

AA216/CME345: PROJECTION-BASED MODEL ORDER REDUCTION

Projection-Based Model Order Reduction

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Outline

- 1 Solution Approximation
- 2 Orthogonal and Oblique Projections
- 3 Galerkin and Petrov-Galerkin Projections
- 4 Equivalent High-Dimensional Model
- 5 Error Analysis
- 6 Preservation of Model Stability

└ Solution Approximation

└ High-Dimensional Model

■ Ordinary Differential Equation (ODE)

$$\frac{d\mathbf{w}}{dt}(t) = \mathbf{f}(\mathbf{w}(t), t) \quad (1)$$

- $\mathbf{w} \in \mathbb{R}^N$: State variable
- initial condition: $\mathbf{w}(0) = \mathbf{w}_0$

■ Output equation

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{w}(t), t) \quad (2)$$

- $\mathbf{y} \in \mathbb{R}^q$: Vector of output variables (typically $q \ll N$)

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- Note the absence of a parameter dependence for now

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└ Low Dimensionality of Trajectories

- In many cases, the trajectories of the solutions computed using High-Dimensional Models (HDMs) are contained in **low-dimensional subspaces**

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- Let \mathcal{S} denote such a subspace and let $k_{\mathcal{S}} = \dim(\mathcal{S})$
- The state variable – or simply, the state – can be written exactly as a linear combination of vectors spanning \mathcal{S}

$$\mathbf{w}(t) = q_1(t)\mathbf{v}_1 + \cdots + q_{k_{\mathcal{S}}}(t)\mathbf{v}_{k_{\mathcal{S}}}$$

- $\mathbf{V}_{\mathcal{S}} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_{k_{\mathcal{S}}}] \in \mathbb{R}^{N \times k_{\mathcal{S}}}$ is a **time-invariant** basis for \mathcal{S}
- $(q_1(t), \dots, q_{k_{\mathcal{S}}}(t))$ are the **generalized coordinates** for $\mathbf{w}(t)$ in \mathcal{S}
- $\mathbf{q}(t) = [q_1(t) \ \cdots \ q_{k_{\mathcal{S}}}(t)]^T \in \mathbb{R}^{k_{\mathcal{S}}}$ is the **reduced-order state vector**

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- In matrix form, the above expansion can be written as

$$\boxed{\mathbf{w}(t) = \mathbf{V}_{\mathcal{S}} \mathbf{q}(t)}$$

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- Often, the exact basis \mathbf{V}_S is unknown but can be estimated empirically by a **trial basis** $\mathbf{V} \in \mathbb{R}^{N \times k}$, $k < N$

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- Substituting the above subspace approximation in Eq. (1) and in Eq. (2) leads to

$$\begin{aligned}\frac{d}{dt}(\mathbf{V}\mathbf{q}(t)) &= \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) + \mathbf{r}(t) \\ \mathbf{y}(t) &\approx \mathbf{g}(\mathbf{V}\mathbf{q}(t), t)\end{aligned}$$

where $\mathbf{r}(t)$ is the residual due to the subspace approximation

└ Solution Approximation

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- The **residual** $\mathbf{r}(t) \in \mathbb{R}^N$ accounts for the fact that $\mathbf{Vq}(t)$ is not in general an exact solution of Eq. (1)

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and therefore

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- Set of N differential equations in terms of k unknowns

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- Set of N differential equations in terms of k unknowns

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- Over-determined system ($k < N$)

└ Orthogonal and Oblique Projections

└ Orthogonality

- Let \mathbf{w} and \mathbf{y} be two vectors in \mathbb{R}^N

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- Let \mathbf{w} and \mathbf{y} be two vectors in \mathbb{R}^N

- \mathbf{w} and \mathbf{y} are **orthogonal** to each other with respect to the canonical inner product in \mathbb{R}^N if and only if

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- Let \mathbf{V} be a matrix in $\mathbb{R}^{N \times k}$

- \mathbf{V} is an **orthogonal (orthonormal) matrix** if and only if

$$\boxed{\mathbf{V}^T \mathbf{V} = \mathbf{I}_k}$$

└ Orthogonal and Oblique Projections

└ Projections

Definition

A matrix $\Pi \in \mathbb{R}^{N \times N}$ is a **projection** matrix (or projective matrix, idempotent matrix) if

$$\Pi^2 = \Pi$$

- Some direct consequences
 - $\text{range}(\Pi)$ is invariant under the action of Π

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 - Π is diagonalizable (follows from the previous consequence)

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 - $\text{range}(\Pi)$ is invariant under the action of Π
 - 0 and 1 are the only possible eigenvalues of Π
 - Π is diagonalizable (follows from the previous consequence)
 - let k be the rank of Π : then, there exists a basis \mathbf{X} such that

$$\Pi = \mathbf{X} \begin{bmatrix} \mathbf{I}_k & \\ & \mathbf{0}_{N-k} \end{bmatrix} \mathbf{X}^{-1}$$

(follows from the two previous consequences)

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- decompose \mathbf{X} as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{bmatrix}, \text{ where } \mathbf{X}_1 \in \mathbb{R}^{N \times k} \text{ and } \mathbf{X}_2 \in \mathbb{R}^{N \times (N-k)}$$

then, $\forall \mathbf{w} \in \mathbb{R}^N$

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- Π defines the **projection onto \mathcal{S}_1 parallel to \mathcal{S}_2**

$$\mathbb{R}^N = \mathcal{S}_1 \oplus \mathcal{S}_2$$

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└ Orthogonal Projections

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- Consider the case where $\mathcal{S}_2 = \mathcal{S}_1^\perp$
- Let $\mathbf{V} \in \mathbb{R}^{N \times k}$ be an **orthogonal** matrix whose columns span \mathcal{S}_1 , and let $\mathbf{w} \in \mathbb{R}^N$: The orthogonal projection of \mathbf{w} onto the subspace \mathcal{S}_1 is

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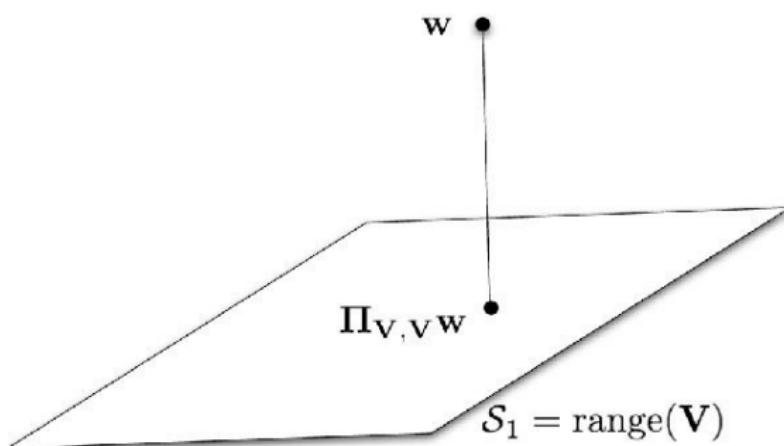
- special case #2: If \mathbf{w} is orthogonal to \mathcal{S}_1 – that is, $\mathbf{V}^T \mathbf{w} = \mathbf{0}$

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└ Orthogonal and Oblique Projections

└ Orthogonal Projections

$$\Pi_{V,V}w = VV^T w$$



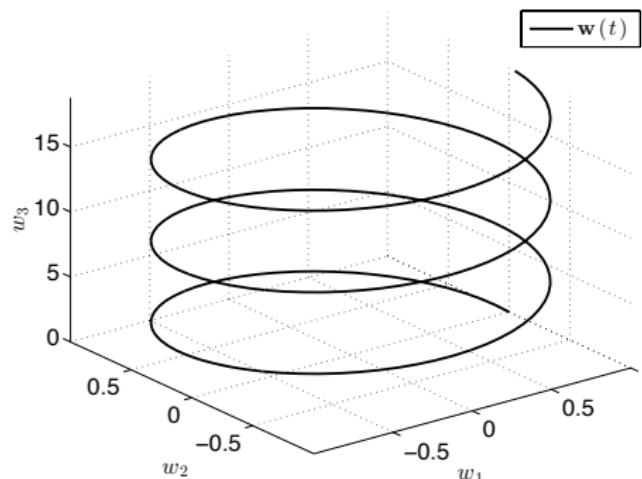
└ Orthogonal and Oblique Projections

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- Example: Helix in 3D ($N = 3$)

- let $\mathbf{w}(t) \in \mathbb{R}^3$ define a curve parameterized by $t \in [0, 6\pi]$ as follows

$$\mathbf{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

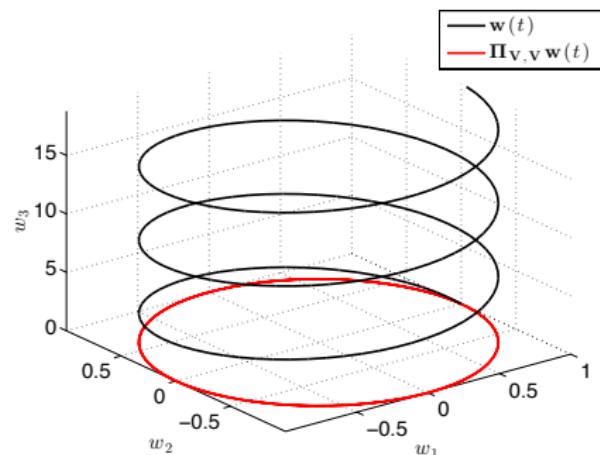


└ Orthogonal and Oblique Projections

└ Orthogonal Projections

- Orthogonal projection onto
 - $\text{range}(\mathbf{V}) = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$

$$\Pi_{\mathbf{V}, \mathbf{V}} \mathbf{w}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix}$$

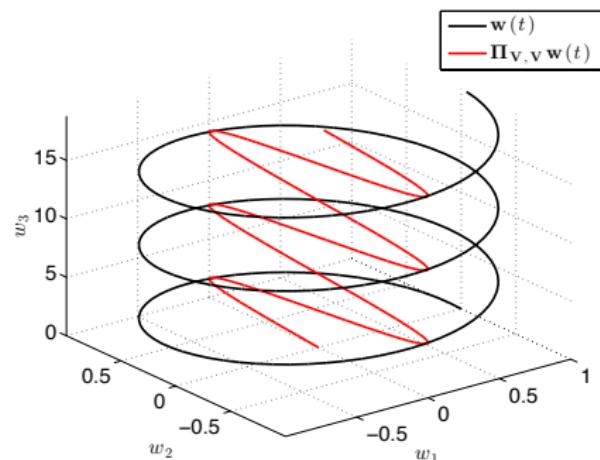


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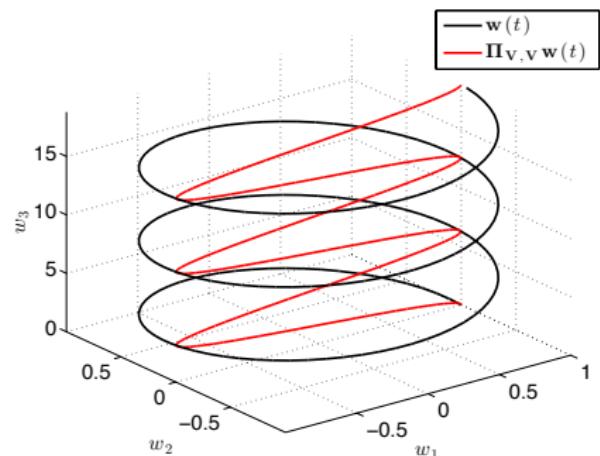


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- The following is the **general case**, where \mathcal{S}_2 may be distinct from \mathcal{S}_1^\perp
- Let $\mathbf{V} \in \mathbb{R}^{N \times k}$ and $\mathbf{W} \in \mathbb{R}^{N \times k}$ be two **full-column rank** matrices whose columns span respectively \mathcal{S}_1 and \mathcal{S}_2^\perp

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- The projection of $\mathbf{w} \in \mathbb{R}^N$ onto the subspace \mathcal{S}_1 parallel to \mathcal{S}_2 is

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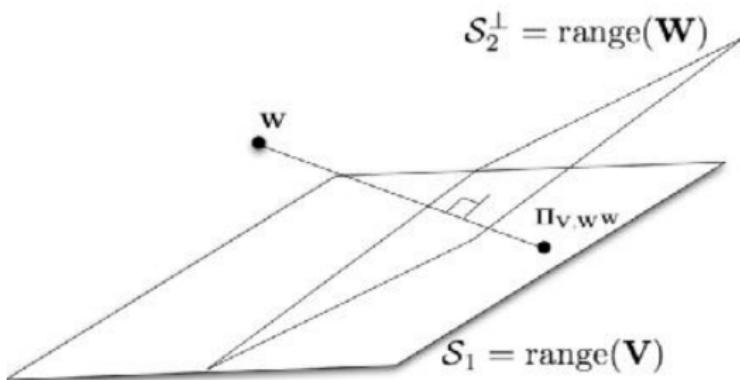
$$\boldsymbol{\Pi}_{\mathbf{V}, \mathbf{W}} \mathbf{w} = \mathbf{V} \underbrace{(\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T}_{\mathbf{I}} \mathbf{V} \mathbf{q} = \mathbf{V}\mathbf{q}$$

- special case #2: If $\mathbf{w} \in \mathcal{S}_2$ – that is, \mathbf{w} is orthogonal to \mathcal{S}_2^\perp – then $\mathbf{W}^T \mathbf{w} = \mathbf{0}$ and $\boldsymbol{\Pi}_{\mathbf{V}, \mathbf{W}} \mathbf{w} = \mathbf{V}(\mathbf{W}^T \mathbf{V})^{-1} \underbrace{\mathbf{W}^T \mathbf{w}}_0 = \mathbf{0}$

└ Orthogonal and Oblique Projections

└ Oblique Projections

$$\Pi_{V,W}w = V(V^T V)^{-1} V^T w$$



└ Orthogonal and Oblique Projections

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■ Example: Helix in 3D

■ bases

$$\mathbf{V} = [\mathbf{e}_1 \ \mathbf{e}_2], \ \mathbf{W} = [\mathbf{e}_1 + \mathbf{e}_3 \ \mathbf{e}_2 + \mathbf{e}_3]$$

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$$\mathbf{V} = [\mathbf{e}_1 \ \mathbf{e}_2], \ \mathbf{W} = [\mathbf{e}_1 + \mathbf{e}_3 \ \mathbf{e}_2 + \mathbf{e}_3]$$

■ projection matrix

$$\mathbf{\Pi}_{\mathbf{V},\mathbf{W}} = \mathbf{V}(\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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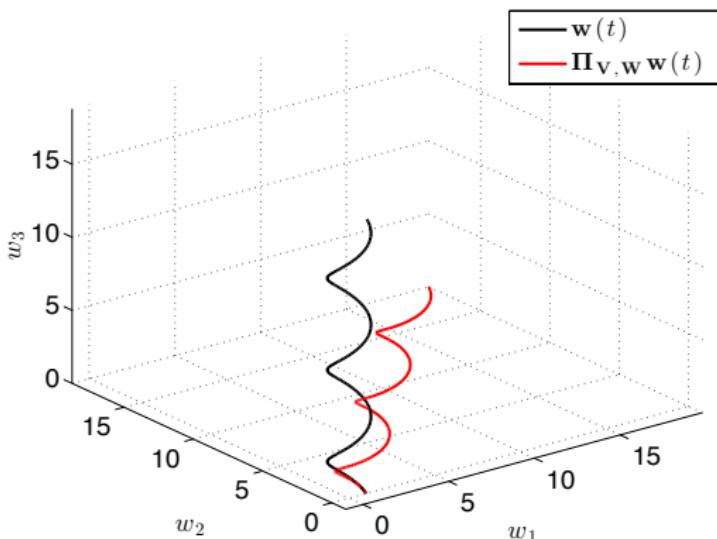
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■ projected helix equation

$$\mathbf{\Pi}_{\mathbf{V}, \mathbf{W}} \mathbf{w}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix} = \begin{bmatrix} \cos(t) + t \\ \sin(t) + t \\ 0 \end{bmatrix}$$

└ Orthogonal and Oblique Projections

└ Oblique Projections



└ Galerkin and Petrov-Galerkin Projections

└ Projection-Based Model Order Reduction

- Start from a HDM for the problem of interest

$$\begin{aligned}\frac{d\mathbf{w}}{dt}(t) &= \mathbf{f}(\mathbf{w}(t), t) \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{w}(t), t) \\ \mathbf{w}(0) &= \mathbf{w}_0\end{aligned}$$

- $\mathbf{w} \in \mathbb{R}^N$: Vector of state variables
- $\mathbf{y} \in \mathbb{R}^q$: Vector of output variables (typically $q \ll N$)
- $\mathbf{f}(\cdot, \cdot) \in \mathbb{R}^N$: Completes the specification of the HDM-based problem

└ Galerkin and Petrov-Galerkin Projections

└ Projection-Based Model Order Reduction

- The goal is to construct a **Projection-based Reduced-Order Model (PROM)**

$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= \mathbf{f}_r(\mathbf{q}(t), t) \\ \mathbf{y}(t) &\approx \mathbf{g}_r(\mathbf{q}(t), t)\end{aligned}$$

where

- $\mathbf{q} \in \mathbb{R}^k$: **Vector of reduced-order state variables**, $k \ll N$
- $\mathbf{y} \in \mathbb{R}^q$: Vector of output variables
- $\mathbf{f}_r(\cdot, \cdot) \in \mathbb{R}^k$: Completes the description of the PROM
- The discussion of the initial condition is deferred to later

└ Galerkin and Petrov-Galerkin Projections

└ Requirements

- A Projection-based Model Order Reduction (PMOR) method should
 - be computationally tractable

└ Galerkin and Petrov-Galerkin Projections

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└ Requirements

- A Projection-based Model Order Reduction (PMOR) method should
 - be computationally tractable
 - be applicable to a large class of dynamical systems
 - minimize a certain measure of the error between the solution computed using the HDM and that computed using the PROM (error criterion)
 - preserve as many properties of the HDM as possible

└ Galerkin and Petrov-Galerkin Projections

└ Petrov-Galerkin Projection

- Recall the residual $\mathbf{r}(t) \in \mathbb{R}^N$ introduced by approximating $\mathbf{w}(t)$ as $\mathbf{V}\mathbf{q}(t)$

$$\mathbf{V} \frac{d\mathbf{q}}{dt}(t) = \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) + \mathbf{r}(t) \Leftrightarrow \mathbf{r}(t) = \mathbf{V} \frac{d\mathbf{q}}{dt}(t) - \mathbf{f}(\mathbf{V}\mathbf{q}(t), t)$$

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- Constrain this residual to be orthogonal to a subspace \mathcal{W} defined by a **test basis** $\mathbf{W} \in \mathbb{R}^{N \times k}$ – that is, compute $\mathbf{q}(t)$ such that

$$\mathbf{W}^T \mathbf{r}(t) = \mathbf{0}$$

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- This leads to the *descriptive form* of the governing equations of the **Petrov-Galerkin** PROM

$$\mathbf{W}^T \mathbf{V} \frac{d\mathbf{q}}{dt}(t) = \mathbf{W}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t), t)$$

└ Galerkin and Petrov-Galerkin Projections

└ Petrov-Galerkin Projection

- Assume that $\mathbf{W}^T \mathbf{V}$ is non-singular: In this case, the PROM can be re-written in the *non-descriptive form*

$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) \\ \mathbf{y}(t) &\approx \mathbf{g}(\mathbf{V}\mathbf{q}(t), t)\end{aligned}$$

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- After the above reduced-order equations have been solved, the subspace approximation of the high-dimensional state vector can be reconstructed, if needed, as follows

$$\mathbf{w}(t) \approx \mathbf{V}\mathbf{q}(t)$$

└ Galerkin and Petrov-Galerkin Projections

└ Galerkin Projection

- If $\mathbf{W} = \mathbf{V}$, the projection method is called a **Galerkin** projection and the resulting PROM is called a **Galerkin** PROM

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- If $\mathbf{W} = \mathbf{V}$, the projection method is called a **Galerkin** projection and the resulting PROM is called a **Galerkin** PROM
- If in addition \mathbf{V} is orthogonal, the reduced-order equations become

$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= \mathbf{V}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) \\ \mathbf{y}(t) &\approx \mathbf{g}(\mathbf{V}\mathbf{q}(t), t)\end{aligned}$$

- └ Galerkin and Petrov-Galerkin Projections
- └ Linear Time-Invariant Systems

- Special case: **Linear Time-Invariant (LTI)** systems

$$\begin{aligned}\mathbf{f}(\mathbf{w}(t), t) &= \mathbf{Aw}(t) + \mathbf{Bu}(t) \\ \mathbf{g}(\mathbf{w}(t), t) &= \mathbf{Cw}(t) + \mathbf{Du}(t)\end{aligned}$$

- $\mathbf{u} \in \mathbb{R}^{\text{in}}$: Vector of input variables

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$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T (\mathbf{A} \mathbf{V} \mathbf{q}(t) + \mathbf{B} \mathbf{u}(t)) \\ \mathbf{y}(t) &= \mathbf{C} \mathbf{V} \mathbf{q}(t) + \mathbf{D} \mathbf{u}(t)\end{aligned}$$

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- reduced-order LTI operators

$$\begin{aligned}\mathbf{A}_r &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{A} \mathbf{V} \in \mathbb{R}^{k \times k}, \quad k \ll N \\ \mathbf{B}_r &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{B} \in \mathbb{R}^{k \times \text{in}} \\ \mathbf{C}_r &= \mathbf{C} \mathbf{V} \in \mathbb{R}^{q \times k} \\ \mathbf{D}_r &= \mathbf{D} \in \mathbb{R}^{q \times \text{in}}\end{aligned}$$

└ Galerkin and Petrov-Galerkin Projections

└ Initial Condition

- High-dimensional initial condition

$$\mathbf{w}(0) = \mathbf{w}_0 \in \mathbb{R}^N$$

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- Alternative: use an affine approximation $\mathbf{w}(t) = \mathbf{w}(0) + \mathbf{V}\mathbf{q}(t)$ (see Homework #1)

└ Equivalent High-Dimensional Model

- Question: Which HDM would produce the same solution as that given by the Petrov-Galerkin PROM? (this notion will prove to be useful for the stability analysis of a PROM)

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- the associated initial condition is

$$\tilde{\mathbf{w}}(0) = \mathbf{V}\mathbf{q}(0) = \mathbf{V}(\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{w}(0)$$

└ Equivalent High-Dimensional Model

- Recall the projector $\Pi_{V,W}$

$$\Pi_{V,W} = V(W^T V)^{-1} W^T$$

Definition

Equivalent HDM

$$\begin{aligned}\frac{d\tilde{w}}{dt}(t) &= \Pi_{V,W} f(\tilde{w}(t), t) \\ \tilde{y}(t) &= g(\tilde{w}(t), t)\end{aligned}$$

with the initial condition

$$\tilde{w}(0) = \Pi_{V,W} w(0)$$

The equivalent dynamical function is

$$\tilde{f}(\cdot, \cdot) = \Pi_{V,W} f(\cdot, \cdot)$$

└ Equivalent High-Dimensional Model

└ Equivalence Between Two Projection-Based Reduced-Order Models

- Consider the Petrov-Galerkin PROM

$$\begin{aligned}\frac{d\mathbf{q}}{dt}(t) &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t), t) \\ \mathbf{y}(t) &\approx \mathbf{g}(\mathbf{V}\mathbf{q}(t), t) \\ \mathbf{q}(0) &= (\mathbf{W}^T \mathbf{V})^{-1} \mathbf{W}^T \mathbf{w}(0)\end{aligned}$$

Lemma

Choosing two different bases \mathbf{V}' and \mathbf{W}' that respectively span the same subspaces \mathcal{V} and \mathcal{W} results in the same reconstructed solution $\mathbf{w}(t)$

In other words, subspaces are more important than bases ...

└ Equivalent High-Dimensional Model

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■ Consequences

- given a HDM, a corresponding PROM is uniquely defined by its associated Petrov-Galerkin projector $\Pi_{V,W}$

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- given a HDM, a corresponding PROM is uniquely defined by its associated Petrov-Galerkin projector $\Pi_{V,W}$
- this projector is itself uniquely defined by the two subspaces

$$\mathcal{W} = \text{range}(\mathbf{W}) \text{ and } \mathcal{V} = \text{range}(\mathbf{V})$$

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- hence

$$\text{PROM} \Leftrightarrow (\mathcal{W}, \mathcal{V})$$

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- \mathcal{W} and \mathcal{V} belong to the **Grassmann manifold** $\mathcal{G}(k, N)$, which is the set of all subspaces of dimension k in \mathbb{R}^N

- └ Error Analysis
- └ Definition

- Question: Can we characterize the **error** of the solution computed using a PROM relative to the solution obtained using the HDM?

$$\begin{aligned}\mathcal{E}_{\text{PROM}}(t) &= \mathbf{w}(t) - \tilde{\mathbf{w}}(t) \\ &= \mathbf{w}(t) - \mathbf{V}\mathbf{q}(t)\end{aligned}$$

- assume here a Galerkin projection and an associated orthogonal basis
 - $\mathbf{V}^T \mathbf{V} = \mathbf{I}_k$
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 - $\mathbf{V}^T \mathbf{V} = \mathbf{I}_k$
 - projector $\mathbf{\Pi}_{\mathbf{V}, \mathbf{V}} = \mathbf{V}\mathbf{V}^T$
- the error vector can be decomposed into **two orthogonal components**

$$\begin{aligned}\mathcal{E}_{\text{PROM}}(t) &= \mathbf{w}(t) - \mathbf{\Pi}_{\mathbf{V}, \mathbf{V}} \mathbf{w}(t) + \mathbf{\Pi}_{\mathbf{V}, \mathbf{V}} \mathbf{w}(t) - \mathbf{V}\mathbf{q}(t) \\ &= (\mathbf{I}_N - \mathbf{\Pi}_{\mathbf{V}, \mathbf{V}}) \mathbf{w}(t) + \mathbf{V} (\mathbf{V}^T \mathbf{w}(t) - \mathbf{q}(t)) \\ &= \mathcal{E}_{\mathbf{V}^\perp}(t) + \mathcal{E}_{\mathbf{V}}(t)\end{aligned}$$

└ Error Analysis

└ Orthogonal Components of the Error Vector

- Error component orthogonal to \mathbf{V}

$$\mathcal{E}_{\mathbf{V}^\perp}(t) = (\mathbf{I}_N - \mathbf{\Pi}_{\mathbf{V}, \mathbf{V}}) \mathbf{w}(t)$$

Interpretation: The exact trajectory does not strictly belong to $\mathcal{V} = \text{range}(\mathbf{V}) \Rightarrow \text{projection error}$

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Interpretation: An “equivalent” but *different* high-dimensional dynamical system is solved $\Rightarrow \text{modeling error}$

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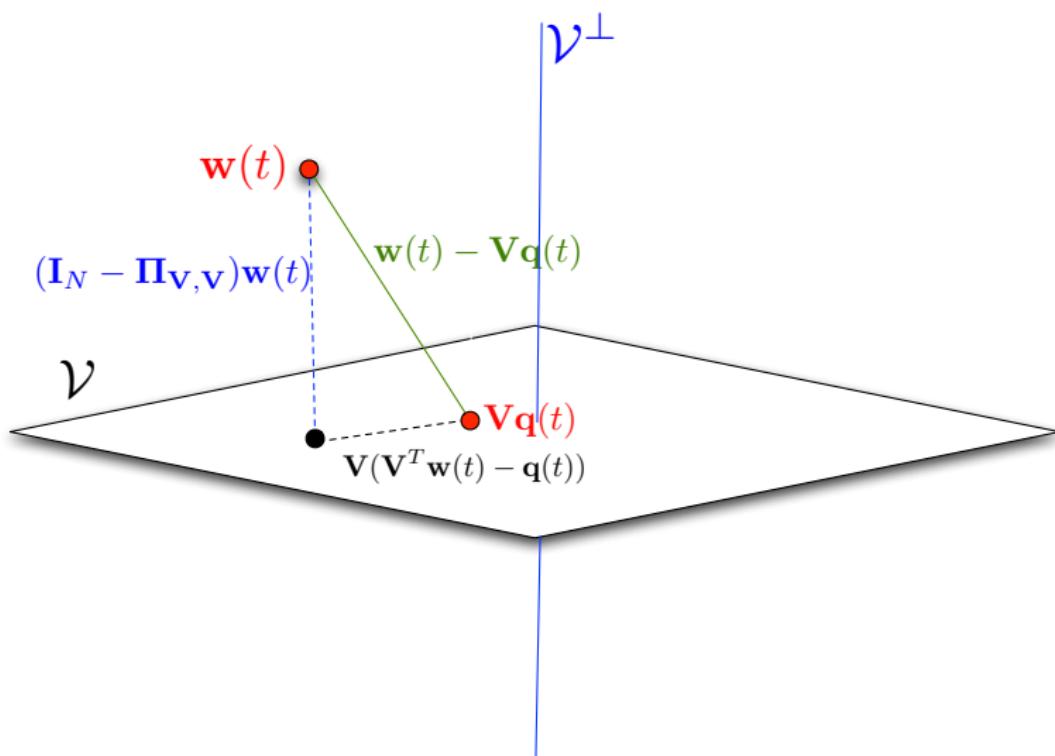
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- Note that $\mathcal{E}_{\mathbf{V}^\perp}(t)$ can be computed without executing the PROM and therefore can provide an **a priori** error estimate

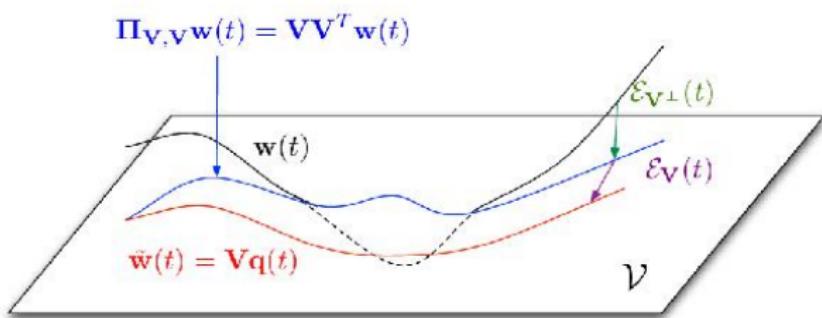
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└ Error Analysis

└ Orthogonal Components of the Error Vector



Adapted from *A New Look at Proper Orthogonal Decomposition*, Rathiman and Petzold, SIAM Journal of Numerical Analysis, Vol. 41, No. 5, 2003.

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- Again, consider the case of an orthogonal Galerkin projection
- One can derive an ODE governing the behavior of the error component lying in \mathcal{V} in terms of that lying in \mathcal{V}^\perp

$$\begin{aligned}\frac{d\mathcal{E}_V}{dt}(t) &= \boldsymbol{\Pi}_{V,V} \left(\mathbf{f}(\mathbf{w}(t), t) - \mathbf{f}(\mathbf{w}(t) - \mathcal{E}_V(t) - \mathcal{E}_{V^\perp}(t), t) \right) \\ \mathcal{E}_V(0) &= \mathbf{0}\end{aligned}$$

- In the case of an **autonomous linear system** (ALS)

$$\frac{d\mathbf{w}}{dt}(t) = \mathbf{Aw}(t)$$

the error ODE has the simple form

$$\frac{d\mathcal{E}_V}{dt}(t) = \boldsymbol{\Pi}_{V,V} (\mathbf{A}\mathcal{E}_V(t)) + \boldsymbol{\Pi}_{V,V} (\mathbf{A}\mathcal{E}_{V^\perp}(t))$$

where $\mathcal{E}_{V^\perp}(t)$ acts as a **forcing term**

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- Then, one can derive the following error bound for an ALS

Theorem

$$\|\mathcal{E}_{PROM}(t)\| \leq (\|F(T, \mathbf{V}^T \mathbf{A} \mathbf{V})\|_2 \|\mathbf{V}^T \mathbf{A} \mathbf{V}^\perp\|_2 + 1) \|\mathcal{E}_{\mathbf{V}^\perp}(t)\|$$

where $\|\cdot\|$ denotes the $\mathcal{L}_2([0, T], \mathbb{R}^N)$ function norm,

$\|f\|_2 = \sqrt{\int_0^T \|f(\tau)\|_2^2 d\tau}$, and $F(T, \mathbf{M})$ denotes the linear operator defined by

$$\begin{aligned} F(T, \mathbf{M}) : \mathcal{L}_2([0, T], \mathbb{R}^N) &\rightarrow \mathcal{L}_2([0, T], \mathbb{R}^N) \\ \mathbf{u} &\mapsto t \mapsto \left(\int_0^t e^{\mathbf{M}(t-\tau)} \mathbf{u}(\tau) d\tau \right) \end{aligned}$$

- Error bounds for the nonlinear case can be found in *A New Look at Proper Orthogonal Decomposition*, Rathiman and Petzold, SIAM Journal of Numerical Analysis, Vol. 41, No. 5, 2003

└ Preservation of Model Stability

- If \mathbf{A} is **symmetric** and the projection is an **orthogonal Galerkin** projection, the stability of the HDM is preserved during the reduction process (Hint: Consider the equivalent HDM and analyze the sign of $\frac{d}{dt} (\tilde{\mathbf{w}}^T \tilde{\mathbf{w}})$ after making the change of variable $\mathbf{y} = (-\mathbf{A})^{1/2} \mathbf{w}$)

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- However, if \mathbf{A} is not symmetric, the stability of the HDM is not preserved: For example, consider a linear HDM characterized by the following unsymmetric matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3.5 \\ 0.6 & -2 \end{bmatrix}$$

- consider next the reduced-order basis \mathbf{V}

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- the eigenvalues of \mathbf{A} are $\{-0.1127, -0.8873\}$ (stable model)
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- $\mathbf{A}_r = [1]$ and therefore the Galerkin PROM is not asymptotically stable