

AA216/CME345: PROJECTION-BASED MODEL ORDER REDUCTION

Projection-Based Model Order Reduction

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Outline

- 1 Solution Approximation
- 2 Orthogonal and Oblique Projections
- 3 Galerkin and Petrov-Galerkin Projections
- 4 Equivalent High-Dimensional Model
- 5 Error Analysis
- 6 Preservation of Model Stability

└ Solution Approximation

└ High-Dimensional Model

■ Ordinary Differential Equation (ODE)

$$\frac{d\mathbf{w}}{dt}(t) = \mathbf{f}(\mathbf{w}(t), t) \quad (1)$$

- $\mathbf{w} \in \mathbb{R}^N$: State variable
- initial condition: $\mathbf{w}(0) = \mathbf{w}_0$

■ Output equation

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{w}(t), t) \quad (2)$$

- $\mathbf{y} \in \mathbb{R}^q$: Vector of output variables (typically $q \ll N$)

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- Note the absence of a parameter dependence for now

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└ Low Dimensionality of Trajectories

- In many cases, the trajectories of the solutions computed using High-Dimensional Models (HDMs) are contained in **low-dimensional subspaces**

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- Let \mathcal{S} denote such a subspace and let $k_{\mathcal{S}} = \dim(\mathcal{S})$
- The state variable – or simply, the state – can be written exactly as a linear combination of vectors spanning \mathcal{S}

$$\mathbf{w}(t) = q_1(t)\mathbf{v}_1 + \cdots + q_{k_{\mathcal{S}}}(t)\mathbf{v}_{k_{\mathcal{S}}}$$

- $\mathbf{V}_{\mathcal{S}} = [\mathbf{v}_1 \cdots \mathbf{v}_{k_{\mathcal{S}}}] \in \mathbb{R}^{N \times k_{\mathcal{S}}}$ is a **time-invariant** basis for \mathcal{S}
- $(q_1(t), \cdots, q_{k_{\mathcal{S}}}(t))$ are the **generalized coordinates** for $\mathbf{w}(t)$ in \mathcal{S}
- $\mathbf{q}(t) = [q_1(t) \cdots q_{k_{\mathcal{S}}}(t)]^T \in \mathbb{R}^{k_{\mathcal{S}}}$ is the **reduced-order state vector**

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- In matrix form, the above expansion can be written as

$$\mathbf{w}(t) = \mathbf{V}_{\mathcal{S}}\mathbf{q}(t)$$

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- Substituting the above subspace approximation in Eq. (1) and in Eq. (2) leads to

$$\begin{aligned}\frac{d}{dt}(\mathbf{V}\mathbf{q}(t)) &= \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) + \mathbf{r}(t) \\ \mathbf{y}(t) &\approx \mathbf{g}(\mathbf{V}\mathbf{q}(t), t)\end{aligned}$$

where $\mathbf{r}(t)$ is the residual due to the subspace approximation

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- Set of N differential equations in terms of k unknowns

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- Set of N differential equations in terms of k unknowns

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- Over-determined system ($k < N$)

└ Orthogonal and Oblique Projections

└ Orthogonality

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- Let \mathbf{w} and \mathbf{y} be two vectors in \mathbb{R}^N
 - \mathbf{w} and \mathbf{y} are **orthogonal** to each other with respect to the canonical inner product in \mathbb{R}^N if and only if

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- Let \mathbf{V} be a matrix in $\mathbb{R}^{N \times k}$
 - \mathbf{V} is an **orthogonal (orthonormal) matrix** if and only if

$$\mathbf{V}^T \mathbf{V} = \mathbf{I}_k$$

Definition

A matrix $\Pi \in \mathbb{R}^{N \times N}$ is a **projection** matrix (or projective matrix, idempotent matrix) if

$$\Pi^2 = \Pi$$

- Some direct consequences
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- $\text{range}(\mathbf{\Pi})$ is invariant under the action of $\mathbf{\Pi}$
- 0 and 1 are the only possible eigenvalues of $\mathbf{\Pi}$
- $\mathbf{\Pi}$ is diagonalizable (follows from the previous consequence)
- let k be the rank of $\mathbf{\Pi}$: then, there exists a basis \mathbf{X} such that

$$\mathbf{\Pi} = \mathbf{X} \begin{bmatrix} \mathbf{I}_k & \\ & \mathbf{0}_{N-k} \end{bmatrix} \mathbf{X}^{-1}$$

(follows from the two previous consequences)

- Consider

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$$\Pi = \mathbf{X} \begin{bmatrix} \mathbf{I}_k & \\ & \mathbf{0}_{N-k} \end{bmatrix} \mathbf{X}^{-1}$$

- decompose \mathbf{X} as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{bmatrix}, \text{ where } \mathbf{X}_1 \in \mathbb{R}^{N \times k} \text{ and } \mathbf{X}_2 \in \mathbb{R}^{N \times (N-k)}$$

then, $\forall \mathbf{w} \in \mathbb{R}^N$

- $\Pi \mathbf{w} \in \text{range}(\Pi) = \text{range}(\mathbf{X}_1) = \mathcal{S}_1$

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- Π defines the **projection onto \mathcal{S}_1 parallel to \mathcal{S}_2**

$$\boxed{\mathbb{R}^N = \mathcal{S}_1 \oplus \mathcal{S}_2}$$

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- Consider the case where $\mathcal{S}_2 = \mathcal{S}_1^\perp$
- Let $\mathbf{V} \in \mathbb{R}^{N \times k}$ be an **orthogonal** matrix whose columns span \mathcal{S}_1 , and let $\mathbf{w} \in \mathbb{R}^N$: The orthogonal projection of \mathbf{w} onto the subspace \mathcal{S}_1 is

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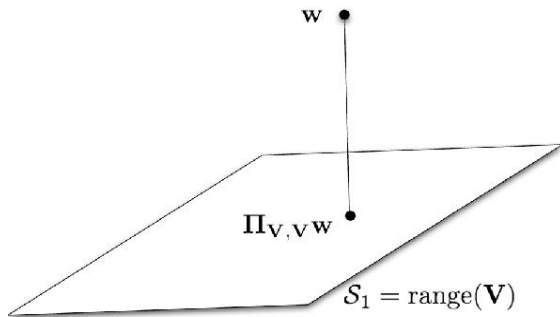
- special case #2: If \mathbf{w} is orthogonal to \mathcal{S}_1 – that is, $\mathbf{V}^T\mathbf{w} = \mathbf{0}$

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└ Orthogonal and Oblique Projections

└ Orthogonal Projections

$$\Pi_{V,V} w = VV^T w$$



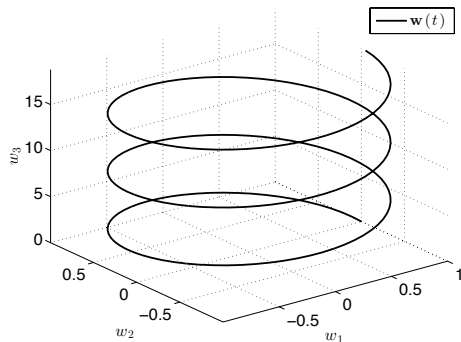
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■ Example: Helix in 3D ($N = 3$)

- let $\mathbf{w}(t) \in \mathbb{R}^3$ define a curve parameterized by $t \in [0, 6\pi]$ as follows

$$\mathbf{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

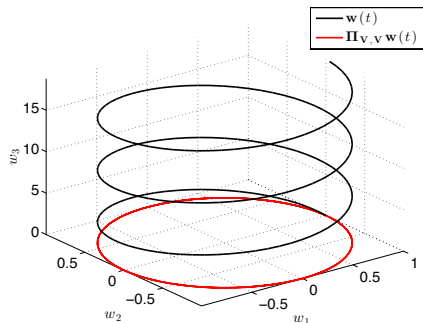


└ Orthogonal and Oblique Projections

└ Orthogonal Projections

- Orthogonal projection onto
 - $\text{range}(\mathbf{V}) = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$

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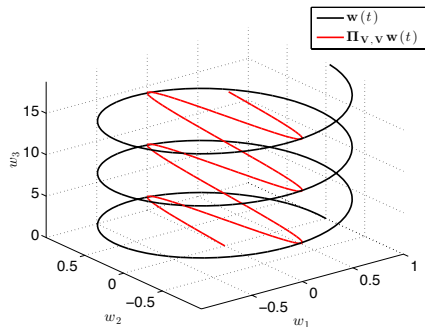


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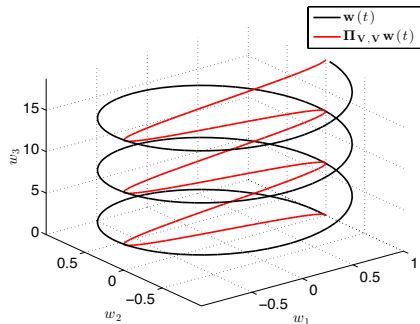


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- The following is the **general case**, where \mathcal{S}_2 may be distinct from \mathcal{S}_1^\perp
- Let $\mathbf{V} \in \mathbb{R}^{N \times k}$ and $\mathbf{W} \in \mathbb{R}^{N \times k}$ be two **full-column rank** matrices whose columns span respectively \mathcal{S}_1 and \mathcal{S}_2^\perp

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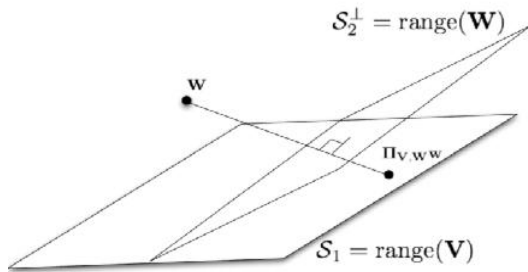
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- special case #2: If $\mathbf{w} \in \mathcal{S}_2$ – that is, \mathbf{w} is orthogonal to \mathcal{S}_2^\perp – then $\mathbf{W}^T \mathbf{w} = \mathbf{0}$ and $\Pi_{\mathbf{V}, \mathbf{W}} \mathbf{w} = \mathbf{V}(\mathbf{W}^T \mathbf{V})^{-1} \underbrace{\mathbf{W}^T \mathbf{w}}_{\mathbf{0}} = \mathbf{0}$

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└ Oblique Projections

$$\Pi_{V,W} w = V(W^T V)^{-1} W^T w$$



└ Orthogonal and Oblique Projections

└ Oblique Projections

■ Example: Helix in 3D

■ bases

$$\mathbf{V} = [\mathbf{e}_1 \ \mathbf{e}_2], \ \mathbf{W} = [\mathbf{e}_1 + \mathbf{e}_3 \ \mathbf{e}_2 + \mathbf{e}_3]$$

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■ projection matrix

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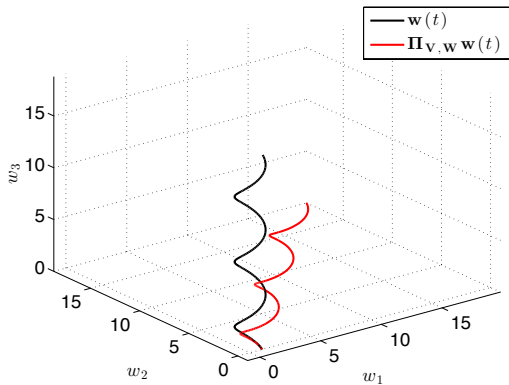
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■ projected helix equation

$$\mathbf{\Pi}_{\mathbf{V},\mathbf{W}} \mathbf{w}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix} = \begin{bmatrix} \cos(t) + t \\ \sin(t) + t \\ 0 \end{bmatrix}$$

└ Orthogonal and Oblique Projections

└ Oblique Projections



- Start from a HDM for the problem of interest

$$\begin{aligned}\frac{d\mathbf{w}}{dt}(t) &= \mathbf{f}(\mathbf{w}(t), t) \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{w}(t), t) \\ \mathbf{w}(0) &= \mathbf{w}_0\end{aligned}$$

- $\mathbf{w} \in \mathbb{R}^N$: Vector of state variables
- $\mathbf{y} \in \mathbb{R}^q$: Vector of output variables (typically $q \ll N$)
- $\mathbf{f}(\cdot, \cdot) \in \mathbb{R}^N$: Completes the specification of the HDM-based problem

- └ Galerkin and Petrov-Galerkin Projections
- └ Projection-Based Model Order Reduction

- The goal is to construct a **Projection-based Reduced-Order Model** (PROM)

$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= \mathbf{f}_r(\mathbf{q}(t), t) \\ \mathbf{y}(t) &\approx \mathbf{g}_r(\mathbf{q}(t), t)\end{aligned}$$

where

- $\mathbf{q} \in \mathbb{R}^k$: **Vector of reduced-order state variables**, $k \ll N$
 - $\mathbf{y} \in \mathbb{R}^q$: Vector of output variables
 - $\mathbf{f}_r(\cdot, \cdot) \in \mathbb{R}^k$: Completes the description of the PROM
- The discussion of the initial condition is deferred to later

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 - preserve as many properties of the HDM as possible

└ Galerkin and Petrov-Galerkin Projections

└ Petrov-Galerkin Projection

- Recall the residual $\mathbf{r}(t) \in \mathbb{R}^N$ introduced by approximating $\mathbf{w}(t)$ as $\mathbf{V}\mathbf{q}(t)$

$$\mathbf{V} \frac{d\mathbf{q}}{dt}(t) = \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) + \mathbf{r}(t) \Leftrightarrow \mathbf{r}(t) = \mathbf{V} \frac{d\mathbf{q}}{dt}(t) - \mathbf{f}(\mathbf{V}\mathbf{q}(t), t)$$

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- Constrain this residual to be orthogonal to a subspace \mathcal{W} defined by a **test basis** $\mathbf{W} \in \mathbb{R}^{N \times k}$ – that is, compute $\mathbf{q}(t)$ such that

$$\mathbf{W}^T \mathbf{r}(t) = \mathbf{0}$$

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- This leads to the *descriptive form* of the governing equations of the **Petrov-Galerkin** PROM

$$\mathbf{W}^T \mathbf{V} \frac{d\mathbf{q}}{dt}(t) = \mathbf{W}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t), t)$$

└ Galerkin and Petrov-Galerkin Projections

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- Assume that $\mathbf{W}^T \mathbf{V}$ is non-singular: In this case, the PROM can be re-written in the *non-descriptive form*

$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) \\ \mathbf{y}(t) &\approx \mathbf{g}(\mathbf{V}\mathbf{q}(t), t)\end{aligned}$$

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- After the above reduced-order equations have been solved, the subspace approximation of the high-dimensional state vector can be reconstructed, if needed, as follows

$$\mathbf{w}(t) \approx \mathbf{V}\mathbf{q}(t)$$

└ Galerkin and Petrov-Galerkin Projections

└ Galerkin Projection

- If $\mathbf{W} = \mathbf{V}$, the projection method is called a **Galerkin** projection and the resulting PROM is called a **Galerkin** PROM

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- If $\mathbf{W} = \mathbf{V}$, the projection method is called a **Galerkin** projection and the resulting PROM is called a **Galerkin** PROM
- If in addition \mathbf{V} is orthogonal, the reduced-order equations become

$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= \mathbf{V}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) \\ \mathbf{y}(t) &\approx \mathbf{g}(\mathbf{V}\mathbf{q}(t), t)\end{aligned}$$

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└ Linear Time-Invariant Systems

- Special case: **Linear Time-Invariant (LTI)** systems

$$\mathbf{f}(\mathbf{w}(t), t) = \mathbf{A}\mathbf{w}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{g}(\mathbf{w}(t), t) = \mathbf{C}\mathbf{w}(t) + \mathbf{D}\mathbf{u}(t)$$

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- reduced-order LTI operators

$$\mathbf{A}_r = (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{A} \mathbf{V} \in \mathbb{R}^{k \times k}, \quad k \ll N$$

$$\mathbf{B}_r = (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{B} \in \mathbb{R}^{k \times \text{in}}$$

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└ Galerkin and Petrov-Galerkin Projections

└ Initial Condition

- High-dimensional initial condition

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- Alternative: use an affine approximation $\mathbf{w}(t) = \mathbf{w}(0) + \mathbf{V} \mathbf{q}(t)$ (see Homework #1)

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- Question: Which HDM would produce the same solution as that given by the Petrov-Galerkin PROM? (this notion will prove to be useful for the stability analysis of a PROM)

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- the associated initial condition is

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└ Equivalent High-Dimensional Model

- Recall the projector $\Pi_{\mathbf{V}, \mathbf{W}}$

$$\Pi_{\mathbf{V}, \mathbf{W}} = \mathbf{V}(\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T$$

Definition

Equivalent HDM

$$\begin{aligned} \frac{d\tilde{\mathbf{w}}}{dt}(t) &= \Pi_{\mathbf{V}, \mathbf{W}} \mathbf{f}(\tilde{\mathbf{w}}(t), t) \\ \tilde{\mathbf{y}}(t) &= \mathbf{g}(\tilde{\mathbf{w}}(t), t) \end{aligned}$$

with the initial condition

$$\tilde{\mathbf{w}}(0) = \Pi_{\mathbf{V}, \mathbf{W}} \mathbf{w}(0)$$

The equivalent dynamical function is

$$\tilde{\mathbf{f}}(\cdot, \cdot) = \Pi_{\mathbf{V}, \mathbf{W}} \mathbf{f}(\cdot, \cdot)$$

└ Equivalent High-Dimensional Model

└ Equivalence Between Two Projection-Based Reduced-Order Models

- Consider the Petrov-Galerkin PROM

$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) \\ \mathbf{y}(t) &\approx \mathbf{g}(\mathbf{V}\mathbf{q}(t), t) \\ \mathbf{q}(0) &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{w}(0)\end{aligned}$$

Lemma

Choosing two different bases \mathbf{V}' and \mathbf{W}' that respectively span the same subspaces \mathcal{V} and \mathcal{W} results in the same reconstructed solution $\mathbf{w}(t)$

In other words, subspaces are more important than bases ...

└ Equivalent High-Dimensional Model

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■ Consequences

- given a HDM, a corresponding PROM is uniquely defined by its associated Petrov-Galerkin projector $\Pi_{V,W}$

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- this projector is itself uniquely defined by the two subspaces

$$\mathcal{W} = \text{range}(\mathbf{W}) \text{ and } \mathcal{V} = \text{range}(\mathbf{V})$$

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- hence

$$\text{PROM} \Leftrightarrow (\mathcal{W}, \mathcal{V})$$

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- \mathcal{W} and \mathcal{V} belong to the **Grassmann manifold** $\mathcal{G}(k, N)$, which is the set of all subspaces of dimension k in \mathbb{R}^N

└ Error Analysis

└ Definition

- Question: Can we characterize the **error** of the solution computed using a PROM relative to the solution obtained using the HDM?

$$\begin{aligned}\mathcal{E}_{\text{PROM}}(t) &= \mathbf{w}(t) - \tilde{\mathbf{w}}(t) \\ &= \mathbf{w}(t) - \mathbf{V}\mathbf{q}(t)\end{aligned}$$

- assume here a Galerkin projection and an associated orthogonal basis
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 - $\mathbf{V}^T \mathbf{V} = \mathbf{I}_k$
 - projector $\Pi_{\mathbf{V}, \mathbf{V}} = \mathbf{V}\mathbf{V}^T$
- the error vector can be decomposed into **two orthogonal components**

$$\begin{aligned}\mathcal{E}_{\text{PROM}}(t) &= \mathbf{w}(t) - \Pi_{\mathbf{V}, \mathbf{V}} \mathbf{w}(t) + \Pi_{\mathbf{V}, \mathbf{V}} \mathbf{w}(t) - \mathbf{V}\mathbf{q}(t) \\ &= (\mathbf{I}_N - \Pi_{\mathbf{V}, \mathbf{V}}) \mathbf{w}(t) + \mathbf{V} \left(\mathbf{V}^T \mathbf{w}(t) - \mathbf{q}(t) \right) \\ &= \mathcal{E}_{\mathbf{V}^\perp}(t) + \mathcal{E}_{\mathbf{V}}(t)\end{aligned}$$

└ Error Analysis

└ Orthogonal Components of the Error Vector

- Error component orthogonal to \mathbf{V}

$$\mathcal{E}_{\mathbf{V}^\perp}(t) = (\mathbf{I}_N - \Pi_{\mathbf{V}, \mathbf{V}}) \mathbf{w}(t)$$

Interpretation: The exact trajectory does not strictly belong to $\mathcal{V} = \text{range}(\mathbf{V}) \Rightarrow$ *projection error*

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$$\mathcal{E}_{\mathbf{V}}(t) = \mathbf{V} (\mathbf{V}^T \mathbf{w}(t) - \mathbf{q}(t))$$

Interpretation: An “equivalent” but *different* high-dimensional dynamical system is solved \Rightarrow *modeling error*

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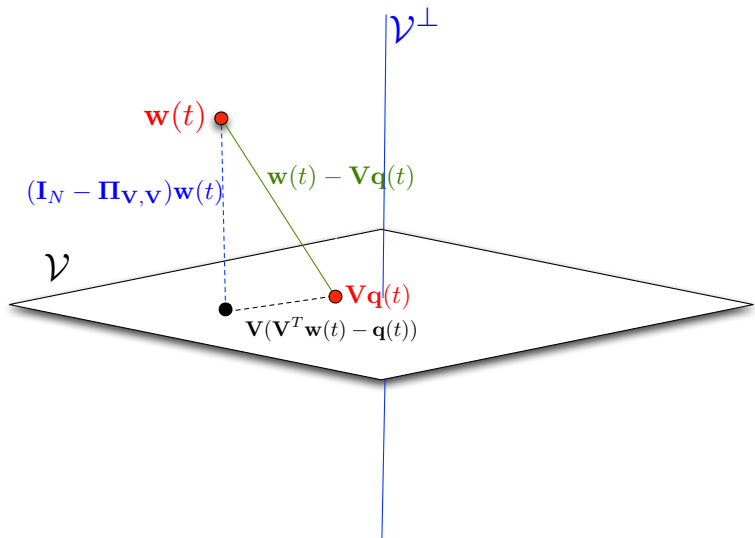
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- Note that $\mathcal{E}_{\mathbf{V}^\perp}(t)$ can be computed without executing the PROM and therefore can provide an **a priori** error estimate

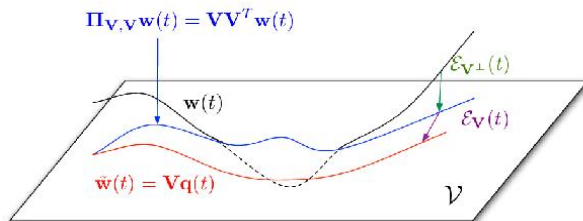
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- Error Analysis

- Orthogonal Components of the Error Vector



Adapted from *A New Look at Proper Orthogonal Decomposition*, Rathiman and Petzold, SIAM Journal of Numerical Analysis, Vol. 41, No. 5, 2003.

└ Error Analysis

└ Orthogonal Components of the Error Vector

- Again, consider the case of an orthogonal Galerkin projection
- One can derive an ODE governing the behavior of the error component lying in \mathcal{V} in terms of that lying in \mathcal{V}^\perp

$$\begin{aligned}\frac{d\mathcal{E}_{\mathbf{V}}}{dt}(t) &= \mathbf{\Pi}_{\mathbf{V},\mathbf{V}}\left(\mathbf{f}(\mathbf{w}(t), t) - \mathbf{f}(\mathbf{w}(t) - \mathcal{E}_{\mathbf{V}}(t) - \mathcal{E}_{\mathbf{V}^\perp}(t), t)\right) \\ \mathcal{E}_{\mathbf{V}}(0) &= \mathbf{0}\end{aligned}$$

- In the case of an **autonomous linear system** (ALS)

$$\frac{d\mathbf{w}}{dt}(t) = \mathbf{A}\mathbf{w}(t)$$

the error ODE has the simple form

$$\frac{d\mathcal{E}_{\mathbf{V}}}{dt}(t) = \mathbf{\Pi}_{\mathbf{V},\mathbf{V}}(\mathbf{A}\mathcal{E}_{\mathbf{V}}(t)) + \mathbf{\Pi}_{\mathbf{V},\mathbf{V}}(\mathbf{A}\mathcal{E}_{\mathbf{V}^\perp}(t))$$

where $\mathcal{E}_{\mathbf{V}^\perp}(t)$ acts as a **forcing term**

- Error Analysis

- Orthogonal Components of the Error Vector

- Then, one can derive the following error bound for an ALS

Theorem

$$\|\mathcal{E}_{PROM}(t)\| \leq (\|F(T, \mathbf{V}^T \mathbf{A} \mathbf{V})\|_2 \|\mathbf{V}^T \mathbf{A} \mathbf{V}^\perp\|_2 + 1) \|\mathcal{E}_{\mathbf{V}^\perp}(t)\|$$

where $\|\cdot\|$ denotes the $\mathcal{L}_2([0, T], \mathbb{R}^N)$ function norm,

$\|f\|_2 = \sqrt{\int_0^T \|f(\tau)\|_2^2 d\tau}$, and $F(T, \mathbf{M})$ denotes the linear operator defined by

$$\begin{aligned} F(T, \mathbf{M}) : \mathcal{L}_2([0, T], \mathbb{R}^N) &\rightarrow \mathcal{L}_2([0, T], \mathbb{R}^N) \\ \mathbf{u} &\mapsto t \mapsto \left(\int_0^t e^{\mathbf{M}(t-\tau)} \mathbf{u}(\tau) d\tau \right) \end{aligned}$$

- Error bounds for the nonlinear case can be found in *A New Look at Proper Orthogonal Decomposition*, Rathiman and Petzold, SIAM Journal of Numerical Analysis, Vol. 41, No. 5, 2003

- If \mathbf{A} is **symmetric** and the projection is an **orthogonal Galerkin** projection, the stability of the HDM is preserved during the reduction process (Hint: Consider the equivalent HDM and analyze the sign of $\frac{d}{dt} (\tilde{\mathbf{w}}^T \tilde{\mathbf{w}})$ after making the change of variable $\mathbf{y} = (-\mathbf{A})^{1/2} \mathbf{w}$)

└ Preservation of Model Stability

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- However, if \mathbf{A} is not symmetric, the stability of the HDM is not preserved: For example, consider a linear HDM characterized by the following unsymmetric matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3.5 \\ 0.6 & -2 \end{bmatrix}$$

- consider next the reduced-order basis \mathbf{V}

$$\mathbf{V} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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- $\mathbf{A}_r = [1]$ and therefore the Galerkin PROM is not asymptotically stable