Problem 1: Projections

1. Which of these matrices are projection matrices? Provide a proof in each case.

\[ M_1 = \begin{bmatrix} 2.4 & 2.8 \\ -1.2 & -1.4 \end{bmatrix} \]
\[ M_2 = \begin{bmatrix} -1.4 & 2.8 \\ -1.2 & 2.4 \end{bmatrix} \]
\[ M_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

2. Consider two full column rank matrices \( X \in \mathbb{R}^{N \times k} \) and \( Y \in \mathbb{R}^{N \times k} \) with \( k \leq N \), and an invertible matrix \( E \in \mathbb{R}^{N \times N} \).

- Which of the matrices below are projection matrices for otherwise general matrices \( X, Y \) and \( E \)?
- If the matrix is a projection matrix, provide a basis for each of the subspaces \( S_1 \) and \( S_2 \) defining the projection.
- If in general the matrix is not a projection matrix, are there specific choices of \( X, Y \) and \( E \) for which it will be a projection matrix? (if so, give one such choice).

Support each of your answer with a proof.

\[ M_4 = EX \left( Y^T EE^T X \right)^{-1} Y^T E^T \]
\[ M_5 = EX \left( Y^T E^T EX \right)^{-1} Y^T E^T \]
\[ M_6 = EX \left( X^T E^T EY \right)^{-1} Y^T E^T \]
\[ M_7 = EX \left( Y^T EX \right)^{-1} Y^T E^T \]
\[ M_8 = X \left( Y^T EX \right)^{-1} Y^T E^T \]
\[ M_9 = EX \left( Y^T EX \right)^{-1} Y^T \]
\[ M_{10} = EX \left( Y^T E^2 X \right)^{-1} Y^T E \]
Problem 2: Model Reduction in an Affine Subspace

Consider the High-Dimensional Model (HDM)
\[
\frac{d}{dt} w(t) = f(w(t), t) \\
y(t) = g(w(t), t) \\
w(0) = w_0
\]

Let \( V \in \mathbb{R}^{N_k} \) and \( W \in \mathbb{R}^{N_k} \) define trial and test bases, respectively.

1. Consider the approximation
   \[
   w(t) \approx w(0) + Vq(t), \quad q(t) \in \mathbb{R}^k
   \]  
   Show that it leads to an approximation of the solution in an affine subspace.

2. Substitute the above approximation in the HDM and write the resulting equation.

3. What is the Reduced-Order Model (ROM) resulting from the Petrov-Galerkin projection?

4. What is the associated initial condition?

5. What is the equivalent HDM for this ROM?

6. What are the advantages of an affine approximation?

7. Consider now the Linear Time-Invariant (LTI) HDM with
   \[
   f(w(t), t) = Aw(t) + Bu(t) \\
g(w(t), t) = Cw(t) + Du(t)
   \]
   What are the reduced LTI operators resulting from a Petrov-Galerkin projection associated with the approximation (1)?

Problem 3: Steady Parameterized Linear PDE

Consider the steady, two-dimensional advection-diffusion equation
\[
\mathcal{U} \cdot \nabla T - \kappa \Delta T = 0, \text{ for } x = (x, y) \in [0, 1] \times [0, 1]
\]
with the boundary conditions
\[
T(x; \mu) = T_D(y; \overline{y}), \text{ for } x = (x, y) \in \Gamma_D = \{0\} \times [0, 1] \\
\nabla T(x; \mu) \cdot n(x) = 0, \text{ for } x \in \Gamma_N = \{1\} \times [0, 1] \cup [0, 1] \times \{1\} \cup \{0, 1\} \times \{0\}
\]
where
\[
T_D(y, t; \overline{y}) = \begin{cases} 
300 & \text{if } y \in [0, \frac{1}{3}] \\
300 + 325 \left( \sin \left( 3\pi |y - \overline{y}| \right) + 1 \right) & \text{if } y \in \left[\frac{1}{3}, \frac{2}{3}\right] \\
300 & \text{if } y \in \left[\frac{2}{3}, 1\right]
\end{cases}
\]  
(2)

and
• \( \mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \), with \( U_1 \in [0, 0.5] \) and \( U_2 = 0 \)

• \( \kappa \in [0, 0.025] \)

• \( \gamma \in [0.4, 0.6] \)

1. Discretize the above problem by finite differences using 75 points in each direction
   (Hint: use upwinding for the advection term and sparse matrices if you choose to use
   Matlab for this purpose). Please attach your code.

2. Compute and plot the solution for 6 different sets of parameters.