^T M x_{p+1} & = 1
\end{cases}
\]

requiring \( M \)-orthogonality leads to

\[
\begin{align*}
  \alpha_p &= x_p^T M K^{-1} M x_p \\
  \beta_{p-1} &= x_p^T M K^{-1} M x_{p-1} \\
  \gamma_p &= \beta_{p-1}
\end{align*}
\]

the reduced-order basis \( X = \begin{bmatrix} x_0 & \cdots & x_p \end{bmatrix} \in \mathbb{R}^{n \times p} \) satisfies

\[ K^{-1} M X = X T + S = \begin{bmatrix} 0 & \cdots & 0 & \gamma_{p+1} x_{p+1} \end{bmatrix} = \gamma_{p+1} x_{p+1} e_p^T \]

where \( T = \text{tridiag} \left( \begin{bmatrix} \gamma_1 & \cdots & \gamma_p \end{bmatrix}, \begin{bmatrix} \alpha_0 & \cdots & \alpha_p \end{bmatrix}, \begin{bmatrix} \gamma_1 & \cdots & \gamma_p \end{bmatrix} \right) \)
Interaction problem

- start with

\[ K^{-1}MX = XT + S \]

- premultiply by \( Y^T = (MX)^T \) associated to the Lanczos vectors \( X \)
- note that \( Y^T S = 0 \)

\[ \Rightarrow Y^T K^{-1}MX = Y^T XT = T \]

- hence, the tridiagonal matrix of the Lanczos coefficients results from the projection of \( D = K^{-1}M \) onto the subspace of Lanczos vectors \( X \). Thus, it has the same eigenvalues as the projection of \( D \) onto the reduced-order basis \( X \).
Subspace Construction Methods

The Lanczos Method

- Difficulties for basic form of the Lanczos method
  - deterioration of the orthogonalization process
  - detection of multiple eigenvalues
  - appearance of parasitic solutions
  - missing eigensolutions