Outline

1. Stability and Accuracy of Time-Integration Operators
2. Newmark’s Family of Methods
3. Explicit Time Integration Using the Central Difference Algorithm
Lagrange’s equations of dynamic equilibrium \((\mathbf{p}(t) = 0)\)

\[
M \ddot{\mathbf{q}} + C \dot{\mathbf{q}} + K \mathbf{q} = 0
\]

\[
\mathbf{q}(0) = \mathbf{q}_0
\]

\[
\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0
\]

First-order form

\[
\begin{pmatrix}
0 & M \\
M & C
\end{pmatrix}
\begin{pmatrix}
\ddot{\mathbf{q}} \\
\dot{\mathbf{q}}
\end{pmatrix}
+ \begin{pmatrix}
-M & 0 \\
0 & K
\end{pmatrix}
\begin{pmatrix}
\dot{\mathbf{q}} \\
\mathbf{q}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\Rightarrow \quad \dot{\mathbf{u}} = \mathbf{A} \mathbf{u}
\]

where \(\mathbf{A} = \mathbf{A}_B^{-1} \mathbf{A}_A\)

Direct time-integration
Stability and Accuracy of Time-Integration Operators

Multistep Time-Integration Methods

- General multistep time-integration method for first-order systems of the form \( \dot{\mathbf{u}} = \mathbf{A}\mathbf{u} \)

\[
\mathbf{u}_{n+1} = \sum_{j=1}^{m} \alpha_j \mathbf{u}_{n+1-j} - h \sum_{j=0}^{m} \beta_j \dot{\mathbf{u}}_{n+1-j}
\]

where \( h = t_{n+1} - t_n \) is the computational time-step, \( \mathbf{u}_n = \mathbf{u}(t_n) \), and

\[
\mathbf{u}_{n+1} = \begin{bmatrix}
\dot{\mathbf{q}}_{n+1} \\
\mathbf{q}_{n+1}
\end{bmatrix}
\]

is the state-vector calculated at \( t_{n+1} \) from the \( m \) preceding state vectors and their derivatives as well as the derivative of the state-vector at \( t_{n+1} \)

- \( \beta_0 \neq 0 \) leads to an implicit scheme — that is, a scheme where the evaluation of \( \mathbf{u}_{n+1} \) requires the solution of a system of equations

- \( \beta_0 = 0 \) corresponds to an explicit scheme — that is, a scheme where the evaluation of \( \mathbf{u}_{n+1} \) does not require the solution of any system of equations and instead can be deduced directly from the results at the previous time-steps
General multistep integration method for first-order systems (continue)

\[ u_{n+1} = \sum_{j=1}^{m} \alpha_j u_{n+1-j} - h \sum_{j=0}^{m} \beta_j \dot{u}_{n+1-j} \]

- trapezoidal rule (implicit)

\[ u_{n+1} = u_n + \frac{h}{2} (\dot{u}_n + \dot{u}_{n+1}) \Rightarrow (\frac{h}{2} A - I) u_{n+1} = -u_n - \frac{h}{2} \dot{u}_n \]

- backward Euler formula (implicit)

\[ u_{n+1} = u_n + h \dot{u}_{n+1} \Rightarrow (hA - I) u_{n+1} = -u_n \]

- forward Euler formula (explicit)

\[ u_{n+1} = u_n + h \dot{u}_n \Rightarrow u_{n+1} = (I + hA) u_n \]
Consider an undamped one-degree-of-freedom oscillator

\[ \ddot{q} + \omega_0^2 q = 0 \]

with \( \omega_0 = \pi \text{ rad/s} \) and the initial displacement

\[ q(0) = 1, \quad \dot{q}(0) = 0 \]

- exact solution

\[ q(t) = \cos \omega_0 t \]

- associated first-order system

\[ \dot{u} = Au \]

where

\[ A = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix} \]

\[ u = [\dot{q}, q]^T \], and initial condition

\[ u(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
Numerical solution

\[ T = 3\text{s}, \ h = \frac{T}{32} \]
Analysis of the characteristic equation of a time-integration method

- consider the first-order system $\dot{u} = Au$
- for this problem, the general multistep method can be written as

$$\mathbf{u}_{n+1} = \sum_{j=1}^{m} \alpha_j \mathbf{u}_{n+1-j} - h \sum_{j=0}^{m} \beta_j \dot{\mathbf{u}}_{n+1-j} \Rightarrow \sum_{j=0}^{m} [\alpha_j I - h \beta_j A] \mathbf{u}_{n+1-j} = 0, \quad \alpha_0 = -1$$

- let $\mu_r$ be the eigenvalues of $A$ and $\mathbf{X}$ be the matrix of associated eigenvectors
- the characteristic equation associated with $\sum_{j=0}^{m} [\alpha_j I - h \beta_j A] \mathbf{u}_{n+1-j} = 0$ is obtained by searching for a solution of the form

$$\mathbf{u}_{n+1-m} = \mathbf{Xa} \quad (\text{decomposition on an eigen basis})$$

$$\mathbf{u}_{(n+1-m)+1} = \lambda \mathbf{u}_{n+1-m} = \lambda \mathbf{Xa} \quad (\text{solution form})$$

$$\vdots$$

$$\mathbf{u}_{n+1} = \lambda \mathbf{u}_n = \cdots = \lambda^k \mathbf{u}_{n+1-k} = \cdots = \lambda^m \mathbf{Xa}$$

where $\lambda \in \mathbb{C}$ is called the solution amplification factor
Analysis of the characteristic equation of a time-integration method (continue)

- Hence

\[ \sum_{j=0}^{m} [\alpha_j I - h \beta_j A] \lambda^{m-j} X a = 0 \]

- Since \( X^{-1} A X = \text{diag}(\mu_r) \), premultiplying the above result by \( X^{-1} \) leads to

\[ \left[ \sum_{j=0}^{m} [\alpha_j I - h \beta_j \text{diag}(\mu_r)] \lambda^{m-j} \right] a = 0 \]

\[ \Rightarrow \sum_{j=0}^{m} [\alpha_j - h \beta_j \mu_r] \lambda^{m-j} = 0, \ r = 1, 2 \]

- hence, the numerical response \( u_{n+1} = \lambda^m X a \) remains bounded if each solution of the above characteristic equation of degree \( m \) satisfies \( |\lambda_k| < 1, \ k = 1, \ldots, m \)
Stability and Accuracy of Time-Integration Operators

Stability Behavior of Numerical Solutions

- Analysis of the characteristic equation of a time-integration method (continue)
  - the stability limit is a circle of unit radius
  - in the complex plane of \( \mu_r h \), the stability limit is therefore given by writing \( \lambda = e^{i\theta}, 0 \leq \theta \leq 2\pi \)

\[
\mu_r h = \frac{\sum_{j=0}^{m} \alpha_j e^{i(m-j)\theta}}{\sum_{j=0}^{m} \beta_j e^{i(m-j)\theta}}
\]

- one-step schemes (\( m = 1 \))

\[
\mu_r h = \frac{\alpha_0 e^{i\theta} + \alpha_1}{\beta_0 e^{i\theta} + \beta_1} = \frac{-e^{i\theta} + \alpha_1}{\beta_0 e^{i\theta} + \beta_1}
\]
Analysis of the characteristic equation of a time-integration method (continue)

- one-step schemes \((m = 1)\) (continue)

\[
\mu_r h = \frac{\alpha_0 e^{i\theta} + \alpha_1}{\beta_0 e^{i\theta} + \beta_1} = \frac{-e^{i\theta} + \alpha_1}{\beta_0 e^{i\theta} + \beta_1}
\]

- **Forward Euler**: \(\alpha_1 = 1, \beta_0 = 0, \beta_1 = -1 \Rightarrow \mu_r h = e^{i\theta} - 1\)
  the solution is unstable in the entire plane except inside the circle of unit radius and center \(-1\)

- **Backward Euler**: \(\alpha_1 = 1, \beta_0 = -1, \beta_1 = 0 \Rightarrow \mu_r h = 1 - e^{-i\theta}\)
  the solution is stable in the entire plane except inside the circle of unit radius and center \(1\)

- **Trapezoidal Rule**: \(\alpha_1 = 1, \beta_0 = -\frac{1}{2}, \beta_1 = -\frac{1}{2} \Rightarrow \mu_r h = \frac{2i\sin \theta}{1 + \cos \theta}\)
  the solution is stable in the entire left-hand plane
Analysis of the characteristic equation of a time-integration method (continue)

- application to the single degree-of-freedom oscillator

\[
\ddot{q} + \omega_0^2 q = 0, \quad \mathbf{A} = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}
\]

- the eigenvalues are \( \mu_r = \pm i\omega_0 \)
- the roots \( \mu_r h \) are located in the unstable region of the forward Euler scheme \( \Rightarrow \) amplification of the numerical solution
- the roots \( \mu_r h \) are located in the stable region of the backward Euler scheme \( \Rightarrow \) decay of the numerical solution
- the roots \( \mu_r h \) are located on the stable boundary of the trapezoidal rule scheme \( \Rightarrow \) the amplitude of the oscillations is preserved
Taylor’s expansion of a function $f$

\[ f(t_n + h) = f(t_n) + hf'(t_n) + \frac{h^2}{2} f''(t_n) + \cdots + \frac{h^s}{s!} f^{(s)}(t_n) + \frac{1}{s!} \int_{t_n}^{t_n+h} f^{(s+1)}(\tau)(t_n + h - \tau)^s \, d\tau \]

Application to the velocities and displacements

\begin{align*}
    f &= \ddot{q}, \ s = 0 \quad \Rightarrow \quad \dddot{q}_{n+1} = \ddot{q}_n + \int_{t_n}^{t_{n+1}} \dddot{q}(\tau) \, d\tau \\
    f &= q, \ s = 1 \quad \Rightarrow \quad q_{n+1} = q_n + h \ddot{q}_n + \int_{t_n}^{t_{n+1}} \dddot{q}(\tau)(t_{n+1} - \tau) \, d\tau
\end{align*}
Taylor expansions of $\ddot{q}_n$ and $\ddot{q}_{n+1}$ around $\tau \in [t_n, t_{n+1}]$

\[\ddot{q}_n = \ddot{q}(\tau) + q^{(3)}(\tau)(t_n - \tau) + q^{(4)}(\tau)\frac{(t_n - \tau)^2}{2} + \cdots \] \hspace{1cm} (1)

\[\ddot{q}_{n+1} = \ddot{q}(\tau) + q^{(3)}(\tau)(t_{n+1} - \tau) + q^{(4)}(\tau)\frac{(t_{n+1} - \tau)^2}{2} + \cdots \] \hspace{1cm} (2)

Combine $(1 - \gamma)$ (1) + $\gamma$ (2) and extract $\ddot{q}(\tau)$

\[\implies \ddot{q}(\tau) = (1 - \gamma)\ddot{q}_n + \gamma\ddot{q}_{n+1} + q^{(3)}(\tau)(\tau - h\gamma - t_n) + O(h^2 q^{(4)})\]

Combine $(1 - 2\beta)$ (1) + $2\beta$ (2) and extract $\ddot{q}(\tau)$

\[\implies \ddot{q}(\tau) = (1 - 2\beta)\ddot{q}_n + 2\beta\ddot{q}_{n+1} + q^{(3)}(\tau)(\tau - 2h\beta - t_n) + O(h^2 q^{(4)})\]
Newmark’s Family of Methods

The Newmark Method

- Substitute the 1st expression of $\ddot{q}(\tau)$ in
  $$\int_{t_n}^{t_{n+1}} \ddot{q}(\tau) d\tau$$

  $$\Rightarrow \int_{t_n}^{t_{n+1}} \ddot{q}(\tau) d\tau = \int_{t_n}^{t_{n+1}} ((1 - \gamma)\ddot{q}_n + \gamma \ddot{q}_{n+1} + q^{(3)}(\tau)(\tau - h\gamma - t_n) + \mathcal{O}(h^2 q^{(4)})) d\tau$$

  $$= (1 - \gamma) h \ddot{q}_n + \gamma h \ddot{q}_{n+1} + \int_{t_n}^{t_{n+1}} q^{(3)}(\tau)(\tau - h\gamma - t_n) d\tau + \mathcal{O}(h^3 q^{(4)})$$

- Apply the mean value theorem

  $$\Rightarrow \int_{t_n}^{t_{n+1}} \ddot{q}(\tau) d\tau = (1 - \gamma) h \ddot{q}_n + \gamma h \ddot{q}_{n+1} + q^{(3)}(\bar{\tau}) \left[\frac{(\tau - h\gamma - t_n)^2}{2}\right]_{t_n}^{t_{n+1}} + \mathcal{O}(h^3 q^{(4)})$$

  $$= (1 - \gamma) h \ddot{q}_n + \gamma h \ddot{q}_{n+1} + \left(\frac{1}{2} - \gamma\right) h^2 q^{(3)}(\bar{\tau}) + \mathcal{O}(h^3 q^{(4)})$$

- Substitute the 2nd expression of $\ddot{q}(\tau)$ in
  $$\int_{t_n}^{t_{n+1}} \ddot{q}(\tau)(t_{n+1} - \tau) d\tau$$

  $$\Rightarrow \int_{t_n}^{t_{n+1}} \ddot{q}(\tau)(t_{n+1} - \tau)d\tau = \left(\frac{1}{2} - \beta\right) h^2 \ddot{q}_n + \beta h^2 \ddot{q}_{n+1} + \left(\frac{1}{6} - \beta\right) h^3 q^{(3)}(\bar{\tau}) + \mathcal{O}(h^4 q^{(4)})$$
In summary

\[ \int_{t_n}^{t_{n+1}} \ddot{q}(\tau) d\tau = (1 - \gamma)h \ddot{q}_n + \gamma h \ddot{q}_{n+1} + r_n \]

\[ \int_{t_n}^{t_{n+1}} \ddot{q}(\tau)(t_{n+1} - \tau) d\tau = \left( \frac{1}{2} - \beta \right) h^2 \ddot{q}_n + \beta h^2 \ddot{q}_{n+1} + r'_n \]

where

\[ r_n = \left( \frac{1}{2} - \gamma \right) h^2 q^{(3)}(\tilde{\tau}) + O(h^3 q^{(4)}) \]

\[ r'_n = \left( \frac{1}{6} - \beta \right) h^3 q^{(3)}(\tilde{\tau}) + O(h^4 q^{(4)}) \]

and \( t_n < \tilde{\tau} < t_{n+1} \)
Hence, the approximation of each of the two previous integral terms by a quadrature scheme leads to

\[
\begin{align*}
\dot{q}_{n+1} &= \dot{q}_n + (1 - \gamma)h\ddot{q}_n + \gamma h\ddot{q}_{n+1} \\
q_{n+1} &= q_n + h\dot{q}_n + h^2 \left(\frac{1}{2} - \beta\right)\ddot{q}_n + h^2\beta\ddot{q}_{n+1}
\end{align*}
\] 

where \( \gamma \) and \( \beta \) are parameters associated with the quadrature scheme.
Newmark’s Family of Methods

The Newmark Method

- Particular values of the parameters $\gamma$ and $\beta$
  - $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$ leads to linearly interpolating $\ddot{q}(\tau)$ in $[t_n, t_{n+1}]$
    \[
    \ddot{q}_{ln}(\tau) = \ddot{q}_n + (\tau - t_n) \left( \frac{\ddot{q}_{n+1} - \ddot{q}_n}{h} \right)
    \]
  - $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ leads to averaging $\ddot{q}(\tau)$ in $[t_n, t_{n+1}]$
    \[
    \ddot{q}_{av}(\tau) = \frac{\ddot{q}_{n+1} + \ddot{q}_n}{2}
    \]
Application to the direct time-integration of $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t)$

- write the equilibrium equation at $t_{n+1}$ and substitute the expressions (3) and (4) into it

$$\implies [\mathbf{M} + \gamma h\mathbf{C} + \beta h^2\mathbf{K}]\ddot{\mathbf{q}}_{n+1} = \mathbf{p}_{n+1} - \mathbf{C}[\dot{\mathbf{q}}_n + (1 - \gamma)h\ddot{\mathbf{q}}_n]$$

$$- \mathbf{K} \left[ \mathbf{q}_n + h\dot{\mathbf{q}}_n + \left(\frac{1}{2} - \beta\right)h^2\ddot{\mathbf{q}}_n \right]$$

- if the time-step $h$ is uniform, $\mathbf{M} + \gamma h\mathbf{C} + \beta h^2\mathbf{K}$ can be factored once
- solve the above system of equations for $\ddot{\mathbf{q}}_{n+1}$
- substitute the result into the expressions (3) and (4) to obtain $\dot{\mathbf{q}}_{n+1}$ and $\mathbf{q}_{n+1}$
Newmark’s Family of Methods

Consistency of a Time-Integration Method

- A time-integration scheme is said to be consistent if
  \[
  \lim_{h \to 0} \frac{u_{n+1} - u_n}{h} = \dot{u}(t_n)
  \]

- The Newmark time-integration method is consistent
  \[
  \lim_{h \to 0} \frac{u_{n+1} - u_n}{h} = \lim_{h \to 0} \left[ \ddot{q}_n + \frac{1}{2} - \beta \right] h \ddot{q}_n + \beta h \ddot{q}_{n+1} = \begin{bmatrix} \ddot{q}_n \\ \ddot{q}_n \end{bmatrix}
  \]

- Consistency is one necessary condition for convergence
A time-integration scheme is said to be stable if there exists an integration time-step $h_0 > 0$ so that for any $h \in [0, h_0]$, a finite variation of the state vector at time $t_n$ induces only a non-increasing variation of the state-vector $\mathbf{u}_{n+j}$ calculated at a subsequent time $t_{n+j}$.

Stability is the other necessary condition for convergence.
Premultiplying Eq. (3) and Eq. (4) by $\mathbf{M}$ and taking into account the equations of equilibrium (1) at $t_n$ and $t_{n+1}$ leads after some algebraic manipulations to

\begin{align*}
\mathbf{M}\ddot{q}_{n+1} &= \mathbf{M}\dot{q}_n + h(1 - \gamma)[-\mathbf{C}\dot{q}_n - \mathbf{K}q_n + \mathbf{p}_n] \\
&+ \gamma h[-\mathbf{C}\dot{q}_{n+1} - \mathbf{K}q_{n+1} + \mathbf{p}_{n+1}] \\
\mathbf{M}q_{n+1} &= \mathbf{M}q_n + h\mathbf{M}\dot{q}_n + \left(\frac{1}{2} - \beta\right)h^2[-\mathbf{C}\dot{q}_n - \mathbf{K}q_n + \mathbf{p}_n] \\
&+ \beta h^2[-\mathbf{C}\dot{q}_{n+1} - \mathbf{K}q_{n+1} + \mathbf{p}_{n+1}] \quad (5)
\end{align*}
Equations (5) can be re-written in matrix form as

$$\mathbf{u}_{n+1} = \mathbf{A}(h)\mathbf{u}_n + \mathbf{g}_{n+1}(h)$$

where $\mathbf{A}$ is the amplification matrix associated with the integration operator

$$\mathbf{A}(h) = \mathbf{H}_1^{-1}(h)\mathbf{H}_0(h), \quad \mathbf{g}_{n+1} = \mathbf{H}_1^{-1}(h)\mathbf{b}_{n+1}(h)$$

$$\mathbf{b}_{n+1} = \begin{bmatrix}
(1 - \gamma)h\mathbf{p}_n + \gamma h\mathbf{p}_{n+1} \\
\left(\frac{1}{2} - \beta\right) h^2 \mathbf{p}_n + \beta h^2 \mathbf{p}_{n+1}
\end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix}
\mathbf{M} + \gamma h\mathbf{C} & \gamma h\mathbf{K} \\
\beta h^2\mathbf{C} & \mathbf{M} + \beta h^2\mathbf{K}
\end{bmatrix}$$

$$\mathbf{H}_0 = -\begin{bmatrix}
-\mathbf{M} + (1 - \gamma)h\mathbf{C} \\
\left(\frac{1}{2} - \beta\right) h^2\mathbf{C} - h\mathbf{M}
\end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix}
(1 - \gamma)h\mathbf{K} \\
-\mathbf{M} + \left(\frac{1}{2} - \beta\right) h^2\mathbf{K}
\end{bmatrix}$$
Effect of an initial disturbance

\[ \delta u_0 = u'_0 - u_0 \]

\[ \Rightarrow \delta u_{n+1} = A(h)\delta u_n = A^2(h)\delta u_{n-1} = \cdots = A(h)^{n+1}\delta u_0 \]

consider the eigenpairs of \( A(h) \)

\[ (\lambda_r, x_r) \]

then

\[ \delta u_{n+1} = A^{n+1}(h) \sum_{s=1}^{2N} a_s x_s = \sum_{s=1}^{2N} a_s \lambda_s^{n+1} x_s \]

where \( N \) is the dimension of the semi-discrete second-order dynamical system

\[ \Rightarrow \delta u_{n+1} \text{ will be amplified by the time-integration operator only if the modulus of an eigenvalue of } A(h) \text{ is greater than unity} \]

\[ \Rightarrow \delta u_{n+1} \text{ will not be amplified by the time-integration operator if all moduli of all eigenvalues of } A(h) \text{ are less than unity} \]
Undamped case

decouple the equations of equilibrium by writing them (for the purpose of analysis) in the modal basis

\[ q = Qy = \sum_{i=1}^{N} y_i q_{a_i} \implies \ddot{y}_i + \omega_i^2 y_i = p_i(t) \]

apply the Newmark scheme to the \( i \)-th modal equation recalled above to obtain the amplification matrix

\[
A(h) = \begin{bmatrix}
1 - \gamma \frac{\omega_i^2 h^2}{1 + \beta \omega_i^2 h^2} & -\omega_i^2 h^2 \left(1 - \frac{\gamma}{2} \frac{\omega_i^2 h^2}{1 + \beta \omega_i^2 h^2}\right) \\
\frac{h}{1 + \beta \omega_i^2 h^2} & 1 - \frac{1}{2} \frac{\omega_i^2 h^2}{1 + \beta \omega_i^2 h^2}
\end{bmatrix}
\]

characteristic equation is

\[
\lambda^2 - \lambda \left(2 - (\gamma + \frac{1}{2})\eta^2\right) + 1 - (\gamma - \frac{1}{2})\eta^2 = 0
\]

where

\[
\eta^2 = \frac{\omega_i^2 h^2}{1 + \beta \omega_i^2 h^2}
\]

characteristic equation has:

- a pair of complex conjugate roots \( \lambda_1 \) and \( \lambda_2 \) if
  \[
  (\gamma + \frac{1}{2})^2 - 4\beta \leq \frac{4}{\omega_i^2 h^2}, \quad i = 1, \cdots, N \text{ (case 1)}
  \]
- two identical real roots if \( (\gamma + \frac{1}{2})^2 \eta^2 - 4 = 0 \) (case 2)
- two distinct real roots if \( (\gamma + \frac{1}{2})^2 \eta^2 > 4 \) (case 3)
Undamped case (continue)

it can be shown that case 1 is the limiting case, in which case

\[ \lambda_{1,2} = \rho e^{\pm i\phi} \]

where

\[ \rho = \sqrt{1 - \left( \gamma - \frac{1}{2} \right) \eta^2} \]

\[ \phi = \arctan \left( \frac{\eta \sqrt{1 - \frac{1}{4} \left( \gamma + \frac{1}{2} \right)^2 \eta^2}}{1 - \frac{1}{2} \left( \gamma + \frac{1}{2} \right) \eta^2} \right) \]

then, the Newmark scheme is stable if

\[ \rho \leq 1 \Rightarrow \gamma \geq \frac{1}{2} \]

and

\[ \left( \gamma + \frac{1}{2} \right)^2 - 4\beta \leq \frac{4}{\omega_i^2 h^2}, \quad i = 1, \cdots, N \]

\[ \Rightarrow \] limitation on the maximum time-step
Undamped case (continue)

- the algorithm is conditionally stable if

\[ \gamma \geq \frac{1}{2} \]

- it is unconditionally stable if furthermore \( \beta \geq \frac{1}{4} \left( \gamma + \frac{1}{2} \right)^2 \) — that is,

\[ \gamma \geq \frac{1}{2} \quad \text{and} \quad \beta \geq \frac{1}{4} \left( \gamma + \frac{1}{2} \right)^2 \]

- the choice \( \gamma = \frac{1}{2} \) and \( \beta = \frac{1}{4} \) leads to an unconditionally stable time-integration operator of maximum accuracy
Undamped case (continue)

\[ \beta = \frac{\gamma (\gamma + \frac{1}{2})^2}{1} \]

\[ (\gamma + \frac{1}{2})^2 - 4\beta \leq 4/\omega^2 h^2 \]

Stability of the Newmark scheme
Damped case ($C \neq 0$)

- consider the case of modal damping
- then, the uncoupled equations of motion are

\[ \ddot{y}_i + 2\xi_i \omega_i \dot{y}_i + \omega_i^2 y_i = p_i(t) \]

where $\xi_i$ is the modal damping coefficient

- consider the case $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$
- an analysis similar to that performed in the undamped case reveals that in this case, the Newmark scheme remains stable as long as $\xi_i < 1$
- in general, damping has a stabilizing effect for moderate values of $\xi_i$
Free-vibration of an undamped linear oscillator

\[ \ddot{y} + \omega^2 y = 0 \quad \text{and} \quad y(0) = y_0, \quad \dot{y}(0) = 0 \]

the above problem has an exact solution \( y(t) = y_0 \cos \omega t \) which can be written in complex discrete form as \( y_{n+1} = e^{i\omega h} y_n \Rightarrow \) the exact amplification factor is \( \rho_{ex} = 1 \) and the exact phase is \( \phi_{ex} = \omega h \)

the numerical solution satisfies \( u_{n+1} = \begin{bmatrix} \dot{y}_{n+1} \\ y_{n+1} \end{bmatrix} = A(h) u_n \)

let \( \lambda_{1,2}(\beta, \gamma) \) be the eigenvalues of \( A(h, \beta, \gamma) \)

when \( (\gamma + \frac{1}{2})^2 - 4\beta \leq \frac{4}{\omega_i^2 h^2} \), \( \lambda_1 \) and \( \lambda_2 \) are complex-conjugate

\[ \lambda_{1,2}(\beta, \gamma) = \rho(\beta, \gamma) e^{\pm i\phi(\beta, \gamma)} \]

where

\[ \rho = \sqrt{1 - \left( \gamma - \frac{1}{2} \right) \eta^2}, \quad \phi = \arctan \left( \frac{\eta \sqrt{1 - \frac{1}{4} (\gamma + \frac{1}{2})^2 \eta^2}}{1 - \frac{1}{2} (\gamma + \frac{1}{2}) \eta^2} \right), \quad \eta^2 = \frac{\omega_i^2 h^2}{1 + \beta \omega_i^2 h^2} \]
Free-vibration of an undamped linear oscillator (continue)

- amplitude error

\[ \rho - \rho_{ex} = \rho - 1 = -\frac{1}{2} \left( \gamma - \frac{1}{2} \right) \omega^2 h^2 + O(h^4) \]

- relative periodicity error

\[ \frac{\Delta T}{T} = \frac{\Delta \frac{1}{\phi}}{\frac{1}{\phi}} = \frac{1}{\phi} - \frac{1}{\phi_{ex}} = \frac{\omega h}{\phi} - 1 = \frac{1}{2} \left( \beta - \frac{1}{12} \right) \omega^2 h^2 + O(h^3) \]
### Newmark’s Family of Methods

#### Amplitude and Periodicity Errors

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>Stability limit $\omega h$</th>
<th>Amplitude error $\rho - 1$</th>
<th>Periodicity error $\frac{\Delta T}{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Purely explicit</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{4}$</td>
<td>$-\frac{\omega^2 h^2}{24}$</td>
</tr>
<tr>
<td>Central difference</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$-\frac{\omega^2 h^2}{24}$</td>
</tr>
<tr>
<td>Fox &amp; Goodwin</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{12}$</td>
<td>2.45</td>
<td>0</td>
<td>$O(h^3)$</td>
</tr>
<tr>
<td>Linear acceleration</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{6}$</td>
<td>3.46</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{24}$</td>
</tr>
<tr>
<td>Average constant</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$\infty$</td>
<td>0</td>
<td>$\frac{\omega^2 h^2}{12}$</td>
</tr>
</tbody>
</table>

**Table:** Time-integration schemes of the Newmark family

- The purely explicit scheme ($\gamma = 0$, $\beta = 0$) is useless
- The Fox & Godwin scheme has asymptotically the smallest phase error but is only conditionally stable
- The average constant acceleration scheme ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$) is the unconditionally stable scheme with asymptotically the highest accuracy
Newmark’s Family of Methods

Total Energy Conservation

- Conservation of total energy
  - dynamic system with scleronomic constraints

\[
\frac{d}{dt} (\mathcal{T} + \mathcal{V}) = -m\mathcal{D} + \sum_{s=1}^{n_s} Q_s \dot{q}_s
\]

- \( \mathcal{T} = \frac{1}{2} \dot{q}^T M \dot{q} \) and \( \mathcal{V} = \frac{1}{2} q^T K q \)

- the dissipation function \( \mathcal{D} \) is a quadratic function of the velocities \( (m = 2) \)

\[
\mathcal{D} = \frac{1}{2} \dot{q}^T \mathcal{C} \dot{q}
\]

- external force component of the power balance

\[
\sum_{s=1}^{n_s} Q_s \dot{q}_s = \dot{q}^T p
\]

- integration over a time-step \([t_n, t_{n+1}]\)

\[
[\mathcal{T} + \mathcal{V}]_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} (\dot{q}^T \mathcal{C} \dot{q} + \dot{q}^T p) dt
\]
Newmark’s Family of Methods

Total Energy Conservation

- Conservation of total energy (continue)
  - note that because $M$ and $K$ are symmetric ($M^T = M$ and $K^T = K$)

$$\begin{align*}
[T + V]_{t_{n+1}} &= [T_{n+1} - T_n] + [V_{n+1} - V_n] \\
&= \frac{1}{2} (\dot{q}_{n+1} - \dot{q}_n)^T M (q_{n+1} + \dot{q}_n) \\
&\quad + \frac{1}{2} (q_{n+1} - q_n)^T K (q_{n+1} + q_n)
\end{align*}$$

- when time-integration is performed using the Newmark algorithm with $
\gamma = \frac{1}{2}, \quad \beta = \frac{1}{4},$
the above variation becomes (see (3) and (4))

$$\begin{align*}
[T + V]_{t_{n+1}} &= \frac{1}{2} (q_{n+1} - q_n)^T (p_n + p_{n+1}) - \frac{h}{4} (\dot{q}_{n+1} + \dot{q}_n)^T C (\dot{q}_{n+1} + \dot{q}_n)
\end{align*}$$

- when applied to a conservative system ($C = 0$ and $p = 0$), preserves the total energy

- for non-conservative systems, $[T + V]_{t_{n+1}} = \int_{t_n}^{t_{n+1}} (-\dot{q}^T C \dot{q} + \dot{q}^T p) dt$ and therefore both terms in the right-hand side of the above formula result from numerical quadrature relationships that are consistent with the time-integration operator

$$\begin{align*}
\int_{t_n}^{t_{n+1}} \dot{q}^T p dt &\approx \left( \int_{t_n}^{t_{n+1}} \dot{q}^T dt \right) \left( \frac{p_n + p_{n+1}}{2} \right) = \frac{1}{2} (q_{n+1} - q_n)^T (p_n + p_{n+1}) \\
\int_{t_n}^{t_{n+1}} \dot{q}^T C \dot{q} dt &\approx \left( \int_{t_n}^{t_{n+1}} \dot{q}^T dt \right) C \left( \frac{\dot{q}_n + \dot{q}_{n+1}}{2} \right) = \frac{1}{2} (q_{n+1} - q_n)^T C \left( \frac{\dot{q}_n + \dot{q}_{n+1}}{2} \right)
\end{align*}$$
Explicit Time Integration Using the Central Difference Algorithm

Algorithm in Terms of Velocities

- Central Difference (CD) scheme = Newmark’s scheme with $\gamma = \frac{1}{2}$, $\beta = 0$

\[ \dot{q}_{n+1} = \dot{q}_n + h_{n+1} \left( \frac{\ddot{q}_n + \ddot{q}_{n+1}}{2} \right) \]  

\[ q_{n+1} = q_n + h_{n+1} \dot{q}_n + \frac{h_{n+1}^2}{2} \ddot{q}_n \]

where $h_{n+1} = t_{n+1} - t_n$

- Equivalent three-step form

  - start with $q_n = q_{n-1} + h_n \dot{q}_{n-1} + \frac{h_n^2}{2} \ddot{q}_{n-1}$
  - divide by $h_n$
  - subtract the result from $q_{n+1}$ divided by $h_{n+1}$
  - account for the relationship (6)

\[ \Rightarrow \ddot{q}_n = \frac{h_n(q_{n+1} - q_n) - h_{n+1}(q_n - q_{n-1})}{h_{n+\frac{1}{2}} h_n h_{n+1}} \]

where $h_{n+\frac{1}{2}} = \frac{h_n + h_{n+1}}{2}$
Explicit Time Integration Using the Central Difference Algorithm

Algorithm in Terms of Velocities

Case of a constant time-step \( h \)

\[
\ddot{q}_n = \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2}
\]

Efficient implementation

- use a lumped mass matrix \( M \)
- initialize: \( \ddot{q}_0 = M^{-1}(p_0 - Kq_0) \) and \( \dot{q}_{1/2} = \dot{q}_0 + \frac{h_1}{2} \ddot{q}_0 \)
- increment the displacement: \( q_n = q_{n-1} + h_n \dot{q}_{n-1/2} \)
- compute the acceleration: \( \ddot{q}_n = M^{-1}(p_n - Kq_n) \)
- increment the velocity at half time-step (formula results from (7))

\[
\dot{q}_{n+1/2} = \dot{q}_{n-1/2} + h_{n+1/2} \ddot{q}_n
\]

Stability condition: \( \omega_{cr} h \leq 2 \) where \( \omega_{cr} \) is the highest frequency contained in the model:
this condition is also known as the Courant condition

\( h_{cr} = \frac{2}{\omega_{cr}} \) is referred to here as the maximum Courant stability time-step
 Explicit Time Integration Using the Central Difference Algorithm

Application Example: the Clamped-Free Bar Excited by an End Load

- Clamped bar subjected to a step load at its free end
- Model made of $N = 20$ finite elements with equal length $l = \frac{L}{N}$

\[
\frac{EA}{ml^2} = 1 \\
\frac{F}{EA} = 1
\]

- Lumped mass matrix
- Eigenfrequencies of the continuous system

\[
\omega_{\text{cont}} = (2r - 1) \frac{\pi}{2} \sqrt{\frac{EA}{ml^2}} = \left( \frac{2r - 1}{N} \right) \frac{\pi}{2} \sqrt{\frac{EA}{ml^2}} = \left( \frac{2r - 1}{N} \right) \frac{\pi}{2}
\]
Explicit Time Integration Using the Central Difference Algorithm

Application Example: the Clamped-Free Bar Excited by an End Load

Finite element stiffness and mass matrices

\[ M = \frac{ml^2}{2} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & \cdot \\ 0 & \cdot & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & \cdot \end{bmatrix} \quad K = \frac{EA}{l} \begin{bmatrix} -1 & 2 & \cdot \\ -1 & 2 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix} \]

Analytical frequencies of the discrete system

\[ \omega_r = 2 \sqrt{\frac{EA}{ml^2}} \sin \left( \left( \frac{2r - 1}{2N} \right) \frac{\pi}{2} \right) = 2 \sin \left( \left( \frac{2r - 1}{2N} \right) \frac{\pi}{2} \right), \quad r = 1, 2, \cdots N \]

\[ \Rightarrow \omega_{cr} < \omega_{cr} (N \to \infty) = 2 \]

Critical time-step for the CD algorithm

\[ \omega_{cr} h_{cr} = 2 \Rightarrow h_{cr} = 1 \]
Explicit Time Integration Using the Central Difference Algorithm

Application Example: the Clamped-Free Bar Excited by an End Load

\[ h = 1, \ h = 0.707 \]
Explicit Time Integration Using the Central Difference Algorithm

Application Example: the Clamped-Free Bar Excited by an End Load

$h = 1.0012$

Node 10

Node 20
Explicit Time Integration Using the Central Difference Algorithm

Restitution of the Exact Solution by the Central Difference Method

For the clamped-free bar example, the CD method computes the exact solution when $h = h_{cr}$

Comparison of the exact solution of the continuous free-vibration bar problem and the analytical expression of the numerical solution

- denote by $q_{j,n}$ the value of the $j$-th d.o.f. at time $t_n$
- if $q_{j,n}$ is not located at the boundary, it satisfies (see (8))

\[
\frac{ml}{h^2}(q_{j,n+1} - 2q_{j,n} + q_{j,n-1}) + \frac{EA}{l}(-q_{j-1,n} + 2q_{j,n} - q_{j+1,n}) = 0
\]

the general solution of the above problem is

\[
q_{j,n} = \sin(j\mu + \phi) \left[ a \cos n\theta + b \sin n\theta \right]
\]

\[\text{spatial component} \quad \text{temporal component}\]  

(9)

comparing the above expression to the exact harmonic solution of the continuous form of this free-vibration problem (which can be derived analytically)

\[
\Rightarrow n\theta = \omega t = \omega nh \Rightarrow \frac{\theta}{h} = \omega_{num}
\]
Explicit Time Integration Using the Central Difference Algorithm

Restitution of the Exact Solution by the Central Difference Method

Comparison of the exact solution of the free-vibration bar problem and the analytical expression of the numerical solution (continue)

- introduce the exact expression for \( q_{j,n} \) in the CD scheme

\[
2[(1 - \cos \mu) - \lambda^2(1 - \cos \theta)]q_{j,n} = 0
\]

where \( \lambda^2 = \left( \frac{ml^2}{EA} \right) \frac{1}{h^2} = \frac{1}{h^2} \Rightarrow 1 - \cos \theta = \frac{1}{\lambda^2} (1 - \cos \mu) \)

- make use of the boundary conditions in space \( (q_{0,n} = 0, \text{ and plug (9) in the last equation in (8)}) \Rightarrow \phi = 0 \) and \( \mu_r = \left( \frac{2r-1}{N} \right) \frac{\pi}{2} \), \( r \in \mathbb{N}^* \)

\[
1 - \cos \theta_r = \frac{1}{\lambda^2} (1 - \cos \mu_r)
\]

- special case \( \lambda^2 = 1 \ (h = h_{cr} = 1) \Rightarrow \theta_r = \mu_r \) and

\[
\omega_{numr} = \frac{\theta_r}{h} = \mu_r = \left( \frac{2r-1}{N} \right) \frac{\pi}{2} \sqrt{\frac{EA}{ml^2}} = \left( \frac{2r-1}{N} \right) \frac{\pi}{2}
\]

\[
\Rightarrow \text{the numerical frequency coincides with the } r\text{-th eigenfrequency of the continuous system}
\]