CME 345: MODEL REDUCTION
Moment Matching

David Amsallem & Charbel Farhat
Stanford University
cfarhat@stanford.edu

These slides are based on the recommended textbook: A.C. Antoulas, “Approximation of Large-Scale Dynamical Systems,” Advances in Design and Control, SIAM, ISBN-0-89871-529-6
Outline

1. Moments of a Function
2. Moment Matching Method
3. Krylov-based Moment Matching Methods
4. Error Bounds
5. Comparison with POD and BPOD in the Frequency Domain
6. Application
\[
\begin{align*}
\frac{d}{dt} \mathbf{w}(t) &= A\mathbf{w}(t) + B\mathbf{u}(t) \\
y(t) &= C\mathbf{w}(t) + D\mathbf{u}(t) \\
w(0) &= \mathbf{w}_0
\end{align*}
\]

- \( \mathbf{w} \in \mathbb{R}^N \): vector state variables
- \( \mathbf{u} \in \mathbb{R}^p \): vector of input variables, typically \( p \ll N \)
- \( \mathbf{y} \in \mathbb{R}^q \): vector output variables, typically \( q \ll N \)
Goal: construct a Reduced-Order Model (ROM)

\[
\frac{dq(t)}{dt} = A_r q(t) + B_r u(t) \\
y(t) = C_r q(t) + D_r u(t)
\]

\( q \in \mathbb{R}^k \): vector of reduced state variables

ROM resulting from Petrov-Galerkin projection

\[
A_r = (W^T V)^{-1} W^T A V \in \mathbb{R}^{k \times k} \\
B_r = (W^T V)^{-1} W^T B \in \mathbb{R}^{k \times p} \\
C_r = CV \in \mathbb{R}^{q \times k} \\
D_r = D \in \mathbb{R}^{q \times p}
\]
Let $h$ denote a general matrix-valued function of time

$$h : t \in \mathbb{R} \mapsto \mathbb{R}^{q \times p}$$

**Example:** impulse response of an LTI system

$$h(t) = Ce^{At}B + D\delta(t)$$

And let $H(s) \in \mathbb{R}^{q \times p}$ denote its Laplace transform

$$H(s) = \int_0^\infty h(t)e^{-st}dt$$

**Example:** impulse response of an LTI system

$$H(s) = C(sI_N - A)^{-1}B + D$$

$H(s)$ is the transfer function associated with the HDM defined by $(A, B, C, D)$ as for each input $U(s)$, it defines the output

$$Y(s) = H(s)U(s)$$
Moments of a Function

Let \( m \in \{0, 1, \cdots, \} \)

The \( m \)-th moment of \( h : t \in \mathbb{R} \mapsto \mathbb{R}^{q \times p} \) at \( s_0 \in \mathbb{C} \) is

\[
\eta_m(s_0) = \int_0^\infty t^m e^{-s_0 t} h(t) \, dt 
\]

Hence, the \( m \)-th moment of \( h \) can be written in terms of the transfer function \( H(s) \) as follows

\[
\eta_m(s_0) = (-1)^m \left. \frac{d^m}{ds^m} H(s) \right|_{s=s_0}
\]

**Example:** impulse response of an LTI system

\[
\eta_0(s_0) = H(s_0) = C(s_0 I_N - A)^{-1} B + D \\
\eta_m(s_0) = m! \ C(s_0 I_N - A)^{-(m+1)} B, \quad \forall m \geq 1
\]
Development of $H(s)$ in Taylor series

$$H(s) = H(s_0) + \frac{d}{ds} H(s) \bigg|_{s=s_0} \frac{(s-s_0)}{1!} + \cdots$$

$$+ \frac{d^m}{ds^m} H(s) \bigg|_{s=s_0} \frac{(s-s_0)^m}{m!} + \cdots$$

$$= \eta_0(s_0) - \eta_1(s_0)\frac{(s-s_0)}{1!} + \cdots + (-1)^m \eta_m(s_0)\frac{(s-s_0)^m}{m!} + \cdots$$

$$= \eta_0(s_0) + \eta_1(s_0)\frac{(s_0-s)}{1!} + \cdots + \eta_m(s_0)\frac{(s_0-s)^m}{m!} + \cdots$$
The Markov parameters of the system defined by $h$ are defined as the coefficients $\eta_m(\infty)$ of the Laurent series expansion of the transfer function at infinity

$$H(s) = \eta_0(\infty) + \frac{1}{s} \eta_1(\infty) + \frac{1}{s^2} \eta_2(\infty) + \cdots + \frac{1}{s^m} \eta_m(\infty) + \cdots$$

Example: impulse response of an LTI system

$$\eta_0(\infty) = \mathbf{D}$$
$$\eta_m(\infty) = \mathbf{C} \mathbf{A}^{m-1} \mathbf{B}, \ \forall m \geq 1$$

Proof: Use the property that for $s \rightarrow \infty$:

$$(s\mathbf{I}_N - \mathbf{A})^{-1} = \frac{1}{s} \mathbf{I}_N + \frac{1}{s^2} \mathbf{A} + \cdots + \frac{1}{s^{m+1}} \mathbf{A}^m + \cdots$$
Moment Matching Method

General Idea

Let \( s_0 \in \mathbb{C} \), and let \( H(s) = C(sI_N - A)^{-1}B + D \) represent the HDM defined by \((A, B, C, D)\)

**Objective:** construct a ROM \((A_r, B_r, C_r, D_r)\) such that the first \( l \) moments \( \{\eta_{r,j}(s_0)\}_{j=0}^{l-1} \) of its transfer function at \( s_0 \),
\[ H_r = C_r(s_0I_r - A_r)^{-1}B_r + D_r \in \mathbb{R}^{q \times p}, \]
match the first \( l \) moments \( \{\eta_j(s_0)\}_{j=0}^{l-1} \) of the transfer function \( H(s) \in \mathbb{R}^{q \times p} \) of the HDM

\[ \eta_{r,j}(s_0) = \eta_j(s_0) \Leftrightarrow H^{(j)}(s_0) = H_r^{(j)}(s_0), \ \forall j = 0, \cdots, l - 1 \]

- the *direct* matching of the moments is in general a numerically unstable procedure
- moment matching is performed best today using an equivalent procedure based on Krylov subspaces

For simplicity, focus is set on the Single Input-Single Output (SISO) \((p = q = 1)\) case throughout the remainder of this chapter

\[ B = b \in \mathbb{R}^N, \ C^T = c^T \in \mathbb{R}^N \]
Theorem

Let $V$ be a right Reduced-Order Basis (ROB) such that

$$\text{range}(V) = \mathcal{K}_k(A, b) = \text{span}\{b, Ab, \cdots, A^{k-1}b\}$$

and $W$ be a left ROB satisfying

$$W^T V = I$$

Then, the ROM obtained by Petrov-Galerkin projection of the HDM $(A, B, C, D)$ using $W$ and $V$ satisfies

$$\eta_{r,j}(\infty) = \eta_j(\infty) \Leftrightarrow H_r^{(j)}(\infty) = H^{(j)}(\infty), \ \forall j = 0, \cdots, k - 1$$
Definition

The order-$k$ Krylov subspace generated by $\mathbf{A} \in \mathbb{R}^{N \times N}$ and $\mathbf{b} \in \mathbb{R}^N$ is

$$\mathcal{K}_k(\mathbf{A}, \mathbf{b}) = \text{span}\{\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{k-1}\mathbf{b}\}$$

Remark: Constructing $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$ requires only the ability to compute the action of the matrix $\mathbf{A}$ onto a vector. In many applications, such a computation can be performed without forming explicitly the matrix $\mathbf{A}$. 
The following lemma is introduced to prove the previous theorem

**Lemma**

*The moments of the transfer function of a ROM do not depend on the underlying left and right ROBs, but only on the subspaces associated with these ROBs.*

**Proof of the Theorem.**

From the above lemma, it follows that \( V \) can be chosen, for example, as follows

\[
V = [v_1, \ldots, v_i, \ldots, v_k] = [b, Ab, \ldots, A^{k-1}b]
\]

\[
W^T V = I_k \Rightarrow AVW^T v_i = AV e_i = Av_i = v_{i+1} = A^i b
\]

\[
\Rightarrow \eta_{r,0}(\infty) = D = \eta_0(\infty)
\]

\[
\eta_{r,1}(\infty) = c_r b_r = cVW^T b = cVW^T v_1 = cVe_1 = cb = \eta_1(\infty)
\]

\[
\eta_{r,j}(\infty) = c_r A^j b_r = cVW^T (AVW^T)^j b = cVW^T (AVW^T)^j v_1
\]

\[
= cVW^T v_{j+1} = cVe_{j+1} = cv_{j+1} = cA^j b = \eta_j(\infty)
\]
Theorem

Let $s_0 \in \mathbb{C}$, $V$ be a right ROB satisfying

$$\text{range}(V) = \mathcal{K}_k \left( (s_0 I_N - A)^{-1}, (s_0 I_N - A)^{-1}b \right)$$

$$= \text{span} \left\{ (s_0 I_N - A)^{-1}b, \cdots, (s_0 I_N - A)^{-k}b \right\}$$

and $W$ be a left ROB satisfying

$$W^T V = I$$

Then, the ROM obtained by Petrov-Galerkin projection of the HDM $(A, B, C, D)$ using $W$ and $V$ satisfies

$$\eta_{r,j}(s_0) = \eta_j(s_0) \iff H_r^{(j)}(s_0) = H^{(j)}(s_0), \ \forall j = 0, \cdots, k - 1$$

This is a more computationally expensive procedure as the computation of each Krylov basis vector requires the solution of a large-scale system of equations.
Theorem

Let \( s_i \in \mathbb{C}, \ i = 1, \cdots, k \), \( V \) be a right ROB satisfying

\[
\text{range}(V) = \text{span}\left\{ (s_1 I_N - A)^{-1} b, \cdots, (s_k I_N - A)^{-1} b \right\}
\]

and \( W \) be a left ROB satisfying

\[
W^T V = I
\]

Then, the ROM obtained by Petrov-Galerkin projection of the HDM \((A, B, C, D)\) using \( W \) and \( V \) satisfies

\[
\eta_{r,0}(s_i) = \eta_0(s_i) \iff H_r(s_i) = H(s_i), \ \forall i = 1, \cdots, k
\]
Theorem

Let \( s_i \in \mathbb{C}, \ i = 1, \ldots, l, \ \mathbf{V} \) be a right ROB satisfying

\[
\text{range}(\mathbf{V}) = \bigcup_{i=1}^{l} \mathcal{K}_k \left( (s_i \mathbf{I}_N - \mathbf{A})^{-1}, (s_i \mathbf{I}_N - \mathbf{A})^{-1} \mathbf{b} \right)
\]

and \( \mathbf{W} \) be a left ROB satisfying

\[
\mathbf{W}^T \mathbf{V} = \mathbf{I}
\]

Then, the ROM obtained by Petrov-Galerkin projection of the HDM \((\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})\) using \( \mathbf{W} \) and \( \mathbf{V} \) satisfies

\[
\eta_{r,j}(s_i) = \eta_j(s_i) \Leftrightarrow \mathbf{H}^{(j)}(s_i) = \mathbf{H}^{(j)}(s_i), \ \forall i = 1, \ldots, l, \ \forall j = 0, \ldots, k - 1
\]
Theorem

Let \( s_i \in \mathbb{C}, \ i = 1, \cdots, 2l \), \( \mathbf{V} \) be a right ROB satisfying

\[
\text{range}(\mathbf{V}) = \bigcup_{i=1}^{l} \mathcal{K}_k \left( (s_i \mathbf{I}_N - \mathbf{A})^{-1}, (s_i \mathbf{I}_N - \mathbf{A})^{-1} \mathbf{b} \right)
\]

and \( \mathbf{W} \) be a left ROB satisfying

\[
\text{range}(\mathbf{W}) = \bigcup_{i=l+1}^{2l} \mathcal{K}_k \left( (s_i \mathbf{I}_N - \mathbf{A}^T)^{-1}, (s_i \mathbf{I}_N - \mathbf{A}^T)^{-1} \mathbf{c}^T \right)
\]

and \( \mathbf{W}^T \mathbf{V} \) is nonsingular

Then, the ROM obtained by Petrov-Galerkin projection of the HDM \( (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \) using \( \mathbf{W} \) and \( \mathbf{V} \) satisfies

\[
\eta_{r,j}(s_i) = \eta_j(s_i) \iff H_r^{(j)}(s_i) = H^{(j)}(s_i), \ \forall i = 1, \cdots, 2l, \ \forall j = 0, \cdots, k-1
\]
Partial realization requires the construction of $\mathcal{K}_k(A, b)$, that is the knowledge of the action of $A$ onto vectors.

Rational interpolation requires the following Krylov space

$$\mathcal{K}_k((s_0 I_N - A)^{-1}, (s_0 I_N - A)^{-1} b).$$

Since the knowledge of the action of $(s_0 I_N - A)^{-1} \in \mathbb{R}^{N \times N}$ is needed, two computationally efficient approaches are possible:

- if $N$ is small enough, an LU factorization of $s_0 I_N - A$ can be performed and $(s_0 I_N - A)^{-1} v$ computed by forward and backward substitution for any vector $v \in \mathbb{R}^N$,

- if $N$ is too large for an LU factorization to be performed, Krylov subspace recycling techniques allowing the reuse of Krylov subspaces for multiple right-hand sides can be used.
\( K_k(A, b) \) can be efficiently constructed using the Arnoldi factorization method.

**Input:** \( A \in \mathbb{R}^{N \times N} \), \( b \in \mathbb{R}^N \)

**Output:** Orthogonal basis \( V_k \in \mathbb{R}^{N \times k} \) for \( K_k(A, b) \)

The following recursion is satisfied:

\[
A V_k = V_k H_k + f_k e_k^T
\]

with \( H_k = V_k^T A V_k \) an upper Hessenberg matrix, \( V_k^T V_k = I_k \) and \( V_k^T f_k = 0 \).
Algorithm:

**Input:** $A \in \mathbb{R}^{N \times N}, \ b \in \mathbb{R}^N$

**Output:** Orthogonal basis $V_k \in \mathbb{R}^{N \times k}$ for $K_k(A, b)$

1. $v_1 = b/\|b\|$;
2. $w = Av_1; \ \alpha_1 = v_1^T w$;
3. $f_1 = w - \alpha_1 v_1$;
4. $V_1 = [v_1]; \ H = [\alpha_1]$;
5. for $j = 1, \cdots, k - 1$ do
6. $\beta_j = \|f_j\|; \ v_{j+1} = f_j/\beta_j$;
7. $V_{j+1} = [V_j, \ v_{j+1}]$;
8. $\hat{H}_j = \begin{bmatrix} H_j \\ \beta_j e_j^T \end{bmatrix}$
9. $w = Av_{j+1}$;
10. $h = V_{j+1}^T w; \ f_{j+1} = w - V_{j+1}h$
11. $H_{j+1} = [\hat{H}_j, \ h]$;
12. end for
Krylov-based Moment Matching Methods

The Two-Sided Lanczos Method for Partial Realization

- $\mathcal{K}_k(A, b)$ and $\mathcal{K}_k(A^T, c^T)$ can be efficiently simultaneously constructed using the Two-sided Lanczos process

**Input:** $A \in \mathbb{R}^{N \times N}$, $b \in \mathbb{R}^N$, $c^T \in \mathbb{R}^N$

**Output:** Bi-orthogonal bases $V_k \in \mathbb{R}^{N \times k}$ and $W_k \in \mathbb{R}^{N \times k}$ for $\mathcal{K}_k(A, b)$ and $\mathcal{K}_k(A^T, c^T)$ respectively satisfying $W_k^T V_k = I_k$

- The following recursions are satisfied:

$$AV_k = V_k T_k + f_k e_k^T,$$

$$A^T W_k = W_k T_k^T + g_k e_k^T,$$

with $T_k = W_k^T A V_k$ a tridiagonal matrix, $W_k^T V_k = I_k$, $W_k^T g_k = 0$ and $V^T f_k = 0$. 
Krylov-based Moment Matching Methods

The Two-Sided Lanczos Method for Partial Realization

Algorithm:

**Input:** \( A \in \mathbb{R}^{N \times N}, \ b \in \mathbb{R}^N, \ c^T \in \mathbb{R}^N \)

**Output:** Bi-orthogonal bases \( V_k \in \mathbb{R}^{N \times k} \) and \( W_k \in \mathbb{R}^{N \times k} \) for \( \mathcal{K}_k(A, b) \) and \( \mathcal{K}_k(A^T, c^T) \) respectively satisfying \( W_k^T V_k = I_k \)

1. \( \beta_1 = \sqrt{|b^T c^T|}, \ \gamma_1 = \text{sign}(b^T c^T) \beta_1 \)
2. \( v_1 = b / \beta_1, \ w_1 = c^T / \gamma_1 \)
3. for \( j = 1, \ldots, k - 1 \) do
   4. \( \alpha_j = w_j^T A v_j; \)
   5. \( r_j = A v_j - \alpha_j v_j - \gamma_j v_{j-1}; \)
   6. \( q_j = A^T w_j - \alpha_j w_j - \beta_j w_{j-1}; \)
   7. \( \beta_{j+1} = \sqrt{|r_j^T q_j|}, \ \gamma_{j+1} = \text{sign}(r_j^T q_j) \beta_{j+1} \)
   8. \( v_{j+1} = r_j / \beta_{j+1}; \)
   9. \( w_{j+1} = q_j / \gamma_{j+1}; \)
4. end for
11. \( V_k = [v_1, \ldots, v_k], \ W_k = [w_1, \ldots, w_k] \)
The $\mathcal{H}_2$ norm of a continuous dynamical system $S = (A, B, C, D)$ is the $L_2$ norm of its associated impulse response $h(\cdot)$. When $A$ is stable and $D = 0$, the norm is bounded and

$$\|S\|_{\mathcal{H}_2} = \left( \int_0^\infty \text{trace} \left( h^T(t)h(t) \right) dt \right)^{1/2}$$

Using Parseval’s theorem, one can obtain the expression in the frequency domain using the transfer function $H(\cdot)$

$$\|S\|_{\mathcal{H}_2} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left( H^*(-i\omega)H(i\omega) \right) d\omega \right)^{1/2}$$

One can also derive the expression of $\|S\|_{\mathcal{H}_2}$ in terms of the reachability and observability gramians $P$ and $Q$.

$$\|S\|_{\mathcal{H}_2} = \sqrt{\text{trace} \left( B^T Q B \right)} = \sqrt{\text{trace} \left( C P C^T \right)}$$
In the SISO case, the transfer function is a rational function. Assuming for simplicity that it has distinct poles $\lambda_i, \ i = 1, \cdots, N$ associated with the residues $h_i$, one can write it as

$$H(s) = \sum_{i=1}^{N} \frac{h_i}{s - \lambda_i}$$

One can then establish the following theorem:

**Theorem**

Let $H_r(\cdot)$ be the transfer function associated with the system $S_r$ resulting from moment matching using the Lanczos procedure of the underlying system $S$. Denoting by $h_{r,i}$ and $\lambda_{r,i}, \ i = 1, \cdots, k$ the respective residues and poles of $H_r(\cdot)$, the following result holds:

$$\|S - S_r\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} h_i (H(-\lambda_i^*) - H_r(-\lambda_i^*)) + \sum_{i=1}^{k} h_{r,i} (H_r(-\lambda_{r,i}) - H(-\lambda_{r,i}))$$
One would like to build ROBs \((V, W)\) of a given dimension \(k\) such that the corresponding reduced system \(S_r\) is \(H_2\)-optimal, i.e. minimizes the following problem

\[
\min_{S_r, \text{rank}(V)=\text{rank}(W)=k} \|S - S_r\|_{H_2}
\]

One can show that a necessary condition is that the ROM matches the first two moments of the HDM at the points \(-\lambda_{r,i}\), mirror images of the poles \(\lambda_{r,i}\) of the reduced transfer function \(H_r(\cdot)\)

\[
H_r(-\lambda_{r,i}) = H(-\lambda_{r,i}), \quad H_r^{(1)}(-\lambda_{r,i}) = H^{(1)}(-\lambda_{r,i}), \quad s = 1, \ldots, k
\]

Unfortunately moment matching ensures that the moments of the transfer function are matched at \(\lambda_{r,i}\), not \(-\lambda_{r,i}\).

The IRKA (Iterative Rational Krylov Approximation) procedure is an iterative procedure to conciliate these two contradicting goals.
POD in the frequency domain (LTI systems):

\[
\text{range}(\mathbf{V}) = \text{span}\{\mathcal{X}(\omega_1), \ldots, \mathcal{X}(\omega_k)\} = \text{span}\\left\{ (j\omega_1 \mathbf{I}_N - \mathbf{A})^{-1} \mathbf{b}, \ldots, (j\omega_k \mathbf{I}_N - \mathbf{A})^{-1} \mathbf{b} \right\}
\]

with \( \omega_1, \ldots, \omega_k \in \mathbb{R}^+ \)

- Rational interpolation with first moment matching at multiple points

\[
\text{range}(\mathbf{V}) = \text{span}\{ (s_1 \mathbf{I}_N - \mathbf{A})^{-1} \mathbf{b}, \ldots, (s_k \mathbf{I}_N - \mathbf{A})^{-1} \mathbf{b} \}
\]

with \( s_1, \ldots, s_k \in \mathbb{C} \)

- Question: would it be possible to extend the two-sided moment matching approach to POD?

- Answer: yes, this is the Balanced POD method
The Balanced POD method generates snapshots for the dual system in addition to the POD snapshots:

\[
S = [(j\omega_1 I_N - A)^{-1} b, \ldots, (j\omega_k I_N - A)^{-1} b] \\
S_{\text{dual}} = [(-j\omega_1 I_N - A^T)^{-1} c^T, \ldots, (-j\omega_k I_N - A^T)^{-1} c^T]
\]

Right and left reduced-order bases can then be computed as

\[
S_{\text{dual}}^T S = U\Sigma Z^T \quad \text{(SVD)} \\
V = SZ_k \Sigma_k^{-1/2} \\
W = S_{\text{dual}} U_k \Sigma_k^{-1/2}
\]

where a subscript \( k \) is relative to the first \( k \) components of the singular value decomposition.

If no truncation occurs, this is equivalent to two-sided moment matching at \( s_i \in \{\omega_1, \ldots, \omega_k\} \).
Application

- Frequency Sweeps

- Structural vibrations and interior noise/acoustics
  - Structural dynamics (Navier)
  - Interior Helmholtz

- Scattering (acoustics and electromagnetics)
  - Exterior Helmholtz
  - Electromagnetics (Maxwell)
  - Aeroacoustics (Helmholtz)
- **Application**

- **Frequency Response Problems**

  - **Structural dynamics**

    \[
    w_s(\omega) = \left(K_s + i\omega D_s - \omega^2 M_s\right)^{-1} f_s(\omega)
    \]

    Rayleigh damping \( D_s = \alpha K_s + \beta M_s \)

  - **Acoustics**

    \[
    w_f(\omega) = \left(K_f - \frac{\omega^2}{c_f^2} M_f + S_a(\omega)\right)^{-1} f_f(\omega)
    \]

  - **Structural (or vibro)-acoustics**

    \[
    w_v(\omega) = \left(K_v - \omega^2 M_v + S_v(\omega)\right)^{-1} f_v(\omega)
    \]

    \[
    = \begin{pmatrix} K_s & C^T \\ 0 & \frac{1}{\rho_f} K_f \end{pmatrix} - \omega^2 \begin{pmatrix} M_s & 0 \\ -C & \frac{1}{\rho_f c_f^2} M_f \end{pmatrix} + \begin{pmatrix} i\omega D_s & 0 \\ 0 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} f_s(\omega) \\ \frac{1}{\rho_f} f_f(\omega) \end{pmatrix}
    \]
Frequency response function $w = w(\omega) \longrightarrow$ problem with multiple left hand sides - very CPU intensive (1,000s of frequencies)
Interpolatory Reduced-Order Model by Krylov-based Moment Matching

- Approximate \( w(\omega) \) by the Galerkin projection \( w \approx \tilde{w} = Vq \)

\[
\tilde{w}(\omega) = V \left( V^H K V + i\omega V^H D V - \omega^2 V^H M V \right)^{-1} V^H f
\]

- If the columns of \( V \) span the solution and its derivatives at some frequency, the projection is **interpolatory**

- Two ways to compute the vectors in \( V \)
  - recursive differentiation with respect to \( \omega \) at the interpolating frequency
  - construction of a Krylov space that spans the derivatives (special cases)

\[
\text{span}\left\{ (K - \omega^2 M + i\omega D)^{-1} f, \right. \\
(K - \omega^2 M + i\omega D)^{-1} M (K - \omega^2 M + i\omega D)^{-1} f, \\
\ldots \\
\left. [(K - \omega^2 M + i\omega D)^{-1} M]^{n-1} (K - \omega^2 M + i\omega D)^{-1} f \right\}
\]
Application

Structural-Acoustic Vibrations

- Frequency sweep analysis of a thick spherical steel shell submerged in water and excited by a point load on its inner surface
- Finite element model using isoparametric cubic elements incorporates with roughly $N = 1,200,000$ dofs
Frequency sweep analysis of a submerged shell

Real part of pressure

Reference frequencies: 1,430 Hz, 2,860 Hz, and 4,290 Hz

Sampling every 12 Hz
Structural-Acoustic Vibrations

Frequency sweep analysis of a submerged shell

- Real part of pressure
- Reference frequencies: 1,430 Hz, 2,860 Hz, and 4,290 Hz
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Application

Structural-Acoustic Vibrations

Frequency sweep analysis of a submerged shell

- Real part of pressure

Reference frequencies: 1,430Hz, 2,860Hz, and 4,290Hz

- 3-points with 32+32+32 vectors

- Sampling every 12Hz
Application

Parameter Selection

- How to choose
  - number of interpolating frequencies
  - location of interpolating frequencies
  - number of derivatives (Krylov vectors)

- Error indicator: relative residual

\[
\frac{\| (K - \omega^2 M + i\omega D) \tilde{w}(\omega) - f \|}{\| f \|}
\]

where

\[
\tilde{w}(\omega) = V \left( V^H K V + i\omega V^H D V - \omega^2 V^H M V \right)^{-1} V^H f
\]
1. Specify the number of derivatives per frequency and an accuracy threshold.

2. Use two interpolations frequencies at the extremities of the frequency band of interest and construct the ROB.

3. Evaluate the residual at some small set of the frequencies in between.

4. Add a frequency where the residual is largest and update the projection.

5. Repeat until the residual is below a threshold at all sampling points.

6. Check at the end the residual at all sampled (or user-specified) frequencies.
Frequency sweep analysis of a submerged shell

- ROM with 16 vectors/frequency
- sampling every \( \Delta f = 12 \text{Hz} \)
- tolerance = 5e-4
- interpolating wavenumbers
Frequency sweep analysis of a submerged shell

- ROM with 16 vectors/frequency
- sampling every $\Delta f = 12\text{Hz}$
- tolerance = 5e-4
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers)

Relative error/residual vs. Frequency [Hz]
Frequency sweep analysis of a submerged shell

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Automatic Residual-Based Adaptivity by a Greedy Approach

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Frequency sweep analysis of a submerged shell

- ROM with 16 vectors/frequency
- sampling every $\Delta f = 12$Hz
- tolerance = $5e^{-4}$
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *
Frequency sweep analysis of a submerged shell

- ROM with 16 vectors/frequency
- sampling every $\Delta f = 12\text{Hz}$
- tolerance = $5\times10^{-4}$
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *
- final residual
Frequency sweep analysis of a submerged shell

- ROM with 16 vectors/frequency
- sampling every $\Delta f = 12\text{Hz}$
- tolerance $= 5e-4$
- interpolating wavenumbers
- sampled residual (at interpolating wavenumbers) *
- final residual
- final error