Convergence analysis of a discontinuous Galerkin method with plane waves and Lagrange multipliers for the solution of Helmholtz problems

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Abstract

We analyze the convergence of a discontinuous Galerkin method (DGM) with plane waves and Lagrange multipliers that was recently proposed by Farhat et al. [3] for solving two-dimensional Helmholtz problems at relatively high wave numbers. We prove that the underlying hybrid variational formulation is well-posed. We also present various *a priori* error estimates that establish the convergence and order of accuracy of the simplest element associated with this method. We prove that, for $k\,(k\,h)^{\frac{2}{3}}$ sufficiently small, the *relative* error in the L^2-norm (resp. in the H^1 semi-norm) is of order $k\,(k\,h)^{\frac{4}{3}}$ (resp. of order $(k\,h)^{\frac{2}{3}}$) for a solution being in $H^{\frac{5}{3}}(\Omega)$. In addition, we establish an a posteriori error estimate that can be used as a practical error indicator when refining the partition of the computational domain.

Key words: acoustic scattering, discontinuous Galerkin, Helmholtz problems, hybrid finite element, $\inf-\sup$ condition, plane waves.

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Introduction

The discontinuous enrichment method (DEM) was developed in [1, 2] for the solution of multi-scale boundary value problems with sharp gradients and rapid oscillations. These are problems for which the standard finite element method (FEM) can become prohibitively expensive. DEM can be described as a discontinuous Galerkin method (DGM) with Lagrange multiplier degrees of freedom (dofs), in which the standard finite element polynomial field is enriched within each element by free-space solutions of the homogeneous partial differential equation to be solved. Usually, these are easily obtained in *analytical* form and are discontinuous across the element interfaces. The Lagrange multiplier dofs are introduced at these interfaces to enforce a weak continuity of the solution. For the Helmholtz equation, the enrichment field can be constructed with plane waves as these are free-space solutions of this equation. In [3], it was shown that for a large class of Helmholtz problems, the polynomial field is not necessary for capturing efficiently the solution. Hence, for these applications, the polynomial field was dropped and DEM was transformed into a DGM with plane wave basis functions. Similar exponential functions were previously introduced in the weak element method (WEM) [4], the partition of unity method (PUM) [5], the ultra weak variational method (UWM) [7], and the least-squares method (LSM) presented in [8], for the solution of the Helmholtz equation. However, unlike WEM, the DGM proposed in [3] is based on a variational framework, and unlike PUM, it is discontinuous. Furthermore, in contrast to LSM, the continuity of the solution at the inter-element boundaries is enforced in DEM by Lagrange multipliers rather than penalty parameters, which increases the robustness and accuracy of the underlying framework of approximation.

In [3], two lower-order rectangular DGM elements with four and eight plane waves, respectively, were constructed and applied to the solution of two-dimensional waveguide problems with $10 \leq kl \leq 100$, where k denotes the wavenumber and l is a characteristic length of the waveguide. The discretization by these elements of such Helmholtz problems was found to require five to seven times fewer dofs than their discretization by the standard $Q2$ element, depending on the desired level of accuracy. In [9], this DGM was extended to exterior Helmholtz problems and was coupled with a second-order absorbing boundary condition. A lower-order quadrilateral element with eight Lagrange multiplier dofs was designed and highlighted with the solution on unstructured meshes of sample acoustic scattering problems with $20 \leq kl \leq 40$, where l denotes a characteristic length of the scatterer. This element was shown to deliver significant improvement over the performance of the standard and comparable $Q2$ element. In [10], two higher-order quadrilateral DGM elements with 16 and 32 plane waves, respectively, were presented. The DGM element with 16 plane waves has a computational complexity that is comparable to that of the standard $Q4$ element and was shown numerically to have the same convergence rate with respect to the mesh size. However, this DGM element was also shown numerically in [9] to deliver the same level of accuracy as $Q4$ using 6 times fewer dofs. All of these performance results highlight the potential of the DGM introduced in [3] and expanded in [9] and [10].

However, no mathematical analysis of this method has been performed yet. The objective of this paper is to fill this gap in the specific context of the two-dimensional low-order element with four plane waves in order to set this DGM method on a firm theoretical basis. The proposed study assumes that the computational domain Ω is a polygonal-shaped domain that can be partitioned into rectangular elements. Note that the computational domain Ω may have reentrant corners, and therefore the considered acoustic scattered field is in $H^{\frac{5}{3}}(\Omega)$ only. We partition the computational domain into *rectangular*-shaped elements and consider the case of the so-called R-4-1 element, that is we approximate *locally* the primal variable by four plane waves and the dual variable by constants

on the edges of interior elements. We must point out that this study cannot be extended–at this time– to higher order elements because it assumes that the normal derivative of the primal variable is constant along the interior edges. This *crucial* property is valid *only* in the case of R-4-1 element. We prove that for $k\,(k\,h)^{\frac{2}{3}}$ small enough, the *relative* error in the L^2-norm (resp. in the H^1 semi-norm) is of order $k\,(k\,h)^{\frac{4}{3}}$ (resp. $(k\,h)^{\frac{2}{3}}$). We recall that in the case of the standard finite element method using P_1 element (see [11, 12]), it has been established that for k^2h small enough, the relative error in the L^2 -norm (resp. in the H^1 semi-norm) is of order $k^3\,h^2$ (resp. $k\,h$). Moreover, if we assume that $k\,h$ is small enough, it has been established in Reference [12], that the relative error for both the L^2 -norm and the H^1 semi-norm are bounded by $k\,(k\,h)^2$. However, all these error estimates have been established assuming that the scattered field is in $H^2(\Omega)$ which is not a realistic assumption for most applications. We must also point out that, to the best of our knowledge, no error estimates have been derived yet in the particular case of $Q4$ finite element when applied to Helmholtz problems.
We also derive a posteriori error estimate that can be used as a practical error indicator when refining the partition of the computational domain. This error estimate reveals that the relative error in the L^2 norm depends on the errors in the approximation of the interior and exterior boundary conditions as well as on the jump across the elements of the partition.
The remainder of this paper is organized as follows. In Section 1, we specify the notations and assumptions used in this paper, state the formulation of a two-dimensional acoustic scattering problem in a bounded domain, and prove that the hybrid problem obtained by applying the DGM introduced above to the solution of the focus Helmholtz problem is well posed in the sense of Hadamard [13]. More specifically, we introduce Theorem 1 to address the issues of existence, uniqueness, and stability of the DGM formulation. Next, we devote Section 3 to the analysis of the discrete solution obtained with a DGM element with four plane waves. More specifically, we recall in Section 3.2 the discrete DGM formulation and announce the main results of this paper. These are existence and uniqueness results, a priori error estimates that are stated in Theorem 2, and a posteriori estimate that is stated in Theorem 3. The proofs of these three sets of fundamental results are detailed in Section 3.3 and Section 3.4. Finally, Section 4 concludes this paper.

1 Preliminaries

We consider throughout this paper the acoustic scattering problem by a *sound-hard* scatterer [14] formulated in a bounded domain as follows

$$\text{(BVP)}\quad \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ \qquad \Delta u + k^2 u = 0 \quad \text{in } \Omega, \\ \qquad \partial_n u = -\partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}} \quad \text{on } \Gamma, \\ \qquad \partial_n u = \mathrm{i}ku \quad \text{on } \Sigma, \end{cases} \tag{1}$$

where u is the scattered field and Ω is the computational domain. Ω is a *bounded* polygonal-shaped domain that can be partitioned into rectangular elements. Γ is its interior boundary and Σ is the exterior boundary. $\mathbf{n}$ is the unitary outward normal vector to the boundaries Γ and Σ and ∂_n is the normal derivative. k is a positive number representing the wavenumber. $\mathbf{d}$ is a unit vector representing the direction of the incident plane wave. The equation on Γ is the Neumann boundary condition that characterizes the sound-hard property of the scatterer. We must point out that the interior Neumann boundary condition on Γ and the exterior condition on Σ are used only for simplicity. The results presented herein apply to all types of admissible boundary conditions. In addition, as it is well-known, one should use *higher order* local absorbing boundary conditions for solving practical problems.

2 The continuous hybrid variational formulation

2.1 Nomenclature and properties

We use throughout this paper the following notations and properties.

- K is a *rectangular*-shaped element of Ω and ∂K is its boundary. $\partial K = \bigcup_{j=1}^{4} T_K^j$ where T_K^j is the jth edge of K with vertices $(\mathbf{s}_j^K, \mathbf{s}_{j+1}^K)$, and $\mathbf{n}_j^K$ its outward unitary normal vector.
- h_j^K is the length of the edge T_K^j and $h_K = \max_{1\leq j\leq 4} h_j^K$.
- $(\mathcal{T}_h)_h$ is a *regular* triangulation of the computational domain $\overline{\Omega}$ into elements K i.e.

$$\exists\, \hat{c} > 0 \;/\; \forall\, h,\; \forall\, K \in \mathcal{T}_h \;;\; h_K^2 \leq \hat{c}|K|$$

  where $|K|$denotes the area of the element K [15]. Note that $(\mathcal{T}_h)_h$ is a quasi-uniform triangulation since its elements K are rectangles.
- $h = \max_{K\in\mathcal{T}_h} h_K$. We also assume that $kh \leq \pi$. This condition means that there is at least two elements per wavelength.
- X is the space of the *primal* variable. X is given by:

$$X = \left\{v \in L^2(\Omega);\; \forall\, K \in \mathcal{T}_h, v_K = v_{|_K} \in H^1(K)\right\} \approx \prod_{K\in\mathcal{T}_h} H^1(K)$$

  and is equipped with the following norm:

$$||v||_X = \left(\sum_{K\in\mathcal{T}_h} ||v_K||^2_{X(K)}\right)^{\frac{1}{2}} \quad , \quad \forall\, v \in X,$$

  where

$$||v_K||_{X(K)} = \left(|v_K|^2_{1,K} + \frac{1}{|K|}||v_K||^2_{0,K}\right)^{\frac{1}{2}}$$

  $||\cdot||_{0,K}$ (resp. $|\cdot|_{1,K}$) is the L^2-norm (resp. semi-nom) on the element K.

- $|\cdot|_{1,\mathcal{T}_h}$ is the semi-norm in the space X defined by:

$$|v|_{1,\mathcal{T}_h} = \left(\sum_{K\in\mathcal{T}_h} |v_K|^2_{1,K} \right)^{\frac{1}{2}} \quad , \quad \forall\, v \in X.$$

- $H^{\frac{1}{2}}(\partial K)$ is the space of the traces of elements of $H^1(K)$ and $H^{-\frac{1}{2}}(\partial K)$ is the dual space of $H^{\frac{1}{2}}(\partial K)$. $H^{\frac{1}{2}}(\partial K)$ is equipped with the following norm:

$$||\lambda||_{\frac{1}{2},\partial K} \;=\; \inf_{w\in W(\lambda)} ||w||_{X(K)} \;=\; ||\Lambda||_{X(K)} \tag{2}$$

  where $W(\lambda) \;=\; \left\{ w \in H^1(K) \;;\; w_{|\partial K} = \lambda \right\}$ and Λ is the *unique* element in $W(\lambda)$ satisfying

$$-\Delta\Lambda + \frac{1}{|K|}\Lambda = 0 \quad \text{a.e. in } K.$$

  It follows from the definition of the norm $||\cdot||_X$ and Eq. (2) that

$$||v||_{\frac{1}{2},\partial K} \;\leq\; ||v||_{X(K)} \;, \quad \forall\, v \in H^1(K). \tag{3}$$

- M is the space of the *dual* variable defined by:

$$M \;=\left\{ \mu \in \prod_{K\in\mathcal{T}_h} H^{-\frac{1}{2}}(\partial K) \;; \quad \forall \lambda \in T,\; \sum_{K\in\mathcal{T}_h} < \mu^K \;,\; \lambda^K >_{-\frac{1}{2}\times\frac{1}{2},\partial K} = 0 \right\}$$

  where $\mu^K = \mu_{|\partial K}$ and the space T is given by:

$$T \;=\; \left\{ \lambda \in \prod_{K\in\mathcal{T}_h} H^{\frac{1}{2}}(\partial K);\; \forall\, K \neq K' \in \mathcal{T}_h,\; \lambda^K = \lambda^{K'} \;\text{ on }\; \partial K \cap \partial K' \right\}$$

  The space M is equipped with the following norm:

$$||\mu||_M = \left(\sum_{K\in\mathcal{T}_h} ||\mu^K||^2_{-\frac{1}{2},\partial K} \right)^{\frac{1}{2}} \quad , \quad \forall \mu \in M$$

  where

$$||\mu^K||_{-\frac{1}{2},\partial K} = \sup_{\lambda\in H^{\frac{1}{2}}(\partial K)} \frac{|< \mu^K \;,\; \lambda >_{-\frac{1}{2}\times\frac{1}{2},\partial K}|}{||\lambda||_{\frac{1}{2},\partial K}} = \sup_{v\in H^1(K)} \frac{|< \mu^K \;,\; v >_{-\frac{1}{2}\times\frac{1}{2},\partial K}|}{||v||_{X(K)}} \tag{4}$$

  and $< . \;,\; . >_{-\frac{1}{2}\times\frac{1}{2},\partial K}$ is the duality product between $H^{-\frac{1}{2}}(\partial K)$ and $H^{\frac{1}{2}}(\partial K)$ [16].

- $\mathcal{M}$ is a subspace of M defined by:

$$\mathcal{M} \;=\; \left\{ \mu \in \prod_{K\in\mathcal{T}_h} L^2(\partial K);\; \mu = 0 \text{ on } \partial\Omega \quad \text{and} \quad \forall\, K \neq K' \in \mathcal{T}_h, \right.$$
$$\left. \mu^K + \mu^{K'} = 0 \;\text{ on }\; \partial K \cap \partial K' \right\}.$$

  Therefore, we have

$$\mathcal{M} = M \cap \prod_{K\in\mathcal{T}_h} L^2(\partial K).$$

2.2 Formulation and mathematical results

We adopt the following hybrid-type variational formulation (VP) for solving the boundary value problem (BVP). Note that the VP is equivalent to BVP as indicated in Remark 1.

$$
\text{(VP)} \quad \begin{cases} \text{Find } (u,\ \lambda) \in X \times M \text{ such that} \\ a(u\ ,\ v) + b(v, \lambda)\ =\ F(v) \qquad \forall\, v \in X, \\ b(u, \mu)\ =\ 0 \qquad \forall\, \mu \in M, \end{cases} \tag{5}
$$

where the *bilinear* forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ and the function F are given by:

$$
a(u\ ,\ v) = \sum_{K\in\mathcal{T}_h} \left(\int_K \nabla u \cdot \nabla \overline{v}\, dx - k^2 \int_K u\overline{v}\, dx - \mathrm{i}k \int_{\partial K \cap \Sigma} u\overline{v}\, dt \right) \quad \forall\, u\, ,\, v \in X,
$$

$$
b(v, \mu) = \sum_{K\in\mathcal{T}_h} < \mu^K\ ,\ \overline{v} >_{-\frac{1}{2}\times\frac{1}{2},\partial K} \qquad \forall\, (v, \mu) \in X \times M,
$$

$$
F(v) = -\sum_{K\in\mathcal{T}_h} \int_{\partial K \cap \Gamma} \overline{v} \partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}\, dt \quad \forall\, v \in X.
$$

Note that the bilinear form $b(\cdot\ ,\ \cdot)$ also satisfies

$$
b(v, \mu) = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu^K \overline{v}\, dt \quad \forall\, (v, \mu) \in X \times \mathcal{M}.
$$

In addition, the bilinear forms $a(\cdot\ ,\ \cdot)$ and $b(\cdot\ ,\ \cdot)$ satisfy the following important properties.

Property 1 *The bilinear forms $a(.\ ,\ .)$ and $b(\cdot\ ,\ \cdot)$ are continuous on $X \times X$ and $X \times M$ respectively. Furthermore, we have:*

i. $a(.\ ,\ .)$ satisfies the Gärding inequality in $H^1(\Omega)$

$$
\Re a(v\ ,\ v) + k^2 ||v||^2_{0,\Omega} = |v|^2_{1,\mathcal{T}_h}\ ; \quad \forall\, v \in X \tag{6}
$$

where $\Re$ designates the real part.

ii. The null space $\mathcal{N}$ corresponding to the bilinear form $b(.\ ,\ .)$ is given

$$
\mathcal{N} = \{ v \in X\ ; \quad b(v, \mu) = 0 \quad \forall\, \mu \in M \} = H^1(\Omega) \tag{7}
$$

iii. The bilinear form $b(.\ ,\ .)$ satisfies the so-called inf-sup *condition [22]:*

$$
\forall \mu \in M\ , \quad \exists \phi \in X\ :\ \sup_{v\in X} \frac{|b(v, \mu)|}{||v||_X} = \frac{|b(\phi, \mu)|}{||\phi||_X} = ||\mu||_M \tag{8}
$$

Proof of Property 1. We prove only the third point since the proof of Eq. (6) and Eq. (7) is straightforward. From the continuity of the bilinear form $b(.\ ,\ .)$ we deduce that

$$\sup_{v\in X}\frac{|b(v,\mu)|}{||v||_X}\leq ||\mu||_M \quad \forall\, \mu\in M. \tag{9}$$

Next, for a fixed $\mu\in M$, we consider the function $\phi\in X$ such that , for every $K\in\mathcal{T}_h$, $\phi_{|K}=\phi_K$ is the *unique* solution of the following variational problem:

$$\int_K \nabla\phi_K\cdot\nabla\overline{v}\,dx+\frac{1}{|K|}\int_K\phi_K\overline{v}\,dx \ = <\mu^K\ ,\ \overline{v}>_{-\frac{1}{2}\times\frac{1}{2},\partial K}\quad \forall\, v\in H^1(K). \tag{10}$$

Hence, using Eq. (3) and Eq. (10), we have

$$||\phi_K||^2_{X(K)} \ = <\mu^K\ ,\ \overline{\phi_K}>_{-\frac{1}{2}\times\frac{1}{2},\partial K}\leq\ ||\mu^K||_{-\frac{1}{2},\partial K}||\phi_K||_{\frac{1}{2},\partial K}\leq\ ||\mu^K||_{-\frac{1}{2},\partial K}||\phi_K||_{X(K)}.$$

Thus, we deduce that $||\phi_K||_{X(K)}\ \leq\ ||\mu^K||_{-\frac{1}{2},\partial K}$, and then $||\phi||_X\ \leq\ ||\mu||_M$.

Moreover, from Eq. (4) and Eq. (10), we have $||\mu^K||_{-\frac{1}{2},\partial K}\ \leq\ ||\phi_K||_{X(K)}$.

Therefore, it follows that $||\phi||_X\ =\ ||\mu||_M$.

On the other hand, from Eq. (10) and the definition of the bilinear form $b(\cdot\ ,\ \cdot)$, we also have

$$b(\phi,\mu)\ =\ \sum_{K\in\mathcal{T}_h}||\phi_K||^2_{X(K)}=||\phi||^2_X=\ ||\phi||_X||\mu||_M$$

which concludes the proof of the *inf-sup* condition given by Eq. (8). □

Remark 1 *The problems BVP and VP are equivalent in the following sense:*

i. If the pair (u,λ) is a solution of VP, then it follows from the second equation of VP that u is in $H^1(\Omega)$. Moreover, using the first equation of VP with test functions $v\in\mathcal{D}(\Omega)$, we deduce that u is the solution of the first equation of BVP. Last, the use of test functions $v\in H^1(\Omega)$ allows to verify that u is satisfies the boundary conditions on Γ and Σ.

ii. If u is the solution of BVP, then from the standard regularity results for Laplace's operator [23] and due to the possible reentrant corners (with a measure angle of $\frac{3\pi}{2}$), it follows that $u\in H^{\frac{5}{3}}(\Omega)$. Thus, $\partial_n u^K\in L^2(\partial K)$ for all $K\in\mathcal{T}_h$ ($\partial_n u^K$ is even in $H^{\frac{1}{6}}(\partial K)$) . Then we set

$$\lambda^K=\begin{cases}-\partial_n u & \text{on}\ \ \partial K\setminus\partial\Omega,\\ \\ 0 & \text{on}\ \ \partial K\cap\partial\Omega.\end{cases} \tag{11}$$

Therefore, the dual variable λ satisfies (11) in the $L^2(\partial K)$ sense, which is the classical *sense. Having that in mind, one can multiply BPV by test functions $v\in X$ and deduce that the pair (u,λ) satisfies VP.*

Next, we prove that the variational problem (VP) is well-posed in the sense of Hadamard [13] . This is main result of this section. It is stated in the following theorem.

Theorem 1 *The variational problem (VP) admits a unique solution $(u, \lambda) \in X \times M$. In addition, u belongs to $H^{\frac{5}{3}}(\Omega)$, and for all $\theta \in [0, \frac{5}{3}]$ there is a positive constant C (C depends on Ω and θ only) such that*

$$|u|_{\theta,\Omega} \leq C\,(1+k)^{\theta}.$$

The proof of this theorem is based on the following intermediate stability result:

Lemma 1 *Let f be in $L^2(\Omega)$. Then, the following boundary value problem*

$$\begin{cases} \Delta U + k^2 U &= f & \text{in } \Omega, \\ \partial_n U &= 0 & \text{on } \Gamma, \\ \partial_n U &= \mathrm{i}k\,U & \text{on } \Sigma \end{cases} \tag{12}$$

has one and only one solution U in $H^{\frac{5}{3}}(\Omega)$. Moreover, for all $\theta \in [0, \frac{5}{3}]$ there is a positive constant C (C depends on Ω and θ only) such that

$$|U|_{\theta,\Omega} \leq C\,(1+k)^{\theta-1}\,||f||_{0,\Omega}. \tag{13}$$

Proof of Lemma 1. First , observe that the variational formulation corresponding to the boundary value problem (12) is given by

$$\begin{cases} \text{Find } U \in H^1(\Omega) \text{ such that} \\ a(U\ ,\ v) \ = -\displaystyle\int_\Omega f\,\overline{v}\,dx \qquad \forall\, v \in H^1(\Omega). \end{cases} \tag{14}$$

From Eq. (6), it follows that the bilinear form $a(.\ ,\ .)$ satisfies the Fredholm alternative on $H^1(\Omega)$. Hence, the uniqueness ensures the existence of the solution U in $H^1(\Omega)$.
Therefore, we need only to prove the uniqueness of the solution of the boundary value problem (12). Let w be the solution of the corresponding homogeneous boundary value problem. The function w satisfies

$$a(w, w) = 0 \quad \text{then} \quad w = 0 \quad \text{on} \quad \Sigma$$

and we deduce that

$$\partial_n w = 0 \quad \text{on} \quad \Gamma \qquad \text{and} \qquad w = \partial_n w = 0 \quad \text{on} \quad \Sigma.$$

Therefore, using the continuation theorem [17, 18], we obtain that $w = 0$ in Ω.

From the standard regularity results for second-order elliptic boundary value problems [23] and due to the possible reentrant corners (with a measure angle of $\frac{3\pi}{2}$), it follows that the solution of problem (12) satisfies $U \in H^{\frac{5}{3}}(\Omega)$, and there is a positive constant C (C depends on Ω only) such that:

$$||U||_{\frac{5}{3},\Omega} \;\leq\; C \left(||\Delta U||_{-\frac{1}{3},\Omega} + ||\partial_n U||_{\frac{1}{6},\partial\Omega} \right). \tag{15}$$

Moreover, using the results established in References [19] and [20], we deduce the existence of a positive constant C (C depends on Ω only) such that:

$$||U||_{0,\Omega} \leq \frac{C}{1+k}\, ||f||_{0,\Omega} \quad \text{and} \quad |U|_{1,\Omega} \leq C\, ||f||_{0,\Omega}. \tag{16}$$

Next, we establish the estimate (13). To do this, we will use the space interpolation results in Reference [21]. First, using boundary conditions in the boundary value problem (12), we deduce that there is a positive constant C (C depends on Ω only) such that:

$$||\partial_n U||_{\frac{1}{6},\partial\Omega} = ||\partial_n U||_{\frac{1}{6},\Sigma} = k\, ||U||_{\frac{1}{6},\Sigma} \leq C\, k\, ||U||_{\frac{2}{3},\Omega}.$$

Therefore, it follows from the space interpolation results in [21] that there is a positive constant C (C depends on Ω only) such that:

$$||\partial_n U||_{\frac{1}{6},\partial\Omega} \;\leq\; C\, k\, ||U||_{0,\Omega}^{\frac{1}{3}}\, |U|_{1,\Omega}^{\frac{2}{3}}.$$

Finally, it follows from Eq. (16) that there exists a positive constant C (C depends on Ω only) such that:

$$||\partial_n U||_{\frac{1}{6},\partial\Omega} \;\leq\; C\,(1+k)^{\frac{2}{3}}\, ||f||_{0,\Omega} \tag{17}$$

Furthermore, from the first equation of the boundary value problem (12), we deduce that

$$||\Delta U||_{0,\Omega} \;\leq\; k^2\, ||U||_{0,\Omega} + ||f||_{0,\Omega}.$$

Hence, it follows from Eq. (16) that there is a positive C (C depends on Ω only) such that

$$||\Delta U||_{0,\Omega} \;\leq\; C\,(1+k)\, ||f||_{0,\Omega}$$

In addition, from the norms properties and Eq. (16), there is a positive C (C depends on Ω only) such that

$$||\Delta U||_{-1,\Omega} \;\leq\; |U|_{1,\Omega} \;\leq\; ||U||_{1,\Omega} \;\leq\; C\, ||f||_{0,\Omega}.$$

Consequently, it follows from these equations and the interpolation space results theorem (see [21]) that there is a positive constant C (C depends on the domain Ω only) such that

$$||\Delta U||_{-\frac{1}{3},\Omega} \;\leq\; C\,(1+k)^{\frac{2}{3}}\, ||f||_{0,\Omega} \tag{18}$$

Estimate (13) is then a direct consequence of Eq. (15), Eq. (17), and Eq. (18). □

Proof of Theorem 1. Since $H^1(\Omega)$ is the null space of the bilinear form $b(.\,,\,.)$ (see Eq. (7)), the VP is reduced to the variational problem

$$a(u\,,\,v) \;=\; F(v) \quad \forall\, v \in H^1(\Omega)$$

From Eq. (6), it follows that the bilinear form $a(.\,,\,.)$ satisfies the Fredholm alternative on $H^1(\Omega)$. Hence, the uniqueness ensures the existence of the solution u in $H^1(\Omega)$. On the other hand, the uniqueness results readily from the solution of the boundary value problem (12). Therefore, the

solution u of the reduced variational problem in the null space $H^1(\Omega)$ of the bilinear form $b(.\,,\,.)$ exists and is unique. Therefore, both existence and uniqueness of the solution of the complete variational problem VP are standard consequences (For example, see Reference [22]) of the inf-sup condition given by Eq. (8).
To prove the stability estimates, we first observe that the pair (u, λ) solution of the variational formulation (VP) satisfies the following *mixed* boundary value problem:

$$\begin{cases} \Delta u + k^2 u &= 0 \qquad \text{in } \Omega, \\ \partial_n u &= -\partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}} \qquad \text{on } \Gamma, \\ \partial_n u &= \mathrm{i}ku \qquad \text{on } \Sigma. \end{cases}$$

and $\forall\, K \in \mathcal{T}_h$, we have

$$\lambda^K = \begin{cases} -\partial_n u & \text{on } \partial K \setminus \partial\Omega, \\ 0 & \text{on } \partial K \cap \partial\Omega. \end{cases}$$

Consequently, if we set

$$U \;=\; u + e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}\,\phi \tag{19}$$

where $\phi \in \mathcal{D}(\overline{\Omega})$ satisfies

$$\phi = 1 \text{ on } \Gamma, \qquad \partial_n\phi = 0 \text{ on } \Gamma, \qquad \phi \;=\; \partial_n\phi = 0 \text{ on } \Sigma,$$

then, it is easy to verify that U is the unique solution of boundary value problem (12) with the right hand-side f given by

$$f \;=\; (2\mathrm{i}k\,\mathbf{d}\cdot\nabla\phi + \Delta\phi)\, e^{ik\mathbf{x}\cdot\mathbf{d}}$$

and there is a positive constant C (C depends on Ω only) such that

$$||f||_{0,\Omega} \,\leq C\,(1+k).$$

Therefore, the proof of Theorem 1's estimate is an immediate consequence of estimate (13) in Lemma 1 which concludes the proof of Theorem 1.□

3 The discrete formulation

3.1 Assumptions, notations, and properties

We adopt throughout this section the following notations and properties.

- $\forall\, K \in \mathcal{T}_h,\ \phi_j^K \;=\; e^{\mathrm{i}k\,\mathbf{n}_j^K\cdot(\mathbf{x}-\mathbf{s}_j^K)};\quad 1 \leq j \leq 4.$
- X_h is the discrete space for the *primal* variable. X_h is given by:

$$X_h \;=\; \left\{ v_h \in X;\ \forall\, K \in \mathcal{T}_h,\ v_{h|_K} \in X_h(K) \right\}$$

where

$$X_h(K) \;=\; \left\{ v_h^K \in H^1(K) \; ; \; v_h^K \;=\; \sum_{j=1}^{4} \alpha_j^K \phi_j^K \quad \text{where } \alpha_j^K \in \mathbb{C} \right\}$$

Note that $X_h \subseteq X$, and therefore X_h is also equipped with the norm $||.||_X$.

- M_h is the discrete space of the *dual* variable. M_h is defined as follows:

$$M_h = \left\{ \mu_h \in \mathcal{M} \; ; \; \forall\, K \in \mathcal{T}_h \;\text{ and }\; \forall\, T_j^K \subset \partial K : \; \mu_j^K = \mu_{|_{T_j^K}} \in \mathbb{C}, \;\; 1 \leq j \leq 4 \right\}$$

- For every $K \in \mathcal{T}_h$, The matrix $B^K = \left(B_{lj}^K \right)_{1 \leq l,j \leq 4}$ represents the elementary matrix corresponding to the bilinear form $b(\cdot\;,\;\cdot)$. Hence, the entries of the matrix B^K are given by:

$$B_{lj}^K \;=\; \frac{1}{h_l^K} \int_{T_l^K} \phi_j^K \, dt, \quad 1 \leq \; l, j \; \leq 4. \tag{20}$$

- $\hat{C}$ designates a generic positive constant. $\hat{C}$ is independent of k, Ω, and the triangulation $\mathcal{T}_h$.

- For a given $K \in \mathcal{T}_h$ and $\forall\, v^K \in H^1(K)$, we have the following two classical inequalities [15]:

$$||v^K||_{0,\partial K} \;\leq\; \hat{C}(\frac{1}{h_K}||v^K||_{0,K}^2 + h_K|v^K|_{1,K}^2)^{\frac{1}{2}}, \tag{21}$$

$$||v^K - \frac{1}{|K|}\int_K v^K dx||_{0,K} \;\leq\; \hat{C} h_K |v^K|_{1,K}. \tag{22}$$

In addition, it follows from combining Eq. (21) (when applied to $v^K - \frac{1}{|K|}\int_K v^K dx$) and Eq. (22) that:

$$||v^K - \frac{1}{|K|}\int_K v^K dx||_{0,\partial K} \;\leq\; \hat{C} h_K^{\frac{1}{2}} |v^K|_{1,K}. \tag{23}$$

3.2 Discrete formulation and announcement of the main results

The discrete variational problem (DVP) corresponding to the variational formulation (VP) can be formulated as follows:

$$(\text{DVP}) \quad \left\{ \begin{array}{lll} \text{Find } (u_h, \lambda_h) \in X_h \times M_h \text{ such that} & & \\ a(u_h, v_h) + b(v_h, \lambda_h) \;=\; F(v_h) & & \forall\, v_h \in X_h, \\ b(u_h, \mu_h) \;=\; 0 & & \forall\, \mu_h \in M_h. \end{array} \right. \tag{24}$$

The next two theorems summarize the main results of this section.

Theorem 2 *The discrete variational problem (DVP) admits a unique solution* $(u_h, \lambda_h) \in X_h \times M_h$*. Moreover, for* $h_0 > 0$ *such that* $k\,(1+k)^{\frac{2}{3}}\,h_0^{\frac{2}{3}}$ *is "sufficiently small" and* $kh_0 \leqslant \pi$*, there is a positive constant* C *(*C *depends on* Ω *only) such that for all* $h \leq h_0$*, we have*

$$\begin{aligned} ||u-u_h||_{0,\Omega} \leq & \quad C(1+k)^{\frac{7}{3}} h^{\frac{4}{3}} \\ |u-u_h|_{1,\mathcal{T}_h} + ||\lambda - \lambda_h||_M \leq & \quad C(1+k)^{\frac{5}{3}} h^{\frac{2}{3}}. \end{aligned} \tag{25}$$

where (u, λ) *is the solution of the continuous variational problem VP(5).*

Theorem 3 *Let* u *be the solution of the continuous variational problem VP(5) and* u_h *be the solution of the discrete variational problem (DVP). We assume that* $kh \leqslant \pi$*, then there exists a constant* $C > 0$ *(*C *depends on* Ω *only) such that*

$$\begin{aligned} ||u-u_h||_{0,\Omega} \leq \hat{C} \quad & \Bigg((\sum_{e\subset\Sigma} h_e ||\partial_n u_h - \mathrm{i}k u_h||^2_{0,e})^{\frac{1}{2}} \\ & + (\sum_{e\subset\Gamma} h_e ||\partial_n u_h + \partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}||^2_{0,e})^{\frac{1}{2}} + (\sum_{e \text{ interior}} h_e^{-1} ||[u_h]||^2_{0,e})^{\frac{1}{2}} \Bigg) \end{aligned} \tag{26}$$

where e *is an edge of* $\mathcal{T}_h$*,* $[u_h]$ *is the jump of* u_h *across the edge* e *and* h_e *is the lenght of* e*.*

Remark 2 *We must point out that it has been reported in [11, 12] that for high frequency regime, the use of* P_1 *finite element method leads to the following estimates:* $|u - u_h|_{1,\Omega} \;\leq\; C\,k^2\,h$ *and* $||u - u_h||_{0,\Omega} \;\leq\; C\,k^3\,h^2$ *when* k^2h *is small enough. These estimates were derived assuming that* $u \in H^2(\Omega)$ *which is not however valid for most problems.*
The a posteriori estimate given by Eq. (26) is a practical tool for a mesh adaptive strategy. This estimate reveals that the L^2 *error depends on how well the jump of the primal variable as well as the interior and exterior boundary conditions are approximated at the element level.*

In order to prove Theorem 2 and Theorem 3, we need first to establish intermediate interpolation results. This is accomplished in Section 3.3. Then, we prove in Section 3.4.1 the existence and the uniqueness of the solution of the discrete variational problem. This result is established as a direct consequence of Proposition 1 and Proposition 2. Section 3.4.2 is devoted to the proof of (25) and (26). The error estimate given by Eq. (25) is established in four steps, each step is formulated as a lemma (see Lemma 7 to Lemma 10). The a posteriori error estimate given by Eq. (26) is established at the end of Section 3.4.2.

The next result, that can be easily established, shows why the existence and the uniqueness of the solution of (DVP) is not a direct consequence of the existence and the uniqueness of the solution of (VP).

Lemma 2 *The null space* $\mathcal{N}_h$ *corresponding to the bilinear form* $b(\cdot\;,\;\cdot)$ *defined by*

$$\mathcal{N}_h = \{v_h \in X_h : \; b(v_h, \mu_h) = 0 \; ; \quad \forall\, \mu_h \in M_h\}$$

satisfies

$$\mathcal{N}_h = \left\{ v_h \in X_h \; ; \quad \int_{\partial K \cap \partial K'} v_h^K \, dt \;=\; \int_{\partial K \cap \partial K'} v_h^{K'} \, dt \;, \quad \forall\, K \neq K' \in \mathcal{T}_h \right\} \tag{27}$$

Remark 3 *Lemma 2 states that $\mathcal{N}_h$ is not a subspace of $\mathcal{N} = H^1(\Omega)$ which is the null space of the bilinear form $b(. \, , \, .)$. Indeed, the trace of an element of $\mathcal{N}_h$ on an edge of an element K is* weakly *continuous in the sense given by (27), while the trace of an element of $\mathcal{N}$ on an edge of an element K is "continuous" almost everywhere. Therefore, the inf-sup condition given by Eq. (8) and then Theorem 1 are no longer valid if we simply replace X and M by X_h and M_h respectively.*

3.3 Mathematical analysis of the interpolation operators

We establish in this section intermediate interpolation results that summarize the main properties of the *projection operator* Π_h from X onto X_h and the *projection operator* P_h from $\mathcal{M}$ onto M_h. These results are obtained in the case of a *rectangular*-shaped partition of the computational domain Ω.

3.3.1 Interpolation operator in X_h

Lemma 3 *For a fixed $K \in \mathcal{T}_h$, we have the following two properties:*

i. The normal derivative $\partial_n \phi_j^K$ is constant on every edge T_l^K $(1 \leq l, j \leq 4)$.

ii. If $kh_K \leq \pi$ then the matrix B^K is invertible and there is a positive constant $\hat{C}$ such that

$$||(B^K)^{-1}||_2 \leq \frac{\hat{C}}{k^2 h_K^2}. \tag{28}$$

Proof of Lemma 3. It follows from the definition of ϕ_j^K (See Section 3.1) that

$$\partial_n \phi_j^K \ = \ \mathrm{i}k \, \mathbf{n}_j^K \cdot \mathbf{n}_l^K \, \phi_j^K \quad \text{on} \quad T_l^K \quad (1 \leq l, j \leq 4).$$

Therefore, since K is a *rectangular*-shaped element, a simple calculation shows that

$$\partial_n \phi_j^K \ = \ \mathrm{i}k \text{ on } T_j^K, \quad \partial_n \phi_j^K \ = \ -\mathrm{i}k \text{ on } T_{j+2}^K \quad \text{and} \quad \partial_n \phi_j^K \ = \ 0 \text{ on } T_{j+1}^K \cup T_{j+3}^K.$$

In addition, it follows from the definition of the elementary matrix B^K (See Eq. (20)) that

$$B^K \ = \ \begin{bmatrix} 1 & b_1 & a_2 & b_1 \\ b_2 & 1 & b_2 & a_1 \\ a_2 & b_1 & 1 & b_1 \\ b_2 & a_1 & b_2 & 1 \end{bmatrix}$$

where $a_j = e^{-\mathrm{i}kh_j^K}$ and $b_j = \dfrac{1 - e^{-\mathrm{i}kh_j^K}}{\mathrm{i}kh_j^K}$, $\quad 1 \leq j \leq 4$.

We set $\Delta = (1 + a_1)(1 + a_2) - 4b_1b_2$. Then, it is easy to verify that $\Delta \neq 0$ for $kh_K \leq \pi$ (which is in fact a sufficient but not necessary condition). This ensures that the matrix B^K is invertible, and we have

$$[B^K]^{-1} = \frac{1}{2} \begin{bmatrix} \frac{1+a_1}{\Delta} + \frac{1}{1-a_2} & -2\frac{b_1}{\Delta} & \frac{1+a_1}{\Delta} - \frac{1}{1-a_2} & -2\frac{b_1}{\Delta} \\ -2\frac{b_2}{\Delta} & \frac{1+a_2}{\Delta} + \frac{1}{1-a_1} & -2\frac{b_2}{\Delta} & \frac{1+a_2}{\Delta} - \frac{1}{1-a_1} \\ \frac{1+a_1}{\Delta} - \frac{1}{1-a_2} & -2\frac{b_1}{\Delta} & \frac{1+a_1}{\Delta} + \frac{1}{1-a_2} & -2\frac{b_1}{\Delta} \\ -2\frac{b_2}{\Delta} & \frac{1+a_2}{\Delta} - \frac{1}{1-a_1} & -2\frac{b_2}{\Delta} & \frac{1+a_2}{\Delta}\ \frac{1}{1-a_1} \end{bmatrix}$$

Finally, one can verify that there is a positive constant $\hat{C}$ and k such that

$$|||[B^K]^{-1}||_2 \leq \frac{\hat{C}}{k^2 h_K^2}. \qquad \square$$

Next, we introduce the sequence of linear operators $(\pi_K)_{K\in\mathcal{T}_h}$ defined as follows:

$$\left| \begin{array}{lll} \pi_K : & H^1(K) & \longrightarrow \mathbb{C}^4 \\ & v^K & \longmapsto \pi_K v^K \end{array} \right.$$

where

$$(\pi_K v^K)_j = \frac{1}{h_j^K} \int_{T_j^K} v^K \, dt \ , \quad 1 \leq j \leq 4. \tag{29}$$

Then, it follows from Eq.(21) that, for any h_K independent vectorial norm $|||.|||$ in $\mathbb{C}^4$, there is a positive constant $\hat{C}$ such that

$$|||\pi_K v^K||| \leq \hat{C} ||v^K||_{X(K)} \ , \quad \forall v^K \in H^1(K). \tag{30}$$

In addition, we have

$$\forall v_h^K \in X_h(K) \ , \quad v_h^K = \sum_{j=1}^{4} \alpha_j^K \phi_j^K \quad \text{where} \quad \alpha_j^K = ([B^K]^{-1} \pi_K v_h^K)_j \ , \quad 1 \leq j \leq 4. \tag{31}$$

The next results states that, for a given $K \in \mathcal{T}_h$, the set of degrees of freedom associated to the planar waves $(\phi_j^K)_{j=1}^4$ is *unisolvent* .

Lemma 4 *For a given $K \in \mathcal{T}_h$ and for any $v_h^K \in X_h(K)$, we have the following equivalence:*

$$\left(\int_{T_l^K} v_h^K \, dt \ = \ 0 \ , \quad 1 \leq l \leq 4 \right) \iff \left(v_h^K \ = \ 0 \quad \text{on} \quad K \right)$$

Proof of Lemma 4. Using Eq. (29) and Eq. (31), it follows that for a given $K \in \mathcal{T}_h$, we have

$$\int_{T_l^K} v_h^K \, dt \ = \ 0 \ , \quad 1 \leq l \leq 4 \iff \pi_K v_h^K \ = \ 0 \iff v_h^K \ = \ 0$$

which proves Lemma 4. $\Box$

Consequently, one can construct a sequence of *local* linear operator $(\Pi_K)_{K\in\mathcal{T}_h}$ as follows:

$$\begin{cases} \Pi_K : \quad H^1(K) & \longrightarrow X_h(K) \\ \qquad\qquad v^K & \longmapsto \Pi_K v^K \end{cases}$$

with

$$\int_{T_j^K} v^K\,dt \;=\; \int_{T_j^K} \Pi_K v^K\,dt\,, \quad 1\le j\le 4. \tag{32}$$

Next, we state three properties of the operator Π_K. These properties are immediate consequences of the definition of Π_K, the inequalities (21)-(22), property (32) of the operator Π_K, and the characterization of elements of $X_h(K)$ with the elementary matrix B^K (see Eq. (31)). Note that The second identity of (33) is obtained by Green's formula using the rectangular shape of K.

Property 2 *The operator Π_K satisfies the following three properties:*

i. $\forall\, K\in\mathcal{T}_h$ *and* $\forall\, v\in H^1(K)$*, we have*

$$\int_{\partial K}(v^K-\Pi_K v^K)\,dt \;=\; 0 \quad , \quad \int_K \nabla(v^K-\Pi_K v^K)\,dx \;=\; 0. \tag{33}$$

ii. There is a positive constant $\hat{C}$ such that

$$\forall\, K\in\mathcal{T}_h\;, \quad ||v^K-\Pi_K v^K||_{0,\partial K} \;\le\; \hat{C}h_K^{\frac{1}{2}}\,|v^K-\Pi_K v^K|_{1,K}\;, \quad \forall\, v^K\in H^1(K). \tag{34}$$

iii. For a given $v^K\in H^1(K)$, we have

$$\pi_K v^K \;=\; \pi_K \;o\; \Pi_K v^K \quad \text{and} \quad \Pi_K v^K \;=\; \sum_{j=1}^{4}\alpha_j^K\phi_j^K \quad \text{with} \quad \alpha_j^K = ([B^K]^{-1}\pi_K v^K)_j. \tag{35}$$

Proof of Property 2. We prove only the second property since the two others are immediate. Using Eq. (33) and the definition of the norm $||.||_{0,\partial K}$, we have

$$\begin{aligned} ||v^K-\Pi_K v^K||_{0,\partial K} &= ||v^K-\Pi_K v^K - \tfrac{1}{|\partial K|}\int_{\partial K}(v^K-\Pi_K v^K)\,dt||_{0,\partial K} \\ &\le \inf_{\beta\in\mathbb{C}} ||v^K-\Pi_K v^K-\beta||_{0,\partial K} \\ &\le ||v^K-\Pi_K v^K - \tfrac{1}{|K|}\int_K (v-\Pi_K v^K)\,dt||_{0,\partial K}. \end{aligned}$$

We then conclude using Eq. (23).

In the next two lemmas, we establish a priori estimates on the operator Π_K.

Lemma 5 *Assume $kh \leq \pi$. Then, there is a positive constant $\hat{C}$ such that $\forall\, K \in \mathcal{T}_h$ and $\forall\, v^K \in H^1(K)$, we have*

$$||v^K - \Pi_K v^K||_{0,K} \;\leq\; \hat{C}\, h_K\, |v^K - \Pi_K v^K|_{1,K} \tag{36}$$

$$k||\Pi_K v^K||_{0,K} + ||\Pi_K v^K||_{X(K)} \leq \hat{C}\, ||v^K||_{X(K)} \tag{37}$$

Proof of Lemma 5. We establish the estimate given by Eq. (36) using Aubin-Nitsche argument [24, 25, 26].
More specifically, consider the following auxiliary boundary value problem

$$\begin{cases} \text{Find } \varphi \in H_0^1(K) \text{ such that} \\ \\ -\Delta\varphi \;=\; v^K - \Pi_K v^K \quad \text{on} \quad K. \end{cases}$$

Since K is a *rectangular-shaped* element, then φ is in fact in $H^2(K) \bigcap H_0^1(K)$ and we have

$$|\varphi|_{2,K} \;=\; ||\Delta\varphi||_{0,K} \;=\; ||v^K - \Pi_K v^K||_{0,K}.$$

It follows that

$$||v^K - \Pi_K v^K||_{0,K}^2 = \int_K \nabla\left(v^K - \Pi_K v^K\right) \cdot \nabla\overline{\varphi}\, dx - \int_{\partial K} \left(v^K - \Pi_K v^K\right) \partial_n \overline{\varphi}\, dt$$

Using Eq. (33), we deduce that

$$\left|\int_K \nabla\left(v^K - \Pi_K v^K\right) \cdot \nabla\overline{\varphi}\, dx\right| \;=\; \left|\int_K \nabla\left(v^K - \Pi_K v^K\right) \cdot \left(\nabla\overline{\varphi} - \frac{1}{|K|}\int_K \nabla\overline{\varphi}\, dx\right) dx\right|.$$

Then,

$$\left|\int_K \nabla\left(v^K - \Pi_K v^K\right) \cdot \nabla\overline{\varphi}\, dx\right| \;\leq\; \left|v^K - \Pi_K v^K\right|_{1,K} ||\nabla\varphi - \frac{1}{|K|}\int_K \nabla\varphi\, dx||_{0,K}.$$

It follows from Eq. (22), that there is a positive constant $\hat{C}$ such that

$$\left|\int_K \nabla\left(v^K - \Pi_K v^K\right) \cdot \nabla\overline{\varphi}\, dx\right| \;\leq\; \hat{C} h_K \left|v^K - \Pi_K v^K\right|_{1,K} |\varphi|_{2,K}$$

Moreover, using Eq. (32) we obtain that

$$\left|\int_{\partial K} \left(v^K - \Pi_K v^K\right) \partial_n \overline{\varphi}\, dt\right| \;=\; \left|\int_{\partial K} \left(v^K - \Pi_K v^K\right) \left(\nabla\overline{\varphi} - \frac{1}{|K|}\int_K \nabla\overline{\varphi}\, dx\right) \cdot \mathbf{n}^K\, dt\right|$$

Hence, we have

$$\left|\int_{\partial K} \left(v^K - \Pi_K v^K\right) \partial_n \overline{\varphi}\, dt\right| \;\leq\; ||v^K - \Pi_K v^K||_{0,\partial K}\, ||\nabla\varphi - \frac{1}{|K|}\int_K \nabla\varphi\, dx||_{0,\partial K}$$

Finally, using the inequality (23) and Eq. (34), it follows that there is positive constant $\hat{C}$ such that

$$\left|\int_{\partial K} \left(v^K - \Pi_K v^K\right) \partial_n \overline{\varphi}\, dt\right| \;\leq\; \hat{C} h_K |v^K - \Pi_K v^K|_{1,K}\, |\varphi|_{2,K}$$

Therefore, Eq. (36) results from :

$$||v^K - \Pi_K v^K||^2_{0,K} \leq \hat{C} h_K |v^K - \Pi_K v^K|_{1,K} \, |\varphi|_{2,K} = \hat{C} h_K |v^K - \Pi_K v^K|_{1,K} \, ||v^K - \Pi_K v^K||_{0,K}.$$

Next, we establish the estimate given by Eq. (37). To do this, we first note that it follows from Eq. (35) that

$$\forall \, v^K \in H^1(K) \; , \quad |||\Pi_K v^K||| \, \leq \, \sum_{j=1}^{4} |\alpha_j^K| \, |||\phi_j^K|||,$$

where $|||.|||$ is any norm in $X_h(K)$. Hence, using Eq. (31), Eq. (30), and Eq. (28), there is a positive constant $\hat{C}$ such that

$$\forall \, v^K \in H^1(K) \; , \quad |||\Pi_K v^K||| \leq \frac{\hat{C}}{k^2 h_K^2} ||v^K||_{X(K)} \max_{1 \leq j \leq 4} |||\phi_j^K|||.$$

On the other hand, it is easy to verify that

$$||\phi_j^K||_{0,K} \, \leq \, h_K \quad \text{and} \quad |\phi_j^K|_{1,K} \, \leq \, k h_K.$$

Consequently, there is a positive constant $\hat{C}$ such that

$$||\Pi_K v^K||_{0,K} \leq \frac{\hat{C}}{k^2 h_K} ||v^K||_{X(K)} \qquad \text{and} \qquad |\Pi_K v^K|_{1,K} \leq \frac{\hat{C}}{k h_K} ||v^K||_{X(K)}.$$

Furtheremore, using Eq. (36), we deduce that

$$||v^K - \Pi_K v^K||_{0,K} \; \leq \; \hat{C}(h_K \, |v^K|_{1,K} + h_K \, |\Pi_K v^K|_{1,K}) \leq \hat{C}(h_K \, |v^K|_{1,K} + \frac{C}{k} ||v^K||_{X(K)})$$

Thus,

$$k||v^K - \Pi_K v^K||_{0,K} \leq \hat{C} \, \left(k h_K \, |v^K|_{1,K} + ||v^K||_{X(K)} \right)$$

and therefore, using the definition of the norm $|| \cdot ||_{X(K)}$, it follows that

$$k||\Pi_K v^K||_{0,K} \leq \hat{C} \, ||v^K||_{X(K)}$$

which concludes the proof of the first part of Eq. (37).

Finally, we establish the second part of the estimate given by Eq. (37). To do this, we observe that $\forall \, v^K \in H^1(K)$, we have

$$\begin{aligned} |v^K - \Pi_K v^K|^2_{1,K} &= \int_K \nabla(v^K - \Pi_K v^K).\nabla \overline{v}^K \, dx \quad + \int_K (v^K - \Pi_K v^K) \Delta \Pi_K \overline{v}^K \, dx, \\ &= \int_K \nabla(v^K - \Pi_K v^K).\nabla \overline{v}^K \, dx \quad - k^2 \int_K (v^K - \Pi_K v^K) \Pi_K \overline{v}^K \, dx, \\ &\leq \quad |v^K - \Pi_K v^K|_{1,K} |v^K|_{1,K} \quad + k^2 ||v^K - \Pi_K v^K||_{0,K} ||\Pi_K v^K||_{0,K}. \end{aligned}$$

Note that there is no boundary terms in the previous equalities because of Lemma 3 and Eq. (32). Using again Eq. (36), we deduce the existence of a positive constant $\hat{C}$ such that

$$|v^K - \Pi_K v^K|_{1,K} \leq |v^K|_{1,K} + \hat{C} k^2 h_K ||\Pi_K v^K||_{0,K}$$

Therefore, using the first part of Eq. (37), we deduce that

$$|v^K - \Pi_K v^K|_{1,K} \leq |v^K|_{1,K} + \hat{C}kh_K||v^K||_{X(K)}$$

Consequently, there is a positive constant $\hat{c}$ such that

$$|\Pi_K v^K|_{1,K} \leq 2|v^K|_{1,K} + \hat{C}kh_K||v^K||_{X(K)} \leq \hat{c}\,||v^K||_{X(K)}$$

Moreover, using Eq. (36), we deduce that there is a positive constant $\hat{C}$ such that

$$||\Pi_K v^K||_{0,K} \ \leq \ ||v^K||_{0,K} + \hat{C}h_K|v^K - \Pi_K v^K|_{1,K}$$

and thus,

$$||\Pi_K v^K||_{0,K} \leq \hat{C}h_K||v^K||_{X(K)}$$

which concludes the proof of Eq. (37) . $\square$

Lemma 6 *Assume $kh \leq \pi$. Then for every $s \in [0,1]$, there is a positive constant $\hat{C}$ such that for all $K \in \mathcal{T}_h$, we have*

$$|v_K - \Pi_K v_K|_{1,K} \leq \hat{C}_2(h_K^s|v_K|_{1+s,K} + k^2h_K||v_K||_{0,K} + k^2h_K^2|v_K|_{1,K}), \qquad \forall v_K \in H^{1+s}(K). \quad (38)$$

Proof of Lemma 6. First, let φ be in $P_1(K)$ where $P_1(K)$ is the space of the affine polynomial functions. Then, using first Eq. (33) and the fact that $\nabla\varphi$ is constant in each triangle, next that functions in X_h satisfy the homogeneous Helmholtz equation in each triangle, we can write:

$$\begin{aligned}
|\varphi - \Pi_K\varphi|_{1,K}^2 \ &= \int_K \nabla(\varphi - \Pi_K\varphi).\nabla(\overline{\varphi} - \Pi_K\overline{\varphi})\,d\mathbf{x} = -\int_K \nabla(\varphi - \Pi_K\varphi).\nabla\Pi_K\overline{\varphi}\,d\mathbf{x} \\
&= \int_K (\varphi - \Pi_K\varphi).\Delta\Pi_K\overline{\varphi}\,d\mathbf{x} - \int_{\partial K} (\varphi - \Pi_K\varphi).\partial_n\Pi_K\overline{\varphi}\,dt \\
&= \int_K (\varphi - \Pi_K\varphi).\Delta\Pi_K\overline{\varphi}\,d\mathbf{x} = -k^2\int_K (\varphi - \Pi_K\varphi).\Pi_K\overline{\varphi}\,d\mathbf{x} \\
&\leq k^2||\varphi - \Pi_K\varphi||_{0,K}||\Pi_K\varphi||_{0,K}.
\end{aligned}$$

From relation (36), we obtain

$$||\varphi - \Pi_K\varphi||_{0,K} \leq \hat{C}h_K|\varphi - \Pi_K\varphi|_{1,K}$$

Moreover, equation (37) gives

$$||\Pi_K\varphi||_{0,K} \leq \hat{C}(||\varphi||_{0,K} + h_K|\varphi|_{1,K}).$$

Hence,

$$|\varphi - \Pi_K\varphi|_{1,K} \leq \hat{C}k^2h_K(||\varphi||_{0,K} + h_K|\varphi|_{1,K}).$$

On the other hand, it follows from equation (37) that for $v_K \in H^1(K)$ and $\varphi \in P_1(K)$, we have

$$|\Pi_K(\varphi - v_K)|_{1,K} \leq \hat{C}(\frac{1}{h_K}||v_K - \varphi||_{0,K} + |v_K - \varphi|_{1,K})$$

and then

$$|v_K - \Pi_K v_K|_{1,K} \leq |v_K - \varphi|_{1,K} + |\varphi - \Pi_K \varphi|_{1,K} + |\Pi_K(\varphi - v_K)|_{1,K}$$

$$\leq \hat{C}(\frac{1}{h_K}||v_K - \varphi||_{0,K} + |v_K - \varphi|_{1,K} + k^2 h_K(||\varphi||_{0,K} + h_K|\varphi|_{1,K})).$$

Furthermore, since $kh_K \leq \pi$, we deduce that

$$|v_K - \Pi_K v_K|_{1,K} \leq \hat{C}(\frac{1}{h_K}||v_K - \varphi||_{0,K} + |v_K - \varphi|_{1,K} + k^2\, h_K||v_K||_{0,K} + k^2\, h_K^2|v_K|_{1,K}).$$

Since $v_K \in H^{1+s}(K)$ with $s \in [0,1]$, we chose φ to be the P_1-polynomial approximation (the Lagrange polynomial interpolation) of v on K if $s \neq 0$, and $\varphi = \frac{1}{|K|}\int_K v\,dx$ if $s = 0$. Therefore, it follows from the standard P_1 interpolation results on K (see [15]) that

$$|v_K - \Pi_K v_K|_{1,K} \leq \hat{C}(h_K^s|v_K|_{1+s,K} + k^2 h_K||v_K||_{0,K} + k^2 h_K^2|v_K|_{1,K}).\Box$$

Next, we introduce the *global* interpolation linear operator Π_h as follows

$$\begin{cases} \Pi_h \;:\; X \longrightarrow X_h \\ \qquad\quad v \longmapsto \Pi_h v \end{cases}$$

with

$$(\Pi_h v)_{|K} \;=\; \Pi_K(v_{|K}) \in X_h(K)\;, \quad \forall\, K \in \mathcal{T}_h.$$

Property 3 *The global interpolation operator $\Pi_h : X \longrightarrow X_h$ satisfies the following four properties:*

i. $\forall v \in H^{1+s}(\Omega)$ *with* $s \in [0,1]$, *we have*

$$||v - \Pi_h v||_{0,\Omega} \leq \hat{C}(h^{1+s}|v|_{1+s,\Omega} + k^2 h^3|v|_{1,\Omega} + k^2 h^2||v||_{0,\Omega}) \tag{39}$$

$$|v - \Pi_h v|_{1,\mathcal{T}_h} \leq \hat{C}(h^s|v|_{1+s,\Omega} + k^2 h^2|v|_{1,\Omega} + k^2 h||v||_{0,\Omega}) \tag{40}$$

ii. $\forall\, v \in H^1(\Omega), \quad \Pi_h v \in \mathcal{N}_h$ *where* $\mathcal{N}_h$ *is the null space of* $b(.\;,\;.)$.

iii. $\forall\, v \in X$ *and* $\forall\, v_h \in X_h$, *we have*

$$\begin{aligned} a(v - \Pi_h v, v_h) \;&=\; -\mathrm{i}k \sum_{K \in \mathcal{T}_h} \int_{\partial K \cap \Sigma} (v - \Pi_h v)\;\overline{v}_h\,dt \\ a(v_h, v - \Pi_h v) \;&=\; -\mathrm{i}k \sum_{K \in \mathcal{T}_h} \int_{\partial K \cap \Sigma} v_h\;(\overline{v} - \Pi_h \overline{v})\;dt \end{aligned} \tag{41}$$

iv. $\forall\, v \in X$ *and* $\forall\, \mu_h \in M_h$, *we have*

$$b(v, \mu_h) \;= b(\Pi_h v, \mu_h) \tag{42}$$

Note that Eqs. (39)–(40) are immediate consequences of Lemma 6, while the two equalities given by Eq. (41) are obtained by Green's formula and using the fact that the plane waves are solutions of the Helmholtz equation.

3.3.2 Interpolation operator in M_h

We introduce here the projection operator P_h for the dual variable λ. P_h is defined as follows:

$$\left\{ \begin{array}{lll} P_h : & \mathcal{M} & \longrightarrow M_h \\ & \mu & \longmapsto P_h\mu \end{array} \right.$$

where

$$\forall\, K \in \mathcal{T}_h \ , \quad P_h\mu_{|_{T_j^K}} \ = \ \frac{1}{h_j^K} \int_{T_j^K} \mu\, dt \ , \quad 1 \leq j \leq 4.$$

Then, the operator P_h satisfies

$$\forall\, K \in \mathcal{T}_h \ , \ \forall\, \mu \in \mathcal{M}, \quad \int_{\partial K} \mu dt \ = \ \int_{\partial K} P_h \mu dt \ . \tag{43}$$

3.4 Proof of Theorem 2

We first prove that the discrete variational problem (DVP) admits a unique solution $(u_h,\ \lambda_h)$ in $X_h \times M_h$, and then we establish the error estimate given by Eq. (25).

3.4.1 Existence and uniqueness

First, we prove that the bilinear form $b(\cdot\ ,\ \cdot)$ satisfies the inf-sup condition [22]. This result is stated in Proposition 1. Then, we prove in Proposition 2 the uniqueness of the solution of the *homogeneous* problem corresponding to the variational problem (DVP). The existence and uniqueness of the discrete variational problem (DVP) is then a direct consequence of Proposition 1 and Proposition 2.

Proposition 1 *Assume $kh \leq \pi$. Then, there is a positive constant γ independent of k and h such that*

$$\gamma ||\mu_h||_M \leq \sup_{v_h \in X_h} \frac{|b(v_h, \mu_h)|}{||v_h||_X} \leq ||\mu_h||_M \quad \forall\, \mu_h \in M_h.$$

Proof of Proposition 1. From Eq. (9), we deduce that

$$\forall\, \mu_h \in M_h \ , \quad \sup_{v_h \in X_h} \frac{|b(v_h, \mu_h)|}{||v_h||_X} \leq ||\mu_h||_M$$

In addition, it follows from Eq. (8) that

$$\forall\, \mu_h \in M_h \ , \ \exists\, \phi \in X \ , \quad \sup_{v \in X} \frac{|b(v, \mu_h)|}{||v||_X} = \frac{|b(\phi, \mu_h)|}{||\phi||_X} = ||\mu_h||_M$$

Therefore, it follows from Eq. (42) that

$$||\mu_h||_M = \frac{|b(\Pi_h\phi, \mu_h)|}{||\Pi_h\phi||_X} \frac{||\Pi_h\phi||_X}{||\phi||_X}$$

Since $kh \leq \pi$, it follows from Eq. (37) that there is a positive constant $\hat{C}$ such that

$$||\mu_h||_M \;\leq\; \hat{C} \sup_{v_h \in X_h} \frac{|b(v_h, \mu_h)|}{||v_h||_X}$$

which concludes the proof of Proposition 1.□

Proposition 2 *Assume $kh \leq \pi$. Then, the only solution of the following homogeneous discrete variational problem*

$$\begin{cases} \text{Find } u_h \in \mathcal{N}_h \text{ such that} \\ \\ a(u_h \ , \ v_h) \ = \ 0 \ , \quad \forall\, v_h \in \mathcal{N}_h. \end{cases}$$

is the trivial one.

Proof of Proposition 2. Let $u_h \in \mathcal{N}_h$ such that $a(u_h, v_h) = 0\ \forall\, v_h \in \mathcal{N}_h$, then $a(u_h, u_h) = 0$ which implies:

$$u_h \ = \ 0 \quad \text{on} \quad \Sigma \quad \text{and} \quad k\,||u_h||_{0,\Omega} \ = \ |u_h|_{1,\mathcal{T}_h}.$$

In addition, since $u_h \in X_h$, then $\Delta u_h + k^2 u_h = 0$ in every $K \in \mathcal{T}_h$. Therefore, using the integration by parts, it follows that:

$$a(u_h, v_h) \ = \ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \overline{v}_h \partial_n u_h \, dt \ = \ 0 \quad \forall v_h \in \mathcal{N}_h$$

then, we also have $\partial_n u_h \ = \ 0$ on $\Gamma \cup \Sigma$ and $[\partial_n u_h] \ = \ 0$ on $\partial K \cap \partial K' \quad \forall K \neq K' \in \mathcal{T}_h$ where $[\partial_n u_h] \ = \ \partial_n u_h^K + \partial_n u_h^{K'}$ is the jump of the normal derivative of u_h across $\partial K \cap \partial K'$.

To conclude the proof of this proposition, we use a discrete continuation result. We consider first the following property (P):

Let $K \in \mathcal{T}_h$ and T_l^K and T_m^K two adjacent edges of K such that

$$\partial_n u_h^K|_{T_l^K} \ = \ \partial_n u_h^K|_{T_m^K} \ = \ \int_{T_l^K} u_h \, dt \ = \ \int_{T_m^K} u_h \, dt \ = \ 0 \qquad \text{then} \qquad u_h = 0 \quad \text{in} \quad K.$$

Note that property (P) is easy to establish since $u_h \in X_h$ (a sum of four plane waves), and therefore u_h satisfies the Helmholtz equation at the element level K.

Now since, there is at least one element $K \in \mathcal{T}_h$ with two adjacent edges belonging to the boundary Σ, then using property (P) leads to $u_h = 0$ in K. Then, we obtain sequentially that $u_h = 0$ in all the quadrilaterals belonging to the first layer adjacent to the boundary Σ. We repeat this process on the second layer of the quadrilaterals and so on, until the boundary Γ is reached, which proves the uniqueness of the solution u_h.□

3.4.2 A priori error estimates

In the next lemmas, we establish *a priori* estimates in order to prove the error estimate (25) given in Theorem 2 between the exact solution (u, λ) and the discrete solution (u_h, λ_h).
We consider the following notations:

$$\kappa_h = h(1+k) \qquad \text{and} \qquad z_h = u_h - \Pi_h u. \tag{44}$$

Lemma 7 *There is a positive constant $\hat{C}$ independent of k and h such that the solution λ of the variational problem VP(5) satisfies*

$$||\lambda - P_h\lambda||_M \;\leq\; \hat{C}\kappa_h^{\frac{2}{3}}(1+k).$$

Proof of Lemma 7. First, recall that

$$\lambda^K = \begin{cases} -\partial_n u & \text{on} \;\; \partial K \setminus \partial\Omega, \\ \\ \quad 0 & \text{on} \;\; \partial K \cap \partial\Omega. \end{cases}$$

Therefore, using the definition of the operator P_h along with the fact the normal unit vector $\mathbf{n}^K$ is constant on each edge e of K, we deduce that $\forall\, K \in \mathcal{T}_h$, we have

$$\begin{aligned} ||\lambda - P_h\lambda||^2_{0,\partial K} &= \sum_{e\subset K, e \text{ interior}} ||\nabla u.\mathbf{n}^K - \frac{1}{|e|}\int_e \nabla u.\mathbf{n}^K\, dt||^2_{0,e} \\ &\leq \sum_{e\subset K, e \text{ interior}} ||\nabla u - \frac{1}{|e|}\int_e \nabla u\, dt||^2_{0,e} = \sum_{e\subset K, e \text{ interior}} \inf_{\beta\in\mathbb{C}^2} ||\nabla u - \beta||^2_{0,e} \\ &\leq \sum_{e\subset K, e \text{ interior}} ||\nabla u - \frac{1}{|K|}\int_K \nabla u\, dx||^2_{0,e} \leq ||\nabla u - \frac{1}{|K|}\int_K \nabla u\, dx||^2_{0,\partial K}. \end{aligned}$$

Finally, using the classical interpolation results [15], there is a positive constant $\hat{C}$ such that

$$\forall\, K \in \mathcal{T}_h \;, \quad ||\lambda - P_h\lambda||_{0,\partial K} \;\leq\; \hat{C} h_K^{\frac{1}{6}}\, |u|_{\frac{5}{3},K}. \tag{45}$$

In addition, we have from equation (4) that

$$||\lambda - P_h\lambda||_{H^{-\frac{1}{2}}(\partial K)} \;=\; \sup_{v\in H^1(K)} \frac{\left|\int_{\partial K}(\lambda - P_h\lambda)v\, dt\right|}{||v||_{X(K)}}\,.$$

On the other hand, from equation (43), we deduce that

$$|\int_{\partial K}(\lambda - P_h\lambda)v dt| = \left|\int_{\partial K}(\lambda - P_h\lambda)(v - \frac{1}{|K|}\int_K v\, dx)\, dt\right| \;, \quad \forall\, v \in H^1(K).$$

Hence,

$$\left|\int_{\partial K} (\lambda - P_h\lambda) v\, dt\right| \leq ||\lambda - P_h\lambda||_{0,\partial K}\, ||v - \frac{1}{|K|}\int_K v\, dx||_{0,\partial K}\ , \quad \forall\, v \in H^1(K).$$

Using the following classical interpolation results [15], it follows that there is a positive constant $\hat{C}$ such that

$$||v - \frac{1}{|K|}\int_K v\, dx||_{0,\partial K} \leq \hat{C} h_K^{\frac{1}{2}} |v|_{1,K} \leq \hat{C} h_K^{\frac{1}{2}} ||v||_{X(K)}.$$

We then deduce the existence of a positive constant $\hat{C}$ such that

$$\forall\, K \in \mathcal{T}_h\ , \quad ||\lambda - P_h\lambda||_{H^{-\frac{1}{2}}(\partial K)} \ \leq\ \hat{C} h_K^{\frac{1}{2}}\, ||\lambda - P_h\lambda||_{0,\partial K}\ , \quad \forall\, \mu \in \mathcal{M}. \tag{46}$$

Lemma 7 is the consequence of equations (45)-(46) and Theorem 1. □

The next lemma can be viewed as a consistency result.

Lemma 8 *Assume $kh \leq \pi$. Then, there is a positive constant $\hat{C}$ independent of k and h such that $\forall v_h \in X_h$ and $\forall v \in H^1(\Omega)$:*

$$|a(z_h, v_h) + b(v_h, \lambda_h - P_h\lambda)| \ \leq \hat{C}\,(1+k)\kappa_h^{\frac{2}{3}}\ [\kappa_h\ |v_h|_{1,\mathcal{T}_h} + |v - v_h|_{1,\mathcal{T}_h}]\,.$$

Proof of Lemma 8. We have

$$a(z_h, v_h) \ =\ a(u_h - \Pi_h u\ ,\ v_h) \ =\ a(u - \Pi_h u\ ,\ v_h) - a(u - u_h\ ,\ v_h).$$

Moreover, since u satisfies VP, we have

$$a(u, v_h) + b(v_h, \lambda) \ =\ F(v_h)$$

and since u_h satisfies DVP, we have

$$a(u_h, v_h) + b(v_h, \lambda_h) \ =\ F(v_h)$$

Consequently, we obtain

$$a(u - u_h\ ,\ v_h) \ =\ -b(v_h, \lambda - \lambda_h)$$

which leads to

$$a(z_h, v_h) + b(v_h, \lambda_h - P_h\lambda) \ =\ a(u - \Pi_h u\ ,\ v_h) + b(v_h, \lambda - P_h\lambda)$$

Hence, it follows from equation (41) that

$$a(u_h - \Pi_h u\ ,\ v_h) + b(v_h, \lambda_h - P_h\lambda) \ =\ -ik\int_\Sigma (u - \Pi_h u)\overline{v}_h\, dt + b(v_h, \lambda - P_h\lambda) \quad \forall v_h \in X_h. \tag{47}$$

Next, using (32) and following the same proof of Eq. (45) in Lemma 7, we obtain

$$\begin{aligned}\left|\int_\Sigma (u-\Pi_h u)\overline{v}_h\,dt\right| &\leq \sum_{e\subset\Sigma}\int_e |u-\Pi_h u|\,|\,\overline{v}_h-\frac{1}{|e|}\int_e \overline{v}_h\,dt|\,dt \\ &\leq \sum_{\partial K\subset\Sigma} ||u-\Pi_h u||_{0,\partial K}\,||v_h-\frac{1}{|K|}\int_K v_h\,dx||_{0,\partial K}\end{aligned}$$

Hence, using Eq. (23), it follows that there is a positive constant $\hat{C}$ such that

$$\left|\int_\Sigma (u-\Pi_h u)\overline{v}_h\,dt\right| \leq \hat{C}\sum_{K\in\mathcal{T}_h} h_K\,|u-\Pi_h u|_{1,K}\,|v_h|_{1,K}$$

Then, it follows from using Theorem 1 and Lemma 6, that there is a positive constant $\hat{C}$ such that

$$\left|\int_\Sigma (u-\Pi_h u)\overline{v}_h\,dt\right| \leq \hat{C}\,(\kappa_h^{\frac{5}{3}}+\kappa_h^2+\kappa_h^3)\,|v_h|_{1,\mathcal{T}_h}$$

which implies (assuming $kh \leq \pi$) that

$$\left|\int_\Sigma (u-\Pi_h u)\overline{v}_h\,dt\right| \leq \hat{C}\,\kappa_h^{\frac{5}{3}}\,|v_h|_{1,\mathcal{T}_h}. \tag{48}$$

On the other hand, we have $\forall v\in H^1(\Omega)$:

$$\begin{aligned}|b(v_h,\lambda-P_h\lambda)| &= \left|\sum_{e\ \text{interior}}\int_e [\overline{v}_h]\,(\lambda-P_h\lambda)\ dt\right| = \left|\sum_{e\ \text{interior}}\int_e [\overline{v}-\overline{v}_h]\,(\lambda-P_h\lambda)\ dt\right| \\ &= \left|\sum_{e\ \text{interior}}\int_e (\lambda-P_h\lambda)\,.[(\overline{v}-\overline{v}_h)-\frac{1}{|e|}\int_e(\overline{v}-\overline{v}_h)]\,dt\right| \\ &\leq \sum_K ||\lambda-P_h\lambda||_{0,\partial K}\,||(v-v_h)-\frac{1}{|K|}\int_K(v-v_h)\,dx||_{0,\partial K}\end{aligned}$$

Therefore, it follows from using using Eq. (23), that there is a positive constant $\hat{C}$ such that

$$|b(v_h,\lambda-P_h\lambda)| \ \leq\ \hat{C}\sum_{K\in\mathcal{T}_h} h_K^{\frac{1}{2}}\,|v-v_h|_{1,K}\,||\lambda-P_h\lambda||_{0,\partial K}$$

Hence, from Eq. (45) and Theorem 1, we obtain that there is a positive constant $\hat{C}$ such that

$$|b(v_h,\lambda-P_h\lambda)| \ \leq\ \hat{C}\,\kappa_h^{\frac{2}{3}}\,(1+k)\,|v-v_h|_{1,\mathcal{T}_h} \tag{49}$$

We conclude the proof of Lemma 8 by substituting Eq. (48) and Eq. (49) into Eq. (47). □

Remark 4 *We deduce from Lemma 8 that, when $kh \le \pi$, there is a positive constant $\hat{C}$ such that $\forall v_h \in \mathcal{N}_h$ and $\forall v \in H^1(\Omega)$,*

$$|a(z_h, v_h)| \ \le \hat{C}\,(1+k)\,\kappa_h^{\frac{2}{3}}\,[\,\kappa_h |v_h|_{1,\mathcal{T}_h} + |v - v_h|_{1,\mathcal{T}_h}\,]. \tag{50}$$

Lemma 9 *Assume $kh \le \pi$. Then, there is a positive constant C (C depends on Ω only) such that,*

$$||z_h||_{0,\Omega} \ \le C\,\kappa_h^{\frac{2}{3}}[(1+k)\kappa_h^{\frac{2}{3}} \ + |z_h|_{1,\mathcal{T}_h}] \tag{51}$$

Proof of Lemma 9. First observe that z_h belongs to $\mathcal{N}_h$ and let ϕ be the solution of the following boundary value problem (see Lemma 1):

$$-\Delta\overline{\phi} - k^2\overline{\phi} = \overline{z}_h \qquad \text{in} \ \ \Omega,$$

and

$$\partial_n\overline{\phi} = 0 \qquad \text{on} \ \ \Gamma, \qquad\qquad \partial_n\overline{\phi} = \mathrm{i}k\,\overline{\phi} \qquad \text{on} \ \ \Sigma.$$

Hence, it follows from Lemma 1 that $\phi \in H^{\frac{5}{3}}(\Omega)$ and (see Eq. (13)) there is constant $C > 0$ (C depends on Ω only) such that, for every $s \in [0, \frac{5}{3}]$, we have

$$|\phi|_{s,\Omega} \le C\,(1+k)^{s-1}\,||z_h||_{0,\Omega}. \tag{52}$$

In addition, we have

$$||z_h||^2_{0,\Omega} \quad = \quad a(z_h, \phi) - \sum_{e \ \text{interior}} \int_e [z_h]\partial_n\overline{\phi}\,dt. \tag{53}$$

Eq. (53) results from multiplying the boundary value problem introduced in Lemma 9, integrating by parts on Ω, and using the definition of the bilinear form a. The second term of this equality is due to the discontinuity of z_h along the interior edges. Recall that the jump $[\phi]$ along $e \in \partial K \cap \partial K'$ is given by $[\phi] = \phi^K - \phi^{K'}$.
On the other hand, we have

$$|a(z_h, \phi)| \ \le \ |a(z_h, \Pi_h\phi)| + |a(z_h, \phi - \Pi_h\phi)|\,.$$

It follows from Eq. (41) that

$$|a(z_h, \phi)| \ \le \ |a(z_h, \Pi_h\phi)| + k\,\left|\int_\Sigma z_h(\overline{\phi} - \Pi_h\overline{\phi})\,dt\right|. \tag{54}$$

Since $\Pi_h\phi \in \mathcal{N}_h$ (see property *ii* in Property 3), then it follows from Remark 4 that there is a positive constant $\hat{C}$ such that

$$|a(z_h, \Pi_h\phi)| \ \le \ \ \hat{C}\,(1+k)\,\kappa_h^{\frac{2}{3}}\,[\kappa_h|\Pi_h\phi|_{1,\mathcal{T}_h} + |\phi - \Pi_h\phi|_{1,\mathcal{T}_h}]\,.$$

Moreover, it follows from Lemma 6, that there is a positive constant $\hat{C}$ such that

$$|\phi - \Pi_h\phi|_{1,\mathcal{T}_h} \le \ \hat{C}\left\{h^{\frac{2}{3}}\,|\phi|_{\frac{5}{3},\Omega} + k^2h||\phi||_{0,\Omega} + k^2h^2|\phi|_{1,\Omega}\right\}$$

then using relation (52) and the assumption $kh \le \pi$, we obtain

$$|\phi - \Pi_h\phi|_{1,\mathcal{T}_h} \le \ \hat{C}\kappa_h^{\frac{2}{3}}||z_h||_{0,\Omega} \qquad \text{and} \qquad |\Pi_h\phi|_{1,\mathcal{T}_h} \le \ \hat{C}||z_h||_{0,\Omega}.$$

We obtain then:
$$|a(z_h, \Pi_h \phi)| \;\leq\; \hat{C}\,(1+k)\kappa_h^{\frac{4}{3}}||z_h||_{0,\Omega}.$$
For the second part of Eq. (54), we have
$$\left|\int_\Sigma z_h(\overline{\phi} - \Pi_h\overline{\phi})\,dt\right| \;\leq\; \hat{C}\,h\,|\phi - \Pi_h\phi|_{1,\mathcal{T}_h}\,|z_h|_{1,\mathcal{T}_h} \leq \hat{C}h\kappa_h^{\frac{2}{3}}|z_h|_{1,\mathcal{T}_h}||z_h||_{0,\Omega}.$$
Note that the previous inequality was obtained using the same methodology to prove Lemma 5. Hence, we first, we use Eq. (32) when we add the constant $(-\frac{1}{|K|}\displaystyle\int_K z_h\,dt)$ to z_h. Then, we apply Cauchy-Schwartz along with inequalities (21) and (23).
Finally, it follows that there is a positive constant C (C depends on Ω only) such that
$$|a(z_h, \phi)| \;\leq\; C\,[(1+k)\kappa_h^{\frac{4}{3}} + \kappa_h^{\frac{5}{3}}|z_h|_{1,\mathcal{T}_h}]||z_h||_{0,\Omega}. \tag{55}$$

Next, we estimate the term $\left|\displaystyle\sum_{e\ \text{interior}}\int_e [z_h]\partial_n\overline{\phi}\,dt\right|$ in Eq. (53). First, observe that
$$\int_e z_h^K\,dt \;=\; \int_e z_h^{K'}\,dt\;, \quad \forall\, e \in \partial K \cap \partial K' \text{ and } K \neq K' \in \mathcal{T}_h$$
and
$$\begin{aligned}\textstyle\int_{e\in\partial K\cap\partial K'}(z_h^K - z_h^{K'})\partial_n\overline{\phi}\,dt \;=\;& \textstyle\int_e \left(z_h^K - \frac{1}{|e|}\int_e z_h^K\,dt\right)\left(\nabla\phi - \frac{1}{|K|}\int_K \nabla\phi\,dx\right)\cdot \mathbf{n}^K\,dt \\ +\;& \textstyle\int_e \left(z_h^{K'} - \frac{1}{|e|}\int_e z_h^{K'}\,dt\right)\left(\nabla\phi - \frac{1}{|K'|}\int_{K'} \nabla\phi\,dx\right)\cdot \mathbf{n}^{K'}\,dt\end{aligned}$$
Therefore,
$$\left|\sum_{e\ \text{interior}}\int_e [z_h]\partial_n\overline{\phi}\,dt\right| \;\leq\; \sum_{K\in\mathcal{T}_h}\sum_{e\subset K}\int_e \left|z_h - \frac{1}{|e|}\int_e z_h\,dt\right|\left|\nabla\phi - \frac{1}{|K|}\int_K \nabla\phi\,dx\right|\,dt.$$
Hence, it follows that
$$\left|\sum_{e\ \text{interior}}\int_e [z_h]\partial_n\overline{\phi}\,dt\right| \;\leq\; \hat{C}\,h^{\frac{2}{3}}\,|z_h|_{1,\mathcal{T}_h}\,|\phi|_{\frac{5}{3},\Omega} \leq C\kappa_h^{\frac{2}{3}}|z_h|_{1,\mathcal{T}_h}\,||z_h||_{0,\Omega}. \tag{56}$$

We conclude the proof of Lemma 9 by substituting Eq. (55) and Eq.(56) into Eq.(53). □

Lemma 10 *Let h_0 be a positive number such that $k\,h_0^{\frac{2}{3}}(1+k)^{\frac{2}{3}}$ is "sufficiently small". Then, there is a positive constant C (C depends on Ω only) such that for all $h \leq h_0$, we have*
$$||u_h - \Pi_h u||_{0,\Omega} \leq \hat{C}(1+k)\kappa_h^{\frac{4}{3}} \quad \text{and} \quad |u_h - \Pi_h u|_{1,\mathcal{T}_h} \leq \hat{C}(1+k)\kappa_h^{\frac{2}{3}}.$$

Proof of Lemma 10. It follows from the definition of the bilinear form $a(.,.)$ that

$$|a(z_h, z_h)|^2 = \left||z_h|^2_{1,\mathcal{T}_h} - k^2 \, ||z_h||^2_{0,\Omega}\right|^2 + k^2 ||z_h||^4_{0,\Gamma}$$

Moreover, using Remark 4 with $v_h = z_h$ and $v = 0$ along with the fact that $kh \leq \pi$ we obtain

$$|a(z_h, z_h)| \leq \hat{C}\,(1+k)\kappa_h^{\frac{2}{3}} |z_h|_{1,\mathcal{T}_h}.$$

Therefore, we deduce that

$$|z_h|^2_{1,\mathcal{T}_h} \;\leq\; k^2 \, ||z_h||^2_{0,\Omega} + \hat{C}\,(1+k)\kappa_h^{\frac{2}{3}} |z_h|_{1,\mathcal{T}_h}.$$

Then, using Eq. (51) along with Young's inequality, we obtain

$$|z_h|^2_{1,\mathcal{T}_h} \;\leq\; C[k^2(1+k)^2\kappa_h^{\frac{8}{3}} + k^2\kappa_h^{\frac{4}{3}} \, |z_h|^2_{1,\mathcal{T}_h} + \,(1+k)\kappa_h^{\frac{2}{3}} |z_h|_{1,\mathcal{T}_h}].$$

Consequently, we have

$$|z_h|^2_{1,\mathcal{T}_h} \;\leq\; C[k^2(1+k)^2\kappa_h^{\frac{8}{3}} + k^2\kappa_h^{\frac{4}{3}} \, |z_h|^2_{1,\mathcal{T}_h} + \,(1+k)^2\kappa_h^{\frac{4}{3}}].$$

Let us consider h_0 such that $Ck^2(1+k)^{\frac{4}{3}} h_0^{\frac{4}{3}} \leq \frac{1}{2}$ then for every $h \leq h_0$, we have $Ck^2\kappa_h^{\frac{4}{3}} \leq \frac{1}{2}$. We deduce that

$$|z_h|^2_{1,\mathcal{T}_h} \;\leq\; C[k^2(1+k)^2\kappa_h^{\frac{8}{3}} + \,(1+k)^2\kappa_h^{\frac{4}{3}}] \quad \text{then} \quad |z_h|_{1,\mathcal{T}_h} \;\leq\; \hat{C}\,(1+k)\kappa_h^{\frac{2}{3}}.$$

In addition, we obtain from using Eq. (51), that

$$||z_h||_{0,\Omega} \leq \hat{C}(1+k)\kappa_h^{\frac{4}{3}}.$$

which concludes the proof of Lemma 10. □

Proof of the a priori error estimate of Theorem 2. We are now ready to prove the estimate given by Eq. (25).

- From Lemma 6 and Lemma 10, it follows that there is a positive constant C (C depends on Ω only) such that

$$||u - u_h||_{0,\Omega} \;\leq\; ||u - \Pi_h u||_{0,\Omega} + ||u_h - \Pi_h u||_{0,\Omega} \;\leq\; C\,[\kappa_h^{\frac{4}{3}} + (1+k)\kappa_h^{\frac{4}{3}}]$$

and

$$|u - u_h|_{1,\mathcal{T}_h} \;\leq\; |u - \Pi_h u|_{1,\mathcal{T}_h} + |u_h - \Pi_h u|_{1,\mathcal{T}_h} \;\leq\; C\,[\kappa_h^{\frac{2}{3}} + k\kappa_h + (1+k)\kappa_h^{\frac{2}{3}}].$$

Hence, we deduce that

$$||u - u_h||_{0,\Omega} \;\leq\; C\,(1+k)\kappa_h^{\frac{4}{3}} \quad \text{and} \quad |u - u_h|_{1,\mathcal{T}_h} \;\leq\; C\,(1+k)\kappa_h^{\frac{2}{3}}.$$

- Moreover, we deduce from Lemma 8 that there is a positive constant $\hat{C}$ such that

$$|b(v_h, \lambda_h - P_h\lambda)| \;\leq\; \hat{C}\,(1+k)\kappa_h^{\frac{2}{3}} \, |v_h|_{1,\mathcal{T}_h} + |a(z_h, v_h)| \quad \forall v_h \in X_h.$$

On the other hand, it follows from the definition of the bilinear form $a(.\,,\,.)$ that

$$|a(z_h, v_h)| \;\leq\; |z_h|_{1,\mathcal{T}_h}\, |v_h|_{1,\mathcal{T}_h} + k^2\, |\int_\Omega z_h.\overline{v}_h dx| + k\, ||z_h||_{0,\Sigma}\, ||v_h||_{0,\Sigma} \quad \forall v_h \in X_h.$$

Therefore, using the definition of the norm $||\cdot||_X$ and inverse inequality results, we deduce that there is a positive constant $\hat{C}$ such that

$$|a(z_h, v_h)| \;\leq\; \left(|z_h|^2_{1,\mathcal{T}_h} + k^2\, h^2\, ||z_h||^2_{0,\Omega}\right)^{\frac{1}{2}}\, ||v_h||_X + \hat{C}\, k\, ||z_h||_{0,\Sigma}\, h^{\frac{1}{2}}\, ||v_h||_X \quad \forall v_h \in X_h.$$

In addition, it follows from the definition of the bilinear form $a(.\,,\,.)$ and from using Eq. (50) with $v_h = z_h$ and $v = 0$ (see Remark 4) that there is a positive constant $\hat{C}$ such that

$$k||z_h||^2_{0,\Sigma} \;\leq\; |a(z_h, z_h)| \;\leq\; \hat{C}\,(1+k)\kappa_h^{\frac{2}{3}}\, |z_h|_{1,\mathcal{T}_h}$$

Therefore, using Lemma 10, we deduce that there is a positive constant C (C depends on Ω only) such that

$$k^{\frac{1}{2}}||z_h||_{0,\Sigma} \;\leq\; (1+k)\kappa_h^{\frac{2}{3}}.$$

Hence, we deduce that there is a positive constant C (C depends on Ω only) such that

$$|a(z_h, v_h)| \;\leq\; C\,(1+k)\kappa_h^{\frac{2}{3}}\, ||v_h||_X \quad \forall v_h \in X_h.$$

Consequently, it follows Proposition 1 that there is a positive constant C (C depends on Ω only) such that

$$||\lambda_h - P_h\lambda||_M \;\leq\; C\,(1+k)\kappa_h^{\frac{2}{3}}.$$

Finally, we deduce from Lemma 7 that there is a positive constant C (C depends on Ω only) such that

$$||\lambda - \lambda_h||_M \;\leq\; C\,(1+k)\kappa_h^{\frac{2}{3}}.$$

which concludes the proof of the error estimate of Theorem 2. □

Proof of the a posteriori error estimate (26) in Theorem 3. Let ϕ be the solution of the boundary value problem (12) (see Lemma 1) with $f = u - u_h$. Then this solution ϕ belongs to $H^{\frac{5}{3}}(\Omega)$ and for every $s \in [0, \frac{5}{3}]$, there exists a constant $C > 0$ depending only on s and Ω such that

$$|\phi|_{s,\Omega} \leqslant C(1+k)^{s-1}||u - u_h||_{0,\Omega}.$$

Using integration by parts, one can easily verify that

$$\begin{aligned} ||u - u_h||^2_{0,\Omega} \;=\; & \sum_{K\in\mathcal{T}_h} \int_{\partial K\cap\Sigma} \overline{\phi}(\partial_n u_h - \mathrm{i}k\, u_h)\, dt + \sum_{K\in\mathcal{T}_h} \int_{\partial K\cap\Gamma} \overline{\phi}(\partial_n u_h + \partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}})\, dt \\ & + \sum_{e \text{ interior}} \int_e [\partial_n u_h]\overline{\phi}\, dt - \sum_{e \text{ interior}} \int_e [u_h]\partial_n\overline{\phi}\, dt \end{aligned}$$

On the other hand, we also have

$$a(u_h, \Pi_h\phi) \;=\; -\sum_{K\in\mathcal{T}_h} \int_{\partial K\cap\Gamma} \partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}\, \Pi_h\overline{\phi}\, dt$$

Therefore, using integration by parts along with the fact that u_h satisfies the Helmholtz equation at the element level, we have

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Gamma}\partial_n u_h\,\Pi_h\overline{\phi}\,dt+\sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Sigma}(\partial_n u_h-\mathrm{i}ku_h)\,\Pi_h\overline{\phi}\,dt+\sum_{e\ \text{interior}}\int_e[\partial_n u_h]\,\Pi_h\overline{\phi}\,dt$$
$$=\quad -\sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Gamma}\partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}\,\Pi_h\overline{\phi}\,dt$$

Consequently, using the fact that for every interior edge e, we have $\int_e[\partial_n u_h]\,\overline{\phi}\,dt=\int_e[\partial_n u_h]\,\Pi_h\overline{\phi}\,dt$, we deduce that

$$\begin{aligned}||u-u_h||^2_{0,\Omega} &= \sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Sigma}\left(\overline{\phi}-\Pi_h\overline{\phi}\right)(\partial_n u_h-\mathrm{i}k\,u_h)\,dt\\ &+\sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Gamma}\left(\overline{\phi}-\Pi_h\overline{\phi}\right)(\partial_n u_h+\partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}})\,dt-\sum_{e\ \text{interior}}\int_e[u_h]\partial_n\overline{\phi}\,dt\end{aligned}\tag{57}$$

Next, we estimate each integral in the right-hand side of Eq. (57) to deduce the a posteriori estimate given by Eq. (26) in Theorem 3.

- First, we estimate: $I_1=\left|\sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Sigma}\left(\overline{\phi}-\Pi_h\overline{\phi}\right)(\partial_n u_h-\mathrm{i}k\,u_h)\,dt\right|$.

We have

$$I_1\leqslant\left(\sum_{e\subset\Sigma}h_e||\partial_n u_h-\mathrm{i}k\,u_h||^2_{0,e}\right)^{\frac{1}{2}}\left(\sum_{e\subset\Sigma}h_e^{-1}||\phi-\Pi_h\phi||^2_{0,e}\right)^{\frac{1}{2}}\leqslant\hat{C}\left(\sum_{e\subset\Sigma}h_e||\partial_n u_h-\mathrm{i}k\,u_h||^2_{0,e}\right)^{\frac{1}{2}}|\phi-\Pi_h\phi|_{1,\mathcal{T}_h}.$$

Therefore, assuming that $kh\leqslant\pi$, it follows from the properties of the operator Π (see Eq. (40) in Property 3) that there is a positive constant $\hat{C}$ such that:

$$I_1\leq\hat{C}_1\left(\sum_{e\subset\Sigma}h_e||\partial_n u_h-\mathrm{i}k\,u_h||^2_{0,e}\right)^{\frac{1}{2}}\left(h^{\frac{2}{3}}|\phi|_{\frac{5}{3},\Omega}+|\phi|_{1,\Omega}+k\|\phi\|_{0,\Omega}\right).$$

We deduce from the a priori estimate on $|\phi|_{s,\Omega}$ that there is a positive constant $\hat{C}_1$ such that:

$$I_1\leq\hat{C}_1\left(\sum_{e\subset\Sigma}h_e||\partial_n u_h-\mathrm{i}k\,u_h||^2_{0,e}\right)^{\frac{1}{2}}||u-u_h||_{0,\Omega}.$$

- Similarly, there is also a positive constant $\hat{C}_2$ such that:

$$I_2=\left|\sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Gamma}\left(\overline{\phi}-\Pi_h\overline{\phi}\right)(\partial_n u_h+\partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}})\,dt\right|\leq\hat{C}\left(\sum_{e\subset\Gamma}h_e||\partial_n u_h+\partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}||^2_{0,e}\right)^{\frac{1}{2}}|\phi-\Pi_h\phi|_{1,\mathcal{T}_h}$$

Then, there is there is a positive constant denoted again by $\hat{C}_2$ such that

$$I_2\leq\hat{C}_2\left(\sum_{e\subset\Gamma}h_e||\partial_n u_h+\partial_n e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}||^2_{0,e}\right)^{\frac{1}{2}}||u-u_h||_{0,\Omega}.$$

- Last, we estimate: $I_3 = \left| \sum_{e \text{ interior}} \int_e [u_h] \partial_n \overline{\phi}\, dt \right|.$

Consider an interior edge $e = \partial K(e) \bigcap \partial K'(e)$, then

$$\int_e [u_h] \partial_n \overline{\phi}\, dt = \int_e [u_h] \nabla \overline{\phi}.\mathbf{n}\, dt = \int_e [u_h]\, (\nabla \overline{\phi} - \beta).\mathbf{n}\, dt \quad \forall \boldsymbol{\beta} \in \mathbb{C}^2.$$

We then obtain

$$|\int_e [u_h] \partial_n \overline{\phi}\, dt| \leq ||[u_h]||_{0,e} \inf_{\boldsymbol{\beta} \in \mathbb{C}^2} ||\nabla \phi - \boldsymbol{\beta}\, ||_{0,e}.$$

On the other hand, since there is a positive constant $\hat{C}$ such that

$$\inf_{\boldsymbol{\beta} \in \mathbb{C}^2} ||\nabla \phi - \boldsymbol{\beta}\, ||_{0,e} \leq \hat{C}\, h_e^{\frac{1}{6}}\, |\phi|_{\frac{5}{3}, K(e)},$$

it follows that

$$I_3 \leq \hat{C} \sum_{e \text{ interior}} h_e^{\frac{1}{6}}\, ||[u_h]||_{0,e} |\phi|_{\frac{5}{3}, K(e)} \leq \hat{C} \left(\sum_{e \text{ interior}} h_e^{-1}\, ||[u_h]||_{0,e}^2 \right)^{\frac{1}{2}} h^{\frac{2}{3}} |\phi|_{\frac{5}{3}, \Omega}.$$

Then, there is a positive constant $\hat{C}_3$ such that

$$I_3 \leq \hat{C}_3 \left(\sum_{e \text{ interior}} h_e^{-1}\, ||[u_h]||_{0,e}^2 \right)^{\frac{1}{2}} ||u - u_h||_{0,\Omega}.$$

4 Conclusion

A discontinuous Galerkin method (DGM) with plane waves and Lagrange multipliers was recently proposed by Farhat et al. [3] for solving two-dimensional Helmholtz problems at relatively high wave numbers. In many previous papers, this method was shown numerically to offer a significant potential for wave propagation problems including acoustic scattering. However, it lacked a formal convergence theory. This paper is a first step toward filling this gap. Indeed, it is proved that the hybrid variational formulation underlying this DGM is well-posed in the sense of Hadamard. In addition, a priori error estimates proved for the so-called R-4-1 element, that is the simplest two-dimensional element associated with this discretization method, establish the convergence of this element and reveal its formal order of accuracy. Furthermore, a posteriori error estimate was derived and that can be used as a practical error indicator when refining the partition of the computational domain. Higher order elements will be analyzed in future research.

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