Problem 1: Cross-Gramian

In addition to the infinite reachability and controllability gramians, one can define a third one, called cross-gramian and defined as the solution $\mathcal{X}$ to the Sylvester equation

$$A\mathcal{X} + \mathcal{X}A + BC = 0$$

1. Show that

$$\mathcal{X} = \int_0^{\infty} e^{At}BCe^{At}dt$$

is solution of the above equation.

2. Definition: An LTI system $(A, B, C, D)$ is called symmetric if $D = D^T$ and there exists a matrix $\Psi = \Psi^T \in \mathbb{R}^{N \times N}$ such that

$$A\Psi = \Psi A^T$$
$$B = \Psi C^T$$

Show that for a symmetric system, $A^k\Psi = \Psi(A^T)^k$, $\forall k = 0, 1, \ldots$.

3. Show that $e^{At}\Psi = \Psi e^{A^Tt}$.

4. Give the relation between $\mathcal{P}$, $\mathcal{X}$ and $\Psi$ for symmetric systems.

5. From now on, assume that $\Psi$ is nonsingular. Show that $\Psi^{-1}e^{At} = e^{A^Tt}\Psi^{-1}$.

6. Use the result of the previous question to establish the relation between $\mathcal{Q}$, $\Psi$ and $\mathcal{X}$ for symmetric systems.
   (Hint: start from the expression for $Q$ and express it in terms of $X$ and $A$.)

7. Conclude that $\mathcal{X}^2 = \mathcal{P}\mathcal{Q}$ in the case of symmetric systems.

8. What is $\mathcal{X}^2$ when the dynamical system is balanced?
Problem 2: Moment Matching

Consider an LTI system \((A, B, C, D)\) for which \(D = 0\). The goal of this problem is to prove the following theorem which was covered in chapter 7.

**Theorem:** Let \(s_i \in \mathbb{C}, i = 1, \cdots, 2l\), \(V\) be a right ROB satisfying

\[
\text{range}(V) = \bigcup_{i=1}^{l} K_k \left( (s_i I_N - A)^{-1}, (s_i I_N - A)^{-1}b \right)
\]

and \(W\) be a left ROB satisfying

\[
\text{range}(W) = \bigcup_{i=l+1}^{2l} K_k \left( (s_i I_N - A^T)^{-1}, (s_i I_N - A^T)^{-1}c^T \right)
\]

and \(W^T V\) is nonsingular. Then, the ROM obtained by Petrov-Galerkin projection of the HDM \((A, B, C, D)\) using \(W\) and \(V\) satisfies

\[
\eta_{r,j}(s_i) = \eta_j(s_i) \Leftrightarrow H_r^{(j)}(s_i) = H^{(j)}(s_i), \forall i = 1, \cdots, 2l, \forall j = 0, \cdots, k - 1
\]

1. Show that when \(W^T V \neq I_k\), the moments of the transfer function associated with the ROM

\[
H_r(s) = c_r(sW^T V - A_r)^{-1}b_r = cV(sW^T V - W^T AV)^{-1}W^T b
\]

are given by

\[
\eta_{r,m}(s) = m!cV \left( (sW^T V - W^T AV)^{-1} (W^T V)^m (sW^T V - W^T AV)^{-1}W^T b \right)
\]

(Hint: exploit the identity \(dM^{-1}(s) \over ds = -M^{-1}(s) dM(s) M^{-1}(s)\).)

2. Prove the above theorem.