AA216/CME345: PROJECTION-BASED MODEL ORDER REDUCTION

Balanced Truncation (BT)

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These slides are based on the recommended textbook: A.C. Antoulas, "Approximation of Large-Scale Dynamical Systems," Advances in Design and Control, SIAM, ISBN-0-89871-529-6

Outline

- Consider the following **stable**, high-dimensional, LTI system

$$\begin{aligned} \frac{d\mathbf{w}}{dt}(t) &= \mathbf{A}\mathbf{w}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{w}(t) \\ \mathbf{w}(0) &= \mathbf{w}_0 \end{aligned}$$

  - $\mathbf{w} \in \mathbb{R}^N$: State variables
  - $\mathbf{u} \in \mathbb{R}^{\text{in}}$: Input variables, typically in $\ll N$
  - $\mathbf{y} \in \mathbb{R}^q$: Output variables, typically $q \ll N$

- Recall that the solution $\mathbf{w}(t)$ of the above linear ODE can be written as

$$\mathbf{w}(t) = \phi(t, \mathbf{u}; t_0, \mathbf{w}_0) = e^{\mathbf{A}(t-t_0)}\mathbf{w}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau, \quad \forall t \geq t_0 \tag{1}$$

Definition

For $T < \infty$, a state $\mathbf{w}(T) \in \mathbb{R}^N$ of a dynamical system is said to be **reachable** (or attainable) from an initial state $\mathbf{w}(t_0)$ if there exists an admissible (finite energy) input function $\mathbf{u}(.)$ defined over $[t_0,\ T]$ that drives the system from $\mathbf{w}(t_0)$ to $\mathbf{w}(T)$

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Definition

A state $\mathbf{w} \in \mathbb{R}^N$ of a dynamical system is said to be **controllable** to the zero state if there exists a finite-time admissible control input function $\mathbf{u}(.)$ defined over $[t_0,\ T]$ ($T < \infty$) that drives the system from the state $\mathbf{w}$ to the zero state – that is, $\phi(T, \mathbf{u}; t_0, \mathbf{w}) = \mathbf{0}_N$

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Definition

A state $\mathbf{w} \in \mathbb{R}^N$ of a dynamical system is said to be **unobservable** if for all $t \geq t_0$,

$$\mathbf{y}(t) = \mathbf{C}\phi(t, \mathbf{0}; t_0, \mathbf{w}) = \mathbf{0}_q$$

Definition (R.E. Kalman, 1963)

A *linear dynamical system* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is said to be **completely controllable** at time t_0 if it is not equivalent, for all $t \geq t_0$, to a system of the type

$$\begin{aligned}
\frac{d\mathbf{w}^{(1)}}{dt} &= \mathbf{A}^{(1,1)}\mathbf{w}^{(1)} + \mathbf{A}^{(1,2)}\mathbf{w}^{(2)} + \mathbf{B}^{(1)}\mathbf{u} \\
\frac{d\mathbf{w}^{(2)}}{dt} &= \mathbf{A}^{(2,2)}\mathbf{w}^{(2)} \\
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$$\begin{aligned} \frac{d\mathbf{w}}{dt}^{(1)} &= \mathbf{A}^{(1,1)}\mathbf{w}^{(1)} + \mathbf{A}^{(1,2)}\mathbf{w}^{(2)} + \mathbf{B}^{(1)}\mathbf{u} \\ \frac{d\mathbf{w}}{dt}^{(2)} &= \mathbf{A}^{(2,2)}\mathbf{w}^{(2)} \\ \mathbf{y}(t) &= \mathbf{C}^{(1)}\mathbf{w}^{(1)} + \mathbf{C}^{(2)}\mathbf{w}^{(2)} \end{aligned}$$

- Interpretation: It is not possible to find a coordinate system in which the state variables are separated into two groups, $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$, such that the second group is affected neither by the first group, nor by the inputs to the system

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- This definition can be extended to linear time-variant systems

Definition

A *linear dynamical system* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is said to be **completely observable** at time t_0 if it is not equivalent, for all $\underline{t \leq t_0}$, to any system of the type

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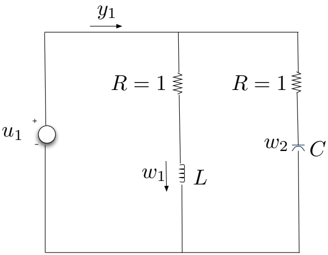

- For $C = L$ and $R = 1$, the equation of the network shown above in terms of the current w_1 flowing through the inductor and the potential w_2 across the capacitor is given by

$$\begin{aligned}\frac{dw_1}{dt} &= -\frac{1}{L}w_1 + u_1\\ \frac{dw_2}{dt} &= -\frac{1}{L}w_2 + u_1\\ y_1 &= \frac{1}{L}w_1 - \frac{1}{L}w_2 + u_1\end{aligned}$$

- Under the change of variable $\bar{w}_1 = (w_1 + w_2)/2$ and $\bar{w}_2 = (w_1 - w_2)/2$, the previous dynamical system becomes

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  - $\bar{w}_1$ is controllable but not observable
  - $\bar{w}_2$ is observable but not controllable
- Hence, this dynamical system is neither completely controllable nor completely observable

Theorem (Kalman, 1961)

Consider a linear dynamical system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. *Then:*
(i) There is a state space coordinate system in which the components of the state vector can be decomposed into four parts

$$\mathbf{w} = [\mathbf{w}^{(a)} \ \mathbf{w}^{(b)} \ \mathbf{w}^{(c)} \ \mathbf{w}^{(d)}]^T$$

(ii) The sizes N_a, N_b, N_c *and* N_d *of these vectors do not depend on the choice of basis*
(iii) The system matrices take the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{(a,a)} & \mathbf{A}^{(a,b)} & \mathbf{A}^{(a,c)} & \mathbf{A}^{(a,d)} \\ 0 & \mathbf{A}^{(b,b)} & 0 & \mathbf{A}^{(b,d)} \\ 0 & 0 & \mathbf{A}^{(c,c)} & \mathbf{A}^{(c,d)} \\ 0 & 0 & 0 & \mathbf{A}^{(d,d)} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}^{(a)} \\ \mathbf{B}^{(b)} \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & \mathbf{C}^{(b)} & 0 & \mathbf{C}^{(d)} \end{bmatrix}$$

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Definition

The **reachable subspace** $\mathbb{W}_{\text{reach}} \subset \mathbb{R}^N$ of a system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is the set containing all reachable states of the system and

$$\mathcal{R}(\mathbf{A}\ \mathbf{B}) = [\mathbf{B}\ \mathbf{AB}\ \cdots\ \mathbf{A}^{N-1}\mathbf{B}]$$

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Definition

The **controllable subspace** $\mathbb{W}_{\text{contr}} \subset \mathbb{R}^N$ of a system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is the set containing all controllable states of the system

Theorem

Given a system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$,

$$\mathbb{W}_{reach} = Im\ \mathcal{R}(\mathbf{A}, \mathbf{B})$$

- Proof
  - recall (1) then set $t_0 = 0$ and $\mathbf{w}(0) = \mathbf{0}$
  - recall that $e^{\mathbf{A}(t-\tau)} = \mathbf{I}_N + \mathbf{A}(t-\tau) + \frac{(\mathbf{A}(t-\tau))^2}{2!} + \cdots$
  - then $\mathbf{w}(t) = \int_0^t \left(\mathbf{I}_N + \mathbf{A}(t-\tau) + \frac{(\mathbf{A}(t-\tau))^2}{2!} + \cdots \right) \mathbf{B}\mathbf{u}(\tau) d\tau$
  - for any finite t, $\int_0^t \frac{(t-\tau)^k}{k!} \mathbf{u}(\tau) d\tau$ acts as an in-long vector multiplying $\mathbf{A}^k\mathbf{B}$ to the right
    $\Rightarrow$ linear combination of $\left\{\mathbf{B}\ \mathbf{A}\mathbf{B}\ \cdots\ \mathbf{A}^{N-1}\mathbf{B}, \mathbf{A}^N\mathbf{B}, \mathbf{A}^{N+1}\mathbf{B}, \cdots\right\}$
  - recall Cayley-Hamilton: $c_N\mathbf{A}^N + c_{N-1}\mathbf{A}^{N-1} + \cdots + c_1\mathbf{A} + c_o\mathbf{I}_N = \mathbf{0}$
    $\Rightarrow$ linear combination of the columns of $[\mathbf{B}\ \mathbf{A}\mathbf{B}\ \cdots\ \mathbf{A}^{N-1}\mathbf{B}]$ □

Corollary

(i) If $\mathcal{R}$ has full rank, $\mathbf{A}\mathbb{W}_{reach} \subset \mathbb{W}_{reach}$
(ii) The system is completely reachable if and only if rank $\mathcal{R}(\mathbf{A}, \mathbf{B}) = N$
(iii) Reachability is basis independent

- Proof
  - only the term $\mathbf{A}^N\mathbf{B} \in \mathbb{R}^{N \times \text{in}}$ requires special attention
    - Cayley-Hamilton: $c_N\mathbf{A}^N + c_{N-1}\mathbf{A}^{N-1} + \cdots + c_1\mathbf{A} + c_o\mathbf{I}_N = \mathbf{0}$

$$\Rightarrow c_N\mathbf{A}^N\mathbf{B} + c_{N-1}\mathbf{A}^{N-1}\mathbf{B} + \cdots + c_1\mathbf{A}\mathbf{B} + c_o\mathbf{B} = \mathbf{0}$$

$$\Rightarrow \mathbf{A}^N\mathbf{B} = -\frac{c_{N-1}}{c_N}\mathbf{A}^{N-1}\mathbf{B} + \cdots - \frac{c_1}{c_N}\mathbf{A}\mathbf{B} - \frac{c_0}{c_N}\mathbf{B} \qquad \square$$

Definition

The **reachability (controllability) Gramian** at time $T < \infty$ is defined as the $N \times N$ symmetric positive semi-definite matrix

$$\mathcal{P}(T) = \int_0^T e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\star e^{\mathbf{A}^\star\tau} d\tau$$

where $\star$ designates the transpose of the complex conjugate

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Definition

The **observability Gramian** at time $T < \infty$ is defined as the $N \times N$ symmetric positive semi-definite matrix

$$\mathcal{Q}(T) = \int_0^T e^{\mathbf{A}^\star \tau} \mathbf{C}^\star \mathbf{C} e^{\mathbf{A}\tau} d\tau$$

Proposition

The columns of $\mathcal{P}(T)$ *span the reachability subspace* $\mathbb{W}_{reach} = Im\ \mathcal{R}(\mathbf{A}, \mathbf{B})$

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Corollary

A system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is reachable if and only if $\mathcal{P}(T)$ is Symmetric Positive Definite (SPD) at some time $T > 0$

- Proof
  - if $\mathcal{P}(T)$ is SPD, define the input $\mathbf{u}(t) = \mathbf{B}^\star e^{\mathbf{A}^\star (T-t)} \mathcal{P}^{-1}\mathbf{w}$, $t \in [0,\ T]$
  - starting from $\mathbf{w}(0) = \mathbf{0}$, the resulting final state is $\mathbf{w}(t) = \int_0^T e^{\mathbf{A}(T-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau = \left(\int_0^T e^{\mathbf{A}(T-\tau)}\mathbf{B}\mathbf{B}^\star e^{\mathbf{A}^\star (T-\tau)} d\tau\right) \mathcal{P}^{-1}\mathbf{w}(T)$
    $= \underbrace{\left(\int_0^T e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\star e^{\mathbf{A}^\star \tau} d\tau\right)}_{\mathcal{P}} \mathcal{P}^{-1}\mathbf{w}(T) = \mathbf{w}(T)$

Theorem

For continuous linear dynamical systems, the notions of controllability and reachability are equivalent – that is,

$$\mathbb{W}_{reach} = \mathbb{W}_{contr}$$

Definition

The **unobservability subspace** $\mathbb{W}_{\text{unobs}} \subset \mathbb{R}^N$ is the set of all unobservable states of the system and the matrix

$$\mathcal{O}(\mathbf{C}, \mathbf{A}) = [\mathbf{C}^\star \ \mathbf{A}^\star\mathbf{C}^\star \ \cdots \ (\mathbf{A}^\star)^i\mathbf{C}^\star \ \cdots]^\star$$

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(iii) Observability is basis independent

Definition

The **infinite reachability (controllability) Gramian** is defined for a **stable** LTI system as the $N \times N$ symmetric positive semi-definite matrix

$$\mathcal{P} = \int_0^{\infty} e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^{\star}e^{\mathbf{A}^{\star}t}dt$$

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The **infinite observability Gramian** is defined for a **stable** LTI system as the $N \times N$ symmetric positive semi-definite matrix

$$\mathcal{Q} = \int_0^\infty e^{\mathbf{A}^\star t}\mathbf{C}^\star \mathbf{C} e^{\mathbf{A}t} dt$$

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  - **infinite reachability Gramian**

$$\boxed{\mathcal{P} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega\mathbf{I}_N - \mathbf{A})^{-1}\mathbf{B}\mathbf{B}^\star(-j\omega\mathbf{I}_N - \mathbf{A}^\star)^{-1} d\omega}$$

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$$\mathcal{Q} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-j\omega\mathbf{I}_N - \mathbf{A}^\star)^{-1}\mathbf{C}^\star\mathbf{C}(j\omega\mathbf{I}_N - \mathbf{A})^{-1}d\omega$$

- The two infinite Gramians are solutions of the following Lyapunov equations
  - infinite reachability Gramian

$$\boxed{\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^\star + \mathbf{B}\mathbf{B}^\star = \mathbf{0}_N}$$

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  - infinite observability Gramian

$$\boxed{\mathbf{A}^\star\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^\star\mathbf{C} = \mathbf{0}_N}$$

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  - $\mathcal{Q}$ measures how observable each direction is
- $\| \ \|_{\mathcal{P}^{-1}}$ and $\| \ \|_{\mathcal{Q}}$ are semi-norms (recall that $\mathcal{P}$ is generally symmetric positive semi-definite and thus $\| \ \|_{\mathcal{P}^{-1}}$ is generally $\| \ \|_{\mathcal{P}^{+}}$)

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- $\| \ \|_{\mathcal{P}^{-1}}$ and $\| \ \|_{\mathcal{Q}}$ are semi-norms (recall that $\mathcal{P}$ is generally symmetric positive semi-definite and thus $\| \ \|_{\mathcal{P}^{-1}}$ is generally $\| \ \|_{\mathcal{P}^{+}}$)
- For a reachable state, the inner product based on $\mathcal{P}^{-1}$ characterizes the minimal energy required to steer the state from $\mathbf{0}$ to $\mathbf{w}$ as $t \to \infty$

$$\|\mathbf{w}\|^2_{\mathcal{P}^{-1}} = \mathbf{w}^T \underbrace{\mathcal{P}^{-1}\mathbf{w}}_{\substack{\textit{homogeneous} \\ \textit{to an input}}} \qquad \left(\leq \int_0^t (\mathbf{Bu}(\tau))^\star \, \mathbf{Bu}(\tau) d\tau \right)$$

- $\mathcal{P}$ and $\mathcal{Q}$ are respective bases for the reachable and observable subspaces
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- The inner product based on $\mathcal{Q}$ indicates the maximal energy produced by observing the output of the system corresponding to an initial state $\mathbf{w}_0$ when no input is applied

$$\|\mathbf{w}\|^2_{\mathcal{Q}} = \mathbf{w}^T \mathcal{Q} \mathbf{w}$$

- If $\mathcal{P}$ is large in some direction $\mathbf{w}$, $\mathbf{w}^T\mathcal{P}^{-1}\mathbf{w}$ is small, $\mathbf{w}$ can be reached using a small control energy, **but $\mathbf{w}^T\mathcal{P}\mathbf{w}$ is large**
- If $\mathcal{Q}$ is large in some direction $\mathbf{w}$, $\mathbf{w}^T\mathcal{Q}\mathbf{w}$ is large and that direction produces a large observation energy
- PMOR strategy: **Eliminate** the states $\mathbf{w}$ that are simultaneously
  - **difficult to reach**, i.e., require a large energy $\|\mathbf{w}\|^2_{\mathcal{P}^{-1}}$ to be reached
  - **difficult to observe**, i.e., produce a small observation energy $\|\mathbf{w}\|^2_{\mathcal{Q}}$
- The above notions are **basis-dependent**
- One would like to consider a basis where both energy measures are **equal** or **balanced** – specifically, a basis where $\mathbf{w}^T\mathcal{P}\mathbf{w}$ and $\mathbf{w}^T\mathcal{Q}\mathbf{w}$ are balanced (see first two bullets)

- Balancing requires changing the basis for the state using a transformation $\mathbf{T} \in \mathrm{GL}(N)$

$$\bar{\mathbf{w}} = \mathbf{T}\mathbf{w}$$

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  - the observability Gramian becomes

$$\boxed{\overline{\mathcal{Q}} = \mathbf{T}^{\star^{-1}}\mathcal{Q}\mathbf{T}^{-1}}$$

- Balancing transformations $\mathbf{T}_{\text{bal}}$ and $\mathbf{T}_{\text{bal}}^{-1}$ can be computed as follows
  1. compute the Cholesky factorization $\mathcal{P} = \mathbf{U}\mathbf{U}^\star$
  2. compute the eigenvalue decomposition of $\mathbf{U}^\star \mathcal{Q} \mathbf{U}$

$$\mathbf{U}^\star \mathcal{Q} \mathbf{U} = \mathbf{K}\boldsymbol{\Sigma}^2\mathbf{K}^\star$$

     where the entries in $\boldsymbol{\Sigma}$ are ordered decreasingly
  3. compute the transformations

$$\begin{aligned} \mathbf{T}_{\text{bal}} &= \boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{K}^\star\mathbf{U}^{-1} \\ \mathbf{T}_{\text{bal}}^{-1} &= \mathbf{U}\mathbf{K}\boldsymbol{\Sigma}^{-\frac{1}{2}} \end{aligned}$$

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- Then, one can check that balancing is achieved
$$\boxed{\mathbf{T}_{\text{bal}}\mathcal{P}\mathbf{T}_{\text{bal}}^\star = \mathbf{T}_{\text{bal}}^{\star^{-1}}\mathcal{Q}\mathbf{T}_{\text{bal}}^{-1} = \mathbf{\Sigma}}$$

Definition (Hankel Singular Values)

$\mathbf{\Sigma} = \text{diag}(\sigma_1, \cdots, \sigma_N)$ contains the N **Hankel singular values of the system** (a Hankel singular value is computed from the Hankel operator or the product of Gramian matrices $(\mathcal{P}\mathcal{Q})$ associated with a LTI system and measures the energy of a corresponding internal state)

- Computing the balancing transformation $\mathbf{T}_{\text{bal}}$ is equivalent to minimizing the following function

$$\boxed{\min_{\mathbf{T}\in GL(N)} f(\mathbf{T}) = \min_{\mathbf{T}\in GL(N)} \text{trace}(\mathbf{T}\mathcal{P}\mathbf{T}^{\star} + \mathbf{T}^{\star^{-1}}\mathcal{Q}\mathbf{T}^{-1})}$$

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- For the optimal transformation $\mathbf{T}_{\text{bal}}$, the function takes the value

$$f(\mathbf{T}_{\text{bal}}) = 2\text{tr}(\boldsymbol{\Sigma}) = 2\sum_{i=1}^{N} \sigma_i$$

where $\{\sigma_i\}_{i=1}^N$ are the Hankel singular values

- Applying the change of variable $\bar{\mathbf{w}} = \mathbf{T}_{\text{bal}}\mathbf{w}$ transforms the given dynamical system into $(\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}})$ where

$$\overline{\mathbf{A}} = \mathbf{T}_{\text{bal}}\mathbf{A}\mathbf{T}_{\text{bal}}^{-1}, \quad \overline{\mathbf{B}} = \mathbf{T}_{\text{bal}}\mathbf{B}, \quad \overline{\mathbf{C}} = \mathbf{C}\mathbf{T}_{\text{bal}}^{-1}$$

- Let $1 \leq k \leq N$; the system can be partitioned in blocks as

$$\overline{\mathbf{A}} = \begin{bmatrix} \overline{\mathbf{A}}_{11} & \overline{\mathbf{A}}_{12} \\ \overline{\mathbf{A}}_{21} & \overline{\mathbf{A}}_{22} \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \overline{\mathbf{B}}_{1} \\ \overline{\mathbf{B}}_{2} \end{bmatrix}, \qquad \overline{\mathbf{C}} = \begin{bmatrix} \overline{\mathbf{C}}_{1} & \overline{\mathbf{C}}_{2} \end{bmatrix}$$

- The subscripts 1 and 2 denote the dimensions k and $N - k$, respectively

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- Then, the following lower-dimensional model of size k

$$(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r) = (\overline{\mathbf{A}}_{11}, \overline{\mathbf{B}}_1, \overline{\mathbf{C}}_1) \in \mathbb{R}^{k \times k} \times \mathbb{R}^{k \times \text{in}} \times \mathbb{R}^{q \times k}$$

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is the PROM constructed by Balanced Truncation (BT)
- The **left and right** ROBs are

$$\mathbf{W} = \mathbf{T}^{\star}_{\text{bal}}(:, 1:k) \quad \text{and} \quad \mathbf{V} = \mathbf{S}_{\text{bal}}(:, 1:k), \text{ respectively,}$$

where $\mathbf{S}_{\text{bal}} = \mathbf{T}^{-1}_{\text{bal}}$

Definition (The Hardy space $\mathcal{H}_\infty$)

The $\mathcal{H}_\infty$-norm associated with a system characterized by a *transfer function* $\mathbf{G}(\cdot)$ is defined as

$$\|\mathbf{G}\|_{\mathcal{H}_\infty} = \sup_{\mathbf{z}\in\mathbb{C}_+} \|\mathbf{G}(z)\|_\infty = \sup_{\mathbf{z}\in\mathbb{C}_+} \sigma_{\max}\left(\mathbf{G}(z)\right)$$

where $z \in \mathbb{C}_+$ if $z \in \mathbb{C}$ and $\mathfrak{Im}(z) > 0$.

Proposition

$$(i) \quad \|\mathbf{G}\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \|\mathbf{G}(i\omega)\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(\mathbf{G}(i\omega)\right)$$

$$(ii) \quad \|\mathbf{G}\|_{\mathcal{H}_\infty} = \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\|\mathbf{y}(\cdot)\|_2}{\|\mathbf{u}(\cdot)\|_2} = \sup_{\mathbf{u}\neq\mathbf{0}} \sqrt{\frac{\int_0^\infty \|\mathbf{y}(t)\|_2^2 dt}{\int_0^\infty \|\mathbf{u}(t)\|_2^2 dt}}$$

- The $\mathcal{H}_\infty$ norm of the error between the HDM- and PROM-based solutions will be used as an error criterion

Theorem (Error Bounds)

The BT procedure yields the following error bound for the output of interest.
Let $\{\bar{\sigma}_i\}_{i=1}^{N_{SV}} \subseteq \{\sigma_i\}_{i=1}^{N}$ *denote the* **distinct** *Hankel singular values of the system and* $\{\bar{\sigma}_i\}_{i=N_k+1}^{N_{SV}}$ *the ones that have been truncated. Then*

$$\|\mathbf{y}(\cdot) - \mathbf{y}_r(\cdot)\|_2 \leq 2 \sum_{i=N_k+1}^{N_{SV}} \bar{\sigma}_i \|\mathbf{u}(\cdot)\|_2$$

Equivalently, the above result can be written in terms of the $\mathcal{H}_\infty$*-norm of the system error as follows*

$$\|\mathbf{G}(\cdot) - \mathbf{G}_r(\cdot)\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=N_k+1}^{N_{SV}} \bar{\sigma}_i$$

where $\mathbf{G}$ *and* $\mathbf{G}_r$ *are the full- and reduced-order transfer functions. Equality holds when* $\bar{\sigma}_{N_k+1} = \bar{\sigma}_{N_{SV}}$ *(all truncated singular values are equal).*

Proof. The proof proceeds in 3 steps:

1. Consider the system error $\mathbf{E}(s) = \mathbf{G}(s) - \mathbf{G}_r(s)$

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3. Use this result to show that in the general case

$$\|\mathbf{E}(\cdot)\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=N_k+1}^{N_{\mathsf{SV}}} \bar{\sigma}_i$$

Theorem (Stability Preservation)

Consider $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r) = (\overline{\mathbf{A}}_{11}, \overline{\mathbf{B}}_1, \overline{\mathbf{C}}_1)$*, a PROM obtained by BT. Then*
(i) $\mathbf{A}_r = \overline{\mathbf{A}}_{11}$ *has no eigenvalues in the open right half plane*

(ii) Furthermore, if the systems $(\overline{\mathbf{A}}_{11}, \overline{\mathbf{B}}_1, \overline{\mathbf{C}}_1)$ *and* $(\overline{\mathbf{A}}_{22}, \overline{\mathbf{B}}_2, \overline{\mathbf{C}}_2)$ *have no Hankel singular values in common,* $\mathbf{A}_r$ *has no eigenvalues on the imaginary axis*

- Because of numerical stability issues, computing the transformations

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  4. Proof: Recall that $\mathbf{\Sigma}$ is always real-valued then compute $\mathbf{T}_{\text{bal}}\mathcal{P}\mathbf{T}_{\text{bal}}^{\star}$ and $\mathbf{T}_{\text{bal}}^{\star^{-1}}\mathcal{Q}\mathbf{T}_{\text{bal}}^{-1}$ using the above SVD

- BT leads to PROMs with quality and stability guarantees; however
  - the computation of a Gramian is intensive as it requires the solution of a Lyapunov equation ($\mathcal{O}(N^3)$ operations)
  - for this reason, BT is in general impractical for large systems – say $N \gtrsim 10^5$ (but monitor progress in the literature if interested)

Recall the theorem underlying the construction of a POD basis

Theorem

Let $\widehat{\mathbf{K}} \in \mathbb{R}^{N \times N}$ be the real symmetric semi-definite positive matrix defined as

$$\hat{\mathbf{K}} = \int_0^T \mathbf{w}(t)\mathbf{w}(t)^T dt$$

Let $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_N \geq 0$ denote the ordered eigenvalues of $\widehat{\mathbf{K}}$ and let $\widehat{\phi}_i \in \mathbb{R}^N$, $i = 1, \cdots, N$, denote the associated eigenvectors

$$\widehat{\mathbf{K}}\widehat{\phi}_i = \hat{\lambda}_i \widehat{\phi}_i, \;\; i = 1, \cdots, N.$$

The subspace $\widehat{\mathcal{V}} = range(\widehat{\mathbf{V}})$ of dimension k minimizing $J(\mathbf{\Pi}_{\mathbf{V},\mathbf{V}})$ is the invariant subspace of $\widehat{\mathbf{K}}$ associated with the eigenvalues $\hat{\lambda}_1, \cdots, \hat{\lambda}_k$

- The response of an LTI system to a single impulse input with a zero initial condition is

$$\mathbf{w}(t) = e^{\mathbf{A}t}\mathbf{B}$$

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- Consequently, the reachability Gramian is

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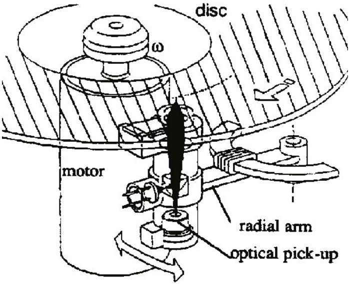
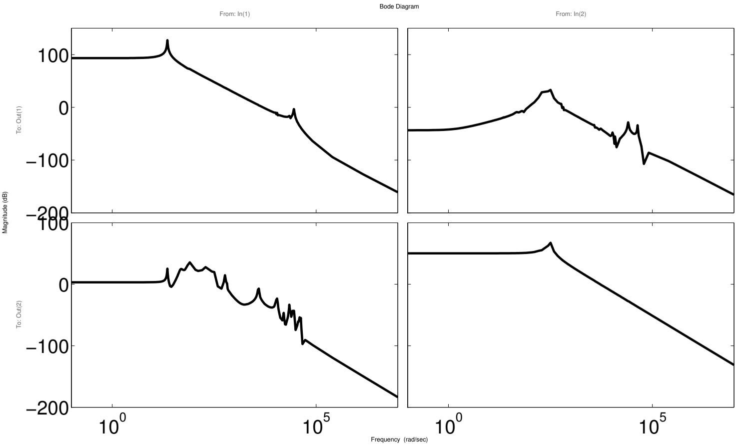
- Unlike the BT method, the POD method does not take into account the observability Gramian to determine the PROM: therefore, every observable state may be truncated



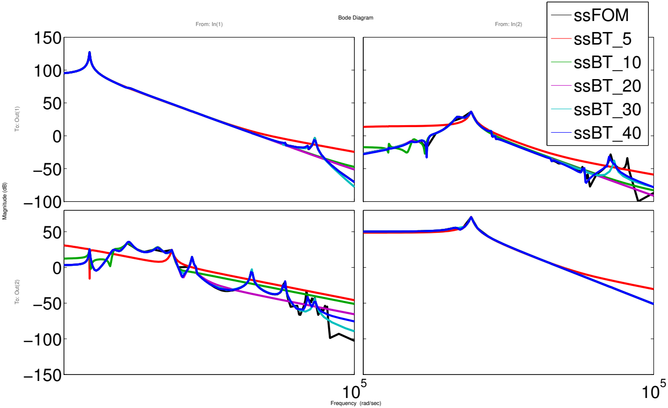
- Objective: model the position of the lens controlled by a swing arm
- System with in $= 2$ inputs
  - control voltage commanding the lens to move up and down to maintain the laser beam's focus on the disc's information layer
  - control voltage commanding the entire swing arm to move radially to keep the laser spot precisely on the data track

  and $q = 2$ outputs
  - focus error signal (degree and direction of vertical misalignment)
  - tracking error signal (degree and direction of radial misalignment)

Bode plots associated with the HDM-based solution ($N = 120$): Each column represents one input and each row represents a different output



Bode plots associated with the PROM-based (BT) solution: Each column represents one input and each row represents a different output



- The Balanced POD (BPOD) method generates two sets of snapshots: The standard POD solution snapshots; and the dual POD snapshots introduced below

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- Next, BPOD computes right and left ROBs as follows

$$\begin{aligned} \mathbf{S}_{\text{dual}}^T \mathbf{S} &= \mathbf{U}\mathbf{\Sigma}\mathbf{Z}^T \quad \text{(SVD)} \\ \mathbf{V} &= \mathbf{S}\mathbf{Z}_k \mathbf{\Sigma}_k^{-1/2} \\ \mathbf{W} &= \mathbf{S}_{\text{dual}} \mathbf{U}_k \mathbf{\Sigma}_k^{-1/2} \end{aligned}$$

  where the subscript k designates the first k terms of the singular value decomposition

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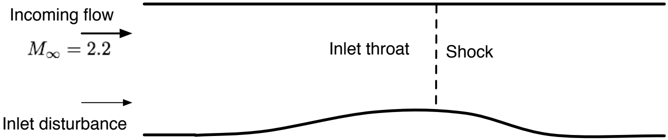
- If no truncation is performed, the result is equivalent to two-sided moment matching at $s_i \in \{\omega_1, \cdots, \omega_{N_{\text{snap}}}\}$ (see later)

- The POD method in the time domain is based solely on the reachability concept
- However, the BPOD method
  - adds the notion of observability in the construction of a PROM
  - is tractable for very large-scale systems
  - provides an approximation to the BT method
  - does not guarantee in general the stability of the resulting PROM

- Supersonic Inlet Problem (part of the Oberwolfach Model Reduction Benchmark Collection repository)



$$\begin{aligned} \mathbf{E}\frac{d\mathbf{w}}{dt}(t) &= \mathbf{A}\mathbf{w}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{w}(t) \end{aligned}$$

- $N = 11\,370$ (2D Euler equations)
- $in = 1$ input (density disturbance of the inlet flow)
- $q = 1$ output (average Mach number at the diffuser throat)

- PMOR in the frequency domain using
  - POD
  - BPOD
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- Plot of the magnitude of the relative error in the transfer function (within the sampled frequency interval) as a function of the dimension k of the constructed PROM