INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING

*Int. J. Numer. Meth. Engng* 2007; **00**:1–6 *Prepared using* nmeauth.cls *[Version: 2002/09/18 v2.02]*

# Strain and stress computations in stochastic finite element methods

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SUMMARY

This paper focuses on the computation of statistical moments of strains and stresses in a random system model where uncertainty is modeled by a Stochastic Finite Element Method (SFEM) based on the Polynomial Chaos Expansion (PCE). It identifies the cases where this objective can be achieved by analytical means using the orthogonality property of the chaos polynomials and those where it requires a numerical integration technique. To this effect, the applicability and efficiency of several numerical integration schemes are considered. These include the Gauss-Hermite quadrature with the direct tensor product — also known as the Kronecker product — Smolyak's approximation of such a tensor product, Monte Carlo sampling, and the Latin Hypercube sampling method. An algorithm for reducing the dimensionality of integration under a direct tensor product is also explored for optimizing

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Contract/grant sponsor: Financial support from the Advanced Simulation and Computing Program of the Department of Energy is gratefully acknowledged.; contract/grant number:

*Received*



*Revised*

the computational cost and complexity. The convergence rate and algorithmic complexity of all of these methods are discussed and illustrated with the non-deterministic linear stress analysis of a plate. 

KEY WORDS: polynomial chaos; stochastic finite element; stress analysis; uncertainty

## 1. INTRODUCTION

The structural properties of a material typically show variability among different samples. The variations usually result from the natural variability in microstructure and from the manufacturing process. This process also causes variations in the geometric properties of the manufactured components. All of these variabilities induce uncertainty in the (numerically) predicted response of the modeled physical system. Additional factors such as uncertainty in external loading, and in some cases insufficient details about the underlying physics such as in the case of modeling the behavior of joints, contribute to magnify the uncertainty in the predicted response. All of these uncertainties can be modeled and their effects can be analyzed using a computational framework based on probability theory.

Among the response quantities, the strains and stresses are often those that are most interesting for an analyst. However, the current literature on probabilistic engineering mechanics focuses on addressing the issue of computing the displacement field. This paper attempts to fill this gap by addressing the computational issues related to the non-deterministic evaluation of strains and stresses. To this effect, the entire problem of uncertainty analysis is formulated in a stochastic finite element framework.

Among the probabilistic methods of uncertainty analysis, Stochastic Finite Element Methods

 

(SFEM) in general and those based on the Polynomial Chaos Expansion (PCE) in particular [26] have gained considerable attention recently. The major advantages of these methods are their ability to handle stochastic processes, their encapsulated representation of random quantities, and their lower computational cost.

Stress evaluation in a stochastic analysis using SFEM has been performed for specific applications such as the computation of the thermal creep stress in concrete [22]. The work presented in this paper considers the more general case in the context of the PCE approach. First, a few specific cases where an orthogonal polynomial chaos expansion can be obtained for stresses are identified. It is shown that in these cases, the computation of the statistical moments of stresses can be performed analytically. In all other cases, an orthogonal expansion such as PCE is not easily obtainable either for strain or for stress quantities and therefore a numerical technique must be used to estimate their statistical moments. Such a numerical estimation often requires a considerable amount of CPU time and programming effort. For example, it is shown in Section 6 of this paper that for a plate problem for which computing the displacement field consumes approximately 90 minutes CPU on a desktop computer, evaluating the mean and standard deviation of the von Mises stress alone requires about $50,000$ quadrature points and 45 minutes CPU. Given that the computation of additional strain and stress fields may be desired, it follows that the post-processing phase of a non-deterministic analysis can often dominate its total CPU time.

The remainder of this paper is organized as follows. First, expressions for various stochastic strain and stress fields and their statistical moments are developed using a PCE for the displacement field. Then, the cases where some of these can be computed analytically are highlighted. Next, those where a numerical integration technique is required are identified

and for this purpose a few algorithms are also examined. These include two probabilistic techniques, namely, Monte Carlo sampling [19, 15] and Latin hypercube sampling [21], and two deterministic techniques, namely, Gauss quadrature on the standard tensor product grid [18, 3] and Smolyak's quadrature [29]. Next, a dimension reduction technique for integrating polynomial expansions and minimizing computational cost is described. Finally, a numerical study is presented and conclusions are offered.

*REMARK.* The Latin Hypercube sampling method was used in [5] and Smolyak's quadrature was used in [2] to estimate directly the statistical moments of the response of a random system without seeking a PCE-like representation of the output. However in these applications, the governing equation was solved at each quadrature point. In this paper, the displacement field is first represented as a PCE, then the chaos coefficients are estimated and finally techniques for estimating the statistical moments of the strains and stresses from these coefficients are developed. Hence unlike in the aforementioned works, the only computational cost incurred by this alternative approach at each quadrature point is that associated with postprocessing the displacement field to obtain strain and stress fields. To the best of the authors' knowledge, the computational aspects of this alternative approach for estimating the statistical moments of strains and stresses have not yet been addressed in the literature.

## 2. STOCHASTIC FINITE ELEMENT METHOD BASED ON THE POLYNOMIAL CHAOS EXPANSION

Let $(\Omega, \mathcal{F}, P)$ denote a probability space, where $\Omega$ is the set of the outcomes $\theta$ of physical experiments, $\mathcal{F}$ is a $\sigma$-algebra in $\Omega$, and $P$ is a probability measure on $\mathcal{F}$. Let $\mathcal{X}$ denote the physical domain of the system. Consider the following equation

$$\mathcal{L}(u,\theta) = f(\theta) \ , \tag{1}$$

which highlights the fact that randomness in the parameters of the underlying physical system induces randomness in $\mathcal{L}$ and $f$. Some of the random parameters can be modeled as random variables $\{\eta_i(\theta)\}_{i=1}^{i=r}$ and some as random processes $\kappa(\boldsymbol{x},\theta)$, where $\boldsymbol{x} \in \mathcal{X}$. For example, a spring stiffness can be modeled as a random variable, whereas the thickness of a plate can be modeled as a random field. The processes $\kappa(\boldsymbol{x},\theta)$ can be discretized using a random basis set $\{\eta_i(\theta)\}_{i=r+1}^{i=s}$ in $L_2(\Omega,\mathcal{F},P)$, where the coefficients of the random variables $\{\eta_i(\theta)\}_{i=r+1}^{i=s}$ turn out to be functions of the parameter $\boldsymbol{x}$. If the covariance function $C(\boldsymbol{x}_1,\boldsymbol{x}_2)$ of the process $\kappa(\boldsymbol{x},\theta)$ is known, the process can be discretized using the Karhunen-Loève expansion [26] as follows

$$\kappa(\boldsymbol{x},\theta) = \sum_{i=0}^{\infty} \sqrt{\lambda_i}\phi_i(\boldsymbol{x})\eta_i(\theta) \ \ , \tag{2}$$

where $\lambda_i$ are the eigenvalues of the covariance kernel $C(\boldsymbol{x}_1,\boldsymbol{x}_2)$ arranged in descending order, $\phi_i$ are the corresponding eigenvectors, and $\eta_i(\theta)$ are zero-mean and orthonormal random variables. For computational convenience, this series is truncated after the first few terms. The set of all random variables $\{\eta_i(\theta)\}_{i=1}^{i=s}$ completely characterizes the uncertainty in the underlying system. These random variables are in turn characterized by their joint probability measure. If this measure is not Gaussian, the random variables can be transformed into a nonlinear function of an independent Gaussian vector $\{\xi_i(\theta)\}_{i=1}^{i=d}$ [25, 27], where the integer $d$ is often referred to as the stochastic dimension of the problem [26, 13]. This new set of independent standard random variables is denoted here by a $d$-dimensional vector $\boldsymbol{\xi}$. Thus, $\mathcal{L}(u,\theta)$ and $f(\theta)$ can now also be written as $\mathcal{L}(u,\boldsymbol{\xi})$ and $f(\boldsymbol{\xi})$. The solution $u(\boldsymbol{x})$ is also function of $\boldsymbol{\xi}$, yielding

the notation $u(\boldsymbol{x}, \boldsymbol{\xi})$. The formulation presented here is also valid when $\boldsymbol{\xi}$ is non-Gaussian and thus is equally applicable to a variety of expansions [7, 23].

Once $\mathcal{L}(u, \boldsymbol{\xi})$ and $f(\boldsymbol{\xi})$ are constructed, the solution $u(\boldsymbol{x}, \boldsymbol{\xi})$ is represented in a PCE [26], where a square-integrable random process is expressed using a set of orthogonal bases as

$$u(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=0}^{\infty} u^{(i)}(\boldsymbol{x}) \psi_i(\boldsymbol{\xi}) \quad . \tag{3}$$

Here, $\psi_i(\boldsymbol{\xi})$ are the Hermite polynomials satisfying

$$\psi_0 \equiv 1 \ , \quad \langle \psi_i \rangle = 0 \ \text{ for } \ i > 0 \qquad \text{and the orthogonality property} \quad \langle \psi_i \psi_j \rangle = \delta_{i,j} \langle \psi_i^2 \rangle \ , \tag{4}$$

where $\langle \cdot \rangle$ denotes the mathematical expectation operator

$$\langle g(\boldsymbol{\xi}) \rangle = \int_{\mathbb{R}^d} g(\boldsymbol{\xi}) p(\boldsymbol{\xi}) d\boldsymbol{\xi} \ , \tag{5}$$

$g(\boldsymbol{\xi})$ is any function of the $d$-dimensional random vector $\boldsymbol{\xi}$ and $p(\boldsymbol{\xi})$ is the joint probability density function (pdf) of $\boldsymbol{\xi}$. $\delta_{i,j}$ denotes the Kronecker delta function and $u^{(i)}(\boldsymbol{x})$ are deterministic coefficients called chaos coefficients. In practice, the above series is truncated after a finite number of terms yielding

$$u(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=0}^{P-1} u^{(i)}(\boldsymbol{x}) \psi_i(\boldsymbol{\xi}) \ . \tag{6}$$

The highest degree of the polynomials retained in this truncated expansion is referred to as the order of the expansion. The index $P$ is determined by the stochastic dimension $d$ and the order of the expansion. For example, in a second-order expansion in two stochastic dimensions $P = 6$ and the polynomials $\psi_i(\xi_1, \xi_2)$ are [26]

$$\begin{aligned}\psi_0(\xi_1,\xi_2) &= 1\ , \qquad \psi_1(\xi_1,\xi_2)=\xi_1\ , \qquad \psi_2(\xi_1,\xi_2)=\xi_2\ ,\\ \psi_3(\xi_1,\xi_2) &= \xi_1^2-1\ , \quad \psi_4(\xi_1,\xi_2)=\xi_1\xi_2\ , \quad \psi_5(\xi_1,\xi_2)=\xi_2^2-1\ .\end{aligned}$$

The chaos coefficients $u^{(i)}(\boldsymbol{x})$ can be computed by minimizing either the error in the solution or the residual of the governing equation [26]. In both cases, a Galerkin approach can be used to find the optimal values of these coefficients. The method consisting in minimizing the error in the solution incurs the evaluation of several numerical integrations. To this end, the Monte Carlo sampling [19, 15] was used in [6], the Latin hypercube sampling [21] was used in [30], and the Gauss quadrature on the standard tensor product grid [18, 3] was used in [14]. To carry out the numerical integrations, it is unfortunately necessary to solve the governing equation at each quadrature point, which is computationally intensive. On the other hand, applying the method consisting in minimizing the residual of the governing equation [26] to a linear statics problem of the form

$$K(\boldsymbol{\xi})u(\boldsymbol{\xi})=f(\boldsymbol{\xi})\ , \quad K(\boldsymbol{\xi})\in\mathbb{R}^{n\times n}, \quad u(\boldsymbol{\xi}),f(\boldsymbol{\xi})\in\mathbb{R}^n\ , \tag{7}$$

where

$$K(\boldsymbol{\xi})=\sum_{i=0}^{L-1}K^{(i)}\psi_i(\boldsymbol{\xi})\ , \qquad f(\boldsymbol{\xi})=\sum_{i=0}^{M-1}f^{(i)}\psi_i(\boldsymbol{\xi})\ , \qquad u(\boldsymbol{\xi})=\sum_{i=0}^{P-1}u^{(i)}\psi_i(\boldsymbol{\xi})\ ,$$
$$P\geq L,M, \quad K^{(i)}\in\mathbb{R}^{n\times n}, \quad f^{(i)},u^{(i)}\in\mathbb{R}^n\ , \tag{8}$$

yields a system of linear deterministic equations of the form

$$\boldsymbol{K}\boldsymbol{u}=\boldsymbol{f}\ , \quad \boldsymbol{K}\in\mathbb{R}^{nP\times nP}, \quad \boldsymbol{u},\boldsymbol{f}\in\mathbb{R}^{nP}\ , \tag{9}$$

where

$$\boldsymbol{K} = \begin{bmatrix} \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_0 \psi_0 \rangle & \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_1 \psi_0 \rangle & \dots & \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_{P-1} \psi_0 \rangle \\ \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_0 \psi_1 \rangle & \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_1 \psi_1 \rangle & \dots & \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_{P-1} \psi_1 \rangle \\ \dots & \dots & \dots & \dots \\ \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_0 \psi_{P-1} \rangle & \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_1 \psi_{P-1} \rangle & \dots & \sum_{i=0}^{L-1} K^{(i)} \langle \psi_i \psi_{P-1} \psi_{P-1} \rangle \end{bmatrix} ,$$

$$\boldsymbol{u} = \begin{Bmatrix} u^{(0)} \\ u^{(1)} \\ . \\ u^{(P-1)} \end{Bmatrix} , \qquad \boldsymbol{f} = \begin{Bmatrix} \langle \psi_0^2 \rangle f^{(0)} \\ \langle \psi_1^2 \rangle f^{(1)} \\ . \\ \langle \psi_{M-1}^2 \rangle f^{(M-1)} \\ . \\ 0 \end{Bmatrix} .$$

Eq. (9) can be solved efficiently by exploiting the sparsity of the matrix $\boldsymbol{K}$ and using an appropriate iterative solver as described in Section 6. Once the coefficients $u^{(i)}$ are estimated, any statistical moment and the pdf of $u(\boldsymbol{x}, \boldsymbol{\xi})$ can be computed using Eq. (6). For example, the mean (the first moment) is

$$\langle u(\boldsymbol{x}, \boldsymbol{\xi}) \rangle = u^{(0)}(\boldsymbol{x}) \ ,$$

and the standard deviation (square root of the second moment about the mean) is

$$stdev\left(u(\boldsymbol{x}, \boldsymbol{\xi})\right) = \sqrt{\left\langle \left( \sum_{i=1}^{P-1} u^{(i)}(\boldsymbol{x}) \psi_i(\boldsymbol{\xi}) \right)^2 \right\rangle} \ ,$$

which, using the orthogonality of the chaos polynomials (4), simplifies to

$$stdev\left(u(\boldsymbol{x},\boldsymbol{\xi})\right) = \sqrt{\sum_{i=1}^{P-1} {u^{(i)}}^2(\boldsymbol{x})\,\langle\psi_i^2(\boldsymbol{\xi})\rangle}\ . \tag{10}$$

## 3. COMPUTATION OF STOCHASTIC STRAINS AND STRESSES

In the linear theory, the strain tensor $\mathcal{E}$ is related to the displacement field $u$ by

$$\mathcal{E} = \frac{1}{2}(\nabla u + \nabla u^T)\ . \tag{11}$$

The components of $\mathcal{E}$ can be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\right) \qquad i,j = 1,2,3\ . \tag{12}$$

For a linear isotropic material, the stress tensor components $\sigma_{ij}$ are derived from the strain tensor components $\varepsilon_{ij}$ using the constitutive relationships

$$\sigma_{ii} = \frac{E}{(1-2\nu)(1+\nu)}\{(1-\nu)\varepsilon_{ii} + \nu\varepsilon_{jj} + \nu\varepsilon_{kk}\} \qquad i \neq j \neq k\ , \tag{13}$$

and

$$\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} \qquad i \neq j\ , \tag{14}$$

where $E$ denotes Young's modulus and $\nu$ denotes Poisson's ratio. The von Mises stress can be computed as

$$\sigma_{VM} = \sqrt{I_1^2 - 3I_2}\ , \tag{15}$$

where

$$I_1 = \sigma_{ii} + \sigma_{jj} + \sigma_{kk} \quad \text{and} \quad I_2 = \sigma_{ii}\sigma_{jj} + \sigma_{jj}\sigma_{kk} + \sigma_{kk}\sigma_{ii} - \sigma_{ij}^2 - \sigma_{jk}^2 - \sigma_{ki}^2\ . \tag{16}$$

In the non-deterministic case, the displacement field can be represented in a form similar to Eq. (6) as

$$u_i(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{p=0}^{P-1} u_i^{(p)}(\boldsymbol{x})\psi_p(\boldsymbol{\xi}) \qquad i = 1,2,3. \tag{17}$$

The corresponding strain tensor components can be written as

$$\varepsilon_{ij}(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{p=0}^{P-1} \varepsilon_{ij}^{(p)}(\boldsymbol{x})\psi_p(\boldsymbol{\xi}) \ , \tag{18}$$

where the coefficients $\varepsilon_{ij}^{(p)}$ are given by

$$\varepsilon_{ii}^{(p)}(\boldsymbol{x}) = \frac{\partial u_i^{(p)}(\boldsymbol{x})}{\partial x_i} \quad \text{and} \quad \varepsilon_{ij}^{(p)}(\boldsymbol{x}) = \frac{1}{2}\left(\frac{\partial u_i^{(p)}(\boldsymbol{x})}{\partial x_j} + \frac{\partial u_j^{(p)}(\boldsymbol{x})}{\partial x_i}\right) \quad \text{for} \quad i \neq j\,; \qquad i,j = 1,2,3\,. \tag{19}$$

For simplicity, the arguments $\boldsymbol{x}$ and $\boldsymbol{\xi}$ are dropped in the remainder of this paper. The mean and standard deviation of the strain components $\varepsilon_{ij}$ are given by

$$\langle \varepsilon_{ij} \rangle = \varepsilon_{ij}^{(0)} \ , \tag{20}$$

and

$$stdev(\varepsilon_{ij}) = \sqrt{\sum_{p=1}^{P-1} \left(\varepsilon_{ij}^{(p)}\right)^2 \langle \psi_p^2 \rangle} \ . \tag{21}$$

The principal strains $\varepsilon_{pr_i}$ $(i = 1,2,3)$ are the eigenvalues of the tensor $\mathcal{E}$. Hence, they are highly nonlinear functions of the strain components $\varepsilon_{ij}$. Therefore, their mean and standard deviations cannot be easily computed analytically, for example, by using the orthogonality property of the chaos polynomials to simplify the expressions as is done in Eqs. (10) and (21). A numerical technique is needed to this effect. The same can be said about the computation of the mean and standard deviation of the von Mises strains.

If both $E$ and $\nu$ are deterministic quantities, then the mean and standard deviation of $\sigma_{ij}$ can be computed directly using the orthogonality property of chaos polynomials to obtain

$$\langle \sigma_{ii} \rangle = \frac{E}{(1-2\nu)(1+\nu)} \{ (1-\nu)\varepsilon_{ii}^{(0)} + \nu\varepsilon_{jj}^{(0)} + \nu\varepsilon_{kk}^{(0)} \} \ , \tag{22}$$

$$stdev(\sigma_{ii}) = \frac{E}{(1-2\nu)(1+\nu)} \sqrt{\sum_{p=1}^{P-1} \{ (1-\nu)\varepsilon_{ii}^{(p)} + \nu\varepsilon_{jj}^{(p)} + \nu\varepsilon_{kk}^{(p)} \}^2 \left\langle \psi_p^2 \right\rangle} \ , i \neq j \neq k \ , \tag{23}$$

$$\langle \sigma_{ij} \rangle = \frac{E}{(1+\nu)} \varepsilon_{ij}^{(0)} \qquad i \neq j \ , \tag{24}$$

$$stdev(\sigma_{ij}) = \frac{E}{(1+\nu)} \sqrt{\sum_{p=1}^{P-1} \left( \varepsilon_{ij}^{(p)} \right)^2 \left\langle \psi_p^2 \right\rangle} \qquad i \neq j \ . \tag{25}$$

In the remainder of this paper, the approach for computing the statistical moments of a post-processing quantity of the non-deterministic solution of a problem based on exploiting the orthogonality property of the chaos polynomials will be referred to as Integration Using Orthogonality (IUO).

The computation of the statistical moments of the von Mises stress is more complicated, primarily because of its highly nonlinear dependence on the stress tensor components, as shown in Eqs. (15) and (16). If both $E$ and $\nu$ are deterministic, only $\left\langle \sigma_{VM}^2 \right\rangle$ can be evaluated directly by IUO; other moments cannot.

If only $E$ is random and admits an $L_E$-term representation of the form

$$E = \sum_{p=0}^{L_E-1} E^{(p)} \psi_p \ ,$$

then from Eqs. (13),(14), and (18) the expressions of the stress components become

$$\sigma_{ii} = \frac{1}{(1-2\nu)(1+\nu)} \sum_{p=0}^{L_E-1} E^{(p)}\psi_p \sum_{q=0}^{P-1} \{(1-\nu)\varepsilon_{ii}^{(q)} + \nu\varepsilon_{jj}^{(q)} + \nu\varepsilon_{kk}^{(q)}\}\psi_q \qquad i \neq j \neq k \ , \quad (26)$$

and

$$\sigma_{ij} = \frac{1}{(1+\nu)} \sum_{p=0}^{L_E-1} E^{(p)}\psi_p \sum_{q=0}^{P-1} \varepsilon_{ij}^{(q)}\psi_q \qquad i \neq j \ . \quad (27)$$

In this case, the IOU approach can be used to compute the mean stress $\langle\sigma_{ij}\rangle$ and gives

$$\langle\sigma_{ii}\rangle = \frac{1}{(1-2\nu)(1+\nu)} \sum_{p=0}^{L_E-1} E^{(p)}\{(1-\nu)\varepsilon_{ii}^{(p)} + \nu\varepsilon_{jj}^{(p)} + \nu\varepsilon_{kk}^{(p)}\} \langle\psi_p^2\rangle \qquad i \neq j \neq k \ , \quad (28)$$

and

$$\langle\sigma_{ij}\rangle = \frac{1}{(1+\nu)} \sum_{p=0}^{L_E-1} E^{(p)}\varepsilon_{ij}^{(p)} \langle\psi_p^2\rangle \qquad i \neq j \ . \quad (29)$$

Note that the upper limit of the above summations are $L_E - 1$ because $P > L_E$ as in Eq. (8).

However, expressions for any nonlinear functions of $\sigma_{ij}$ such as $\sigma_{ij}^2$ or the von Mises stress immediately become complicated to the point where their statistical moments cannot be evaluated by the IUO approach. More specifically, nonlinear functions of $\sigma_{ij}$ contain terms that are products of three or more chaos polynomials $\psi_i$. The orthogonality property of the chaos polynomials does not simplify the evaluation of the statistical moments of such terms. Furthermore, if $\nu$ becomes a random quantity, none of the statistical moments of any stress can be evaluated by the IUO method.

From the above discussion, it follows that only in some cases a few statistical moments of some strains and stresses can be computed analytically using the orthogonality property of the chaos polynomials. These quantities are reported in Table I, along with the conditions under which they can be computed by the IUO approach. The terms in the third column can be evaluated as indicated when the parameters in columns 1 and 2 are deterministic and random,

respectively. For example, if $\nu$ is deterministic and $E$ is random, $\langle \sigma_{ij} \rangle$ can be computed using IUO, but other quantities such as $stdev(\sigma_{ij})$ and $\left\langle \sigma_{VM}^2 \right\rangle$ cannot. If both $E$ and $\nu$ are random, none of the statistical moments of $\sigma_{ii}$ and $\sigma_{VM}$ (except the trivial zero-th moments) can be evaluated directly using IUO.

Table I. Cases where the orthogonality property of the chaos polynomials can be used to compute the statistical moments of the stresses.

| Deterministic | Random | Quantities that can be evaluated by IUO (without requiring an algorithm to re-arrange a polynomial to PCE) |
|---|---|---|
| $E$, $\nu$ | | $\langle \sigma_{ij} \rangle$, $stdev(\sigma_{ij})$, $\left\langle \sigma_{VM}^2 \right\rangle$ $\quad i = 1,2,3;\ j = 1,2,3$ |
| $\nu$ | $E$ | $\langle \sigma_{ij} \rangle$ $\quad i = 1,2,3;\ j = 1,2,3$ |
| | $\nu$, $E$ | – |

Thus for computing the statistical moments of the terms that cannot be evaluated using the orthogonality property of chaos polynomials, a numerical integration procedure is needed. For this purpose, a few techniques are considered next and their computational efficiency and algorithmic complexity are contrasted. These techniques are: (i) Monte Carlo sampling, (ii) Latin hypercube sampling, (iii) Gauss quadrature on the standard tensor product grid, and (iv) Smolyak's quadrature.

## 4. NUMERICAL INTEGRATION

Consider the $d$-dimensional integral

$$I = \int_{\mathcal{D}} g(\boldsymbol{\xi}) p(\boldsymbol{\xi}) d\boldsymbol{\xi} \ , \tag{30}$$

where $\mathcal{D}$ is the domain of integration, $g(\boldsymbol{\xi}) : \mathbb{R}^d \to \mathbb{R}$ is a given integrand, and $p(\boldsymbol{\xi}) : \mathbb{R}^d \to \mathbb{R}$ a given weighting function. Note that here the dimension of the integral, $d$, is chosen to be equal to the stochastic dimension of the random system defined in Section 2. If $p(\boldsymbol{\xi})$ is the joint probability density function of a random vector $\boldsymbol{\xi}$, $I$ gives the statistical moment of the function $g(\boldsymbol{\xi})$.

A numerical technique for evaluating the above integral can be written as

$$\hat{I} = \sum_{i=1}^{N} w_i g(\boldsymbol{\xi}_i) \ , \tag{31}$$

where $\boldsymbol{\xi}_i$ are quadrature points and $w_i$ are corresponding weights constructed to accelerate the convergence of $\hat{I}$ to $I$ as $N$ is increased.

Numerical integration procedures can be grouped into two different classes, namely, deterministic methods [18, 3] and probabilistic ones [19, 15]. In a probabilistic numerical integration method, the points $\boldsymbol{\xi}_i$ are often referred to as realizations of the random vector $\boldsymbol{\xi}$ and $N$ is called the sample size.

The evaluation of multi-dimensional integrals deserves special attention in order to manage the computational cost which increases with the dimension. An earlier survey of this topic can be found in [28]. Here two probabilistic methods, namely, Monte Carlo sampling and Latin hypercube sampling, and two deterministic methods, namely, Gauss quadrature with the standard tensor product and Smolyak's quadrature are examined.

### *4.1. Monte Carlo (MC) sampling*

*Description.* In this numerical method, the quadrature points $\boldsymbol{\xi}_i$ are produced by a random number generator so that $p(\boldsymbol{\xi})$ is their probability density function. The weights $w_i$ are set to

$1/N$[19, 15]. Therefore, the MC approximation of the integral (30) can be written as

$$\hat{I}_{MC} = \frac{\sum_{i=1}^{N} g(\boldsymbol{\xi}_i)}{N} \ . \tag{32}$$

The resulting numerical integration procedure is illustrated here with the computation of the mean of the von Mises stress.

First, realizations of the random vector $\boldsymbol{\xi}$ are generated using a random number generator. Each of them corresponds to a realization of the given random system model. For each of these realizations, the following steps are followed. First, the realization of each strain tensor component is computed using Eq. (18). Then, the realization of each material property such as $E$ and $\nu$ is computed to generate the corresponding realization of the constitutive matrix. Next, these results are used to compute the realizations of the stress tensor components. Then, the realization of the von Mises stress is computed using Eqs. (15) and (16). Thus, $g(\boldsymbol{\xi}_i)$ in Eq. (32) represents in this case the $i^{th}$ realization of the von Mises stress. Finally, the mean value of the von Mises stress is computed over $N$ realizations using Eq. (32).

The MC method illustrated above is straightforward to implement. However, it requires a large number of realizations to produce an accurate estimate of the statistical moments. Thus, it is computationally intensive.

*Convergence.* The error $(I - \hat{I}_{MC})$ in the MC approximation is a random variable with

$$\text{mean} \ = \ \left\langle I - \hat{I}_{MC} \right\rangle = 0 \ , \tag{33}$$

$$\text{and variance} \ = \ \left\langle (I - \hat{I}_{MC})^2 \right\rangle = \frac{(stdev(g))^2}{N} \ , \tag{34}$$

where $stdev(g)$ is the standard deviation of $g$ [19, 15]. The probability distribution of this error converges to the Gaussian distribution.

The zero-mean property is referred to as unbiasedness of the estimator $\hat{I}_{MC}$. The variance (34) represents the strength of the integration error. Thus, the convergence of the MC method is of the order $stdev(g)/O(\sqrt{N})$. This convergence has the advantage of being independent of the dimension of the integration [2], but is considered by many to be slow.

For most physical systems, the variability of the response increases with the number of uncertain parameters characterized by the stochastic dimension $d$. In this case, $stdev(g)$ also increases and according to Eq.(34), the required sample size also increases. Furthermore, the smoothness of the function $g(\boldsymbol{\xi})$ may vary with the stochastic dimension, which may affect convergence [19, 31].

### *4.2. Latin hypercube sampling*

*Description.* The Latin Hypercube Sampling (LHS) method differs from the MC method only in the choice of the quadrature points — that is, the realizations of the random vector $\boldsymbol{\xi}$. LHS [21] chooses the quadrature points as to achieve a faster convergence than the standard MC method. More specifically, LHS is a variance reduction technique [24, 19] that tries to reduce the variance of the estimator $\hat{I}$ so that

$$var(\hat{I}_{LHS}) < var(\hat{I}_{MC}) \ , \tag{35}$$

where $var$ denotes the variance and $\hat{I}_{LHS}$ denotes the integral estimator obtained using the LHS. This implies that

$$var(I - \hat{I}_{LHS}) < var(I - \hat{I}_{MC}) \ . \tag{36}$$

Since the variance of the error is a measure of its strength, a reduction in variance reduces the strength of the error. As a result, for any fixed error level, LHS requires a smaller sample size than MC to estimate the integral.

*Convergence.* Both MC and LHS lead to an unbiased estimate of the integral [21], which means that the mean value of the estimator $\hat{I}$ is the exact value of the integral $I$. It is proved in [21] that if the integrand is monotonic in all its arguments, the variance of the estimator obtained by LHS is lesser than that of the estimator obtained using MC. Furthermore, it is shown in [20] that the monotonicity condition can be relaxed in the asymptotic case — that is, when $N \to \infty$. Thus, LHS converges in principle faster than MC.

### *4.3. Standard Gauss quadrature*

*Description.* This is a deterministic quadrature rule [18, 3]. Recall that $p(\boldsymbol{\xi})$ is the distribution of a Gaussian random vector. Gauss quadrature with a Gaussian distribution as a weighting function is also known as Gauss-Hermite quadrature.

In one dimension, an $m_i$-point quadrature rule for a function $g$ of (one variable) $\xi_i$ can be written as

$$U^i(g) = \sum_{j=1}^{m_i} a_j^i g(\xi_i^j) \ , \tag{37}$$

where $\xi_i^j$ denotes the $j^{th}$ quadrature point and $a_j^i$ denote the associated weight. Such an approximation is exact for a polynomial integrand of degree less or equal to $(2m_i - 1)$. Its extension to a $d$-dimensional integration can be written using the Kronecker product as follows

$$\hat{I} = (U^{i_1} \otimes \cdots \otimes U^{i_d})(g) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_d=1}^{m_{i_d}} (a_{j_1}^{i_1} \otimes \cdots \otimes a_{j_d}^{i_d}) g(\xi_{i_1}^{j_1} \cdots \xi_{i_d}^{j_d}) \ , \tag{38}$$

where $m_{i_1}, \ldots, m_{i_d}$ are the number of quadrature points used for defining the integration rule in the variables $\xi_{i_1}, \ldots, \xi_{i_d}$, respectively. Thus, the computation of the above integral requires $(m_{i_1} \cdots m_{i_d})$ function evaluations. In the particular case corresponding to $m_{i_1} = m_{i_2} = \cdots = m_{i_d} = m$, $m^d$ function evaluations are required. Hence, the computational burden increases significantly with the dimension $d$. This is often referred to as the *curse of dimensionality.*

*Convergence.* The aforementioned property of exact integration of polynomials in one dimension extends to multiple dimensions as follows. A function $g(\boldsymbol{\xi})$ is said to be of smoothness $r$ if its first $r$ mixed derivatives are bounded. This can be written as

$$g : \mathbb{R}^d \to \mathbb{R} \ / \ \max_{|\alpha| \leq r} \|g^{(\alpha)}\|_\infty < \infty \ , \tag{39}$$

where

$$\alpha = \{\alpha_1, \ldots, \alpha_d) \ \text{ is an } n\text{-tuple of non-negative integers,} \qquad |\alpha| = \sum_{i=0}^{d} \alpha_i \ ,$$

and

$$g^{(\alpha)}(\boldsymbol{\xi}) = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_d^{\alpha_d}} g(\boldsymbol{\xi}) \ .$$

For such integrands, the convergence of the Gauss quadrature method is dictated by the error term $\epsilon(N) = \mathcal{O}(N^{-r/d})$ [31, 9].

### *4.4. Smolyak's quadrature*

*Description.* Unlike the Gauss quadrature method, Smolyak's quadrature [29] — also referred to as the Smolyak cubature in the literature — uses recursive contributions of lower order tensor products for estimating the value of the integral $I$, in order to avoid the *curse of dimensionality*. More specifically, this method estimates the value of $I$ as follows

$$\hat{I}_{q,d} = \sum_{q-d+1 \leq |\boldsymbol{i}| \leq q} (-1)^{q-|\boldsymbol{i}|} \binom{d-1}{q-|\boldsymbol{i}|} (U^{i_1} \otimes \cdots \otimes U^{i_d})(g) \ , \tag{40}$$
$$\boldsymbol{i} = \{i_1, \ldots, i_d\}, \quad |\boldsymbol{i}| = i_1 + \cdots + i_d \ .$$

Here, $\hat{I}_{q,d}$ is the quadrature rule, $q \geq d$ is a parameter to be selected, $\boldsymbol{i}$ is a $d$-dimensional index set, $U^{i_k}$ are one-dimensional quadrature rules as in Eq. (37), and $m_{i_j} \neq m_{i_k}$ for $i_j \neq i_k$.

A closer look at Eq. (40) reveals that one-dimensional quadrature rules are permuted in all dimensions. These rules can be chosen as Gauss quadrature rules, with $m_{i_1} = 1, m_{i_2} = 2, \ldots$. Computational cost saving in Smolyak's quadrature compared to the standard Gauss quadrature rule increases as the dimension of the problem, $d$, grows.

*Convergence.* It is proved in [10] that if every univariate quadrature rule $U^i$ in Eq. (37) can integrate any univariate polynomial of degree less or equal to $(2m_i - 1)$ exactly, then Smolyak's quadrature (40) computes exactly the integral of a $d$-variate polynomial of total degree less or equal to $(2M - 1)$, where

$$M = \max_{i_1 \leq i_k \leq i_d} m_{i_k} \ . \tag{41}$$

In general, a bound on the error associated with Smolyak's approximation is given in [29] as

$$|I_d - \hat{I}_{q,d}| \leq \mathcal{O}(q^{(d-1)} \cdot 2^{-r \cdot q}) \|g\| \ ,$$

where

$$\|g\| = \max_{|\alpha| \leq r} \|g^{(\alpha)}\|_\infty \ . \tag{42}$$

Furthermore, if $m_i \leq 2^i$, which is true for the standard Gauss quadrature, this error bound is further refined in [8] as

$$|I_d - \hat{I}_{q,d}| \leq \mathcal{O}(n^{-r} \cdot (\log n)^{(d-1)\cdot(r+1)}) \|g\| \ ,$$

where $n = n(q, d)$ denotes the number of points used by $\hat{I}_{q,d}$. Further error analysis of this method can be found in [12, 8, 17, 16].

## 5. POLYNOMIAL EXPANSION AND DIMENSION REDUCTION

When the integrand $g(\boldsymbol{\xi})$ is a polynomial of degree *deg* that is less than or equal to the stochastic dimension $d$ of the problem, the cost of the numerical integration can be reduced significantly by using a reduced dimension integration as described herein.

In the remainder of this paper, the *effective dimension* of a function is defined as the number of independent random variables $\xi_i$ explicitly present in this function. For example, consider a problem that has stochastic dimension $d = 8$ — that is, there are total eight random variables $(\xi_1, \cdots, \xi_8)$ that completely characterize the uncertainty in the system. In the polynomial representation of a random output quantity such as displacement or strain, consider a term of the form $C\xi_5^4\xi_8$, where $C$ is the deterministic coefficient. According to the definition, the effective dimension of this term is two, since it involves only two independent random variables, $\xi_5$ and $\xi_8$. In this case, the integral

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} C\xi_5^4\xi_8 \; p(\xi_1, \cdots, \xi_8) \; d\xi_1 \cdots d\xi_8 \tag{43}$$

can actually be computed as

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C\xi_5^4\xi_8 \; p(\xi_5, \xi_8) \; d\xi_5 d\xi_8, \tag{44}$$

where $p(\xi_i, \cdots, \xi_j)$ denotes the joint pdf of the variables $\xi_i, \cdots, \xi_j$. Thus in this case, the dimension of the integral is reduced from eight to two, which reduces the complexity of the numerical approximation by a factor four.

The above concept for reducing the dimension of a given integral can be extended to the integration of polynomial expansions. The stochastic dimension remaining as $d$, let a random

quantity such as a stress field be expressed in a polynomial expansion of degree $deg$. If $deg < d$, then any term in this expansion will involve at most $deg$ independent random variables $\xi_i$. Thus, the highest effective dimension of any term in the expansion will be $deg$, and not $d$. In this case, as in Eqs. (43) and (44), any term in the given expansion can be numerically integrated using quadrature points along at most $deg$ directions. This is indeed computationally more efficient than integrating each term using quadrature points along all of the $d$ directions. In the remainder of this paper this reduced dimension integration will be referred to as the Integration in Reduced Dimension (IRD). When such a reduction in dimension is not performed and the integration is carried out instead on the full dimension $d$, the integration will be referred to as Integration in Full Dimension (IFD).

A nomenclature consistent with the existing literature convention is defined next. When a numerical integration technique uses equal number of quadrature points along all directions, the resulting tensor product will be referred to as *isotropic tensor product*. The tensor product using different number of quadrature points along different directions will be referred to as *anisotropic tensor product* [11]. Note that the standard tensor product can either be isotropic or anisotropic, depending upon whether Eq. (38) $m_{i_j}$-s are equal or not. However, the Smolyak's quadrature defined in Eq. (40) is always isotropic.

The steps of implementing the IRD are as follows

- **S1:** Initialize $\hat{I} = 0$.
- **S2:** Consider the set of polynomials of a particular degree $deg$.
- **S3:** Generate the quadrature points on the isotropic tensor product grid in $deg$ dimensions. The underlying univariate quadrature rule must be able to integrate any univariate polynomial of degree less or equal to $deg$ exactly.

**S4:** Evaluate the polynomials and add their contribution to $\hat{I}$. Update $deg = deg + 1$ and go back to step **S2**.

To implement the reduced dimension integration, an algorithm is needed to identify the random variables $\xi_i$-s that are explicitly present in a given term. For example, identifying the indices 5 and 8 in the term $C\xi_5^4\xi_8$. This can be achieved while numerically constructing the PCE bases, as suggested in [26, 13]. However, the degrees of the individual $\xi_i$-s cannot be obtained explicitly during this construction. This justifies why in step **S3** (i) an isotropic tensor product grid is chosen, (ii) the constraint imposed on the underlying univariate quadrature rule is needed. Among all schemes considered in this paper, the Gauss-Hermite quadrature rule in the standard tensor product form is the most appropriate one for performing the IRD, considering its overall computational complexity. Smolyak's quadrature rule is not considered in this case because of its relatively high computational cost for (high-degree) polynomials.

The idea of the reduced dimension integration can be viewed as a combination of the anisotropic and isotropic tensor products. The anisotropic part corresponds to the elimination of a few dimensions. On the other hand, for a given dimension $deg$, only the isotropic tensor product is used in step **S3**.

An example of calculation that can benefit from IRD is the evaluation of $\langle \sigma_{VM}^2 \rangle$ (see Eqs. (15) and (16)). In this case, if $\nu$ is deterministic, $E$ is modeled using a first-order PCE and the displacement field $u$ is expressed in $deg^{th}$-order PCE, the maximum degree of any polynomial in the expression of $\sigma_{VM}^2$ is $2(deg+1)$. If $2(deg+1) \leq d$, where $d$ refers again to the stochastic dimension of the problem, a quadrature rule for $2(deg+1)$ dimensional integration is sufficient for the numerical evaluation of $\langle \sigma_{VM}^2 \rangle$.

A summary and classification of the integration techniques described so far in this paper is

given in Figure 1.

## 6. NUMERICAL EXAMPLES

Two numerical examples are discussed in this Section. In the first one, the numerical integration of a polynomial expansion is considered using three deterministic integration techniques, namely, Gauss quadrature on the standard tensor product grid in IFD mode, Gauss quadrature on the standard tensor product grid in IRD mode, and Smolyak's quadrature. The accuracy and computational costs of all three techniques are compared. In the second example, a plate problem with random material properties is considered and the first two statistical moments of various strain and stress fields are computed using some of the numerical techniques described in this paper. Various computational performance issues are also discussed.

In both examples, the Gauss-Hermite quadrature method is used as the basic univariate integration rule for the deterministic integration techniques — that is, as the $U^i(g)$ function in Eq. (37) and as $U^{i_k}(g)$, $k = 1, \cdots, d$, in Eq. (40).

### *6.1. Numerical integration of a fourth-degree polynomial*

Here, attention is focused on the Gauss quadrature on the standard tensor product grid in IFD mode, the Gauss quadrature on the standard tensor product grid in IRD mode and Smolyak's quadrature. Figure 2 reports the variation with the stochastic dimension of the problem of the number of quadrature points needed by these three deterministic techniques in order to integrate exactly the fourth-order PCE of a random function. An example of such a random function is $\sigma_{ii}^2$ obtained from Eq. (26), when $\nu$ is deterministic and both $E$ and $u$ are represented using first-order PCEs. When the Gauss quadrature on the standard tensor

product in IFD mode and Smolyak's quadrature are used, all of the terms in the expansion must be evaluated at each quadrature point. On the other hand when using the Gauss quadrature with standard tensor product in IRD mode, the numerical integration can be performed term by term using selectively the quadrature points and therefore is computationally less expensive. Thus in Figure 2, the plots for Smolyak's quadrature and IFD are indicative of the actual computational time but the plot for IRD is indicative of an upper bound of the associated computational time.

Figure 2 reveals that for a stochastic dimension greater than 10, the Gauss quadrature on standard tensor product in IRD mode is a significant improvement over that in IFD mode as far as reducing the number of quadrature points needed to compute exactly the PCE. This improvement becomes more significant when the dimension increases. For a stochastic dimension greater than 5, Smolyak's quadrature needs the least number of quadrature points among all three techniques. However, Smolyak's quadrature is the most difficult and the standard tensor product in IFD mode the easiest to implement among the three considered schemes. The main complexity of Smolyak's quadrature stems from the need to develop recursive subroutines for generating the quadrature points.

### *6.2. Numerical estimation of statistical moments of stress and strain fields for a plate problem*

The objective here is to compare the accuracy and computational speed of some of the numerical integration techniques described in this paper when applied to the calculation of a few statistical moments of the stress and strain fields associated with a statically loaded square plate with random material properties. The plate is $2.3m \times 2.3m$ and has a thickness equal to $5mm$. It is made of 20 metal strips joined side by side along the edges. The dimensions of

each strip are $2.3m \times 0.115m \times 5mm$ (see Figure 3). Young's modulus of the strips is assumed to be random and is modeled as

$$E_i = \bar{E}_i + stdev_{E_i}\xi_i \qquad i = 1, \ldots, 20 \ , \tag{45}$$

where $\bar{E}_i$ and $stdev_{E_i}$ are the mean and standard deviation of Young's modulus of the material in strip $i$ and the $\xi_i$s are independent normal random variables. Thus, the stochastic dimension $d$ of this problem is 20. Since the stiffness matrix is linearly dependent on Young's modulus, $L = 20$ in Eq. (8).

The plate is fixed along its four edges. A finite element (FE) model with 400 four-noded quadrilateral elements is constructed. Among the twenty different random materials, ten have a Young's modulus with a mean value equal to $2.0 \times 10^5 MPa$ and ten with a mean value equal to $2.1 \times 10^5 MPa$. The standard deviations for all materials are assumed to be equal to 20% of their respective mean values. The density of the plate material is assumed to be equal to $7860 Kg/m^3$, which is the same as for steel. The plate is subjected to a linear static loading resulting from its self-weight and three concentrated external forces applied at its center: two in-plane forces equal to $400KN$ and $300KN$, respectively, and a transverse force equal to $2KN$. The random displacement field is represented using a second-order PCE as in Eq. (6). The expansion has a total of $P = 231$ chaos polynomials. The chaos coefficients are computed by solving Eq. (9) where $\boldsymbol{K}$ is block-sparse, using a Block-Jacobi preconditioned conjugate gradient (PCG) method.

After estimating the chaos coefficients $u^{(i)}$ of the displacement field, the mean and standard deviation of the von Mises stress — $\bar{\sigma}_{VM}$ and $stdev(\sigma_{VM})$ are computed using MC, LHS and Smolyak's quadrature. The values of these quantities at an arbitrary node of the FE model are plotted in Figures 4 and 5, respectively, as functions of the number of quadrature points used.

It is observed that when a large number of quadrature points are used, the estimates of these statistical moments obtained using these three techniques are almost same. For example, the standard deviation computed by Smolyak's quadrature is plotted in Figure 6. The estimates obtained using MC and LHS look similar and therefore are not plotted separately. All three techniques incur the evaluation of the chaos polynomials at every quadrature point [26, 13]. The random numbers are generated using Matlab [1]. Usually, the computer-generated random numbers do not satisfy the orthonormal properties for a finite sample size. Therefore, these numbers are orthonormalized using a transformation approach[4] and the new numbers are used for MC- and LHS-related computations.

The evaluation of $stdev(\sigma_{VM})$ requires the computation of $< \sigma_{VM}^2 >$ as an intermediate step. From Eq. (15), it follows that $\sigma_{VM}^2$ is a polynomial of degree 6. From Eq. (41), it follows that Smolyak's quadrature rule requires $M = 4$ points for each random variable to compute exactly $< \sigma_{VM}^2 >$. Since the stochastic dimension of this problem is $d = 20$, this implies a total of 135751 quadrature points.

From Figures 4 and 5, the following observations can be made:

1. Whereas Smolyak's quadrature requires 135751 quadrature points to compute $< \sigma_{VM}^2 >$ exactly ($M = 4$), 12341 quadrature points ($M = 3$) suffice in practice to obtain a very good estimate of $< \sigma_{VM}^2 >$ and $stdev(\sigma_{VM})$. This trend was also observed for most of the other FE nodes as well. There were only a few nodes for which increasing $M$ from 3 to 4 noticeably affected the estimation of the integral. Therefore in the expansion of $\sigma_{VM}^2$, the contributions from the terms with higher degree are very small for most of the nodes.
2. All three numerical techniques lead to comparable values of the considered statistical

moments.

Finally, it is noted that the transformation of the machine-generated random numbers mentioned above helped improving the convergence of MC and LHS. For the problem considered here, both methods delivered comparable performances. A possible explanation is that in this case, the integrands are not monotonic functions of the arguments and therefore variance reduction in LHS is not guaranteed [21].

## 7. CONCLUDING REMARKS

The statistical moments of the strains and stresses of a system with uncertain properties can be computed from a given polynomial chaos expansion (PCE) of the displacement field. A few such moments can be computed analytically, using the orthogonality property of the chaos polynomials. However in most cases, a numerical integration algorithm is needed for this purpose. To this effect, either a deterministic or a stochastic technique can be used. The stochastic dimension of the system — that is, the number of basic random variables involved in modeling uncertainty — plays an important role in selecting the numerical integration technique. If this dimension is very low, say in the range of one to five, the computational cost is moderate for most available numerical schemes. The Gauss-Hermite quadrature with the standard tensor product rule, which is perhaps the simplest algorithm to implement, can be used in this case to achieve a good accuracy at relatively low cost. As the stochastic dimension increases, the computational cost of this method becomes prohibitive and other techniques such as Monte Carlo sampling, Latin hypercube sampling, or Smolyak's quadrature become better alternatives. A reduced dimension deterministic integration scheme was also introduced

in this paper for those cases where a polynomial form of the integrand of degree less or equal to the stochastic dimension of the problem is obtainable. When applicable, this scheme reduces significantly the computational cost of the numerical integration. It can be considered as a trade-off between the computational cost of Smolyak's quadrature and the implementational simplicity of the full dimension Gauss-Hermite quadrature with the standard tensor product rule. A roadmap for the choice of a particular integration method is given in Figure 7. The implementations of Monte Carlo and Latin hypercube sampling are similar to that of the Gauss-Hermite quadrature with the standard tensor product, with an added complexity to generate the random numbers. The implementation of Smolyak's quadrature is however more complex. The numerical error associated with sampling-based techniques is in general random and therefore can possibly exceed a desired deterministic tolerance level. However, the probability of this event decreases when the number of quadrature points included in the numerical integration technique is increased. For example, for a simple non-deterministic plate bending problem considered in this paper, both the Monte Carlo and Latin hypercube sampling methods performed as satisfactorily as Smolyak's quadrature method.

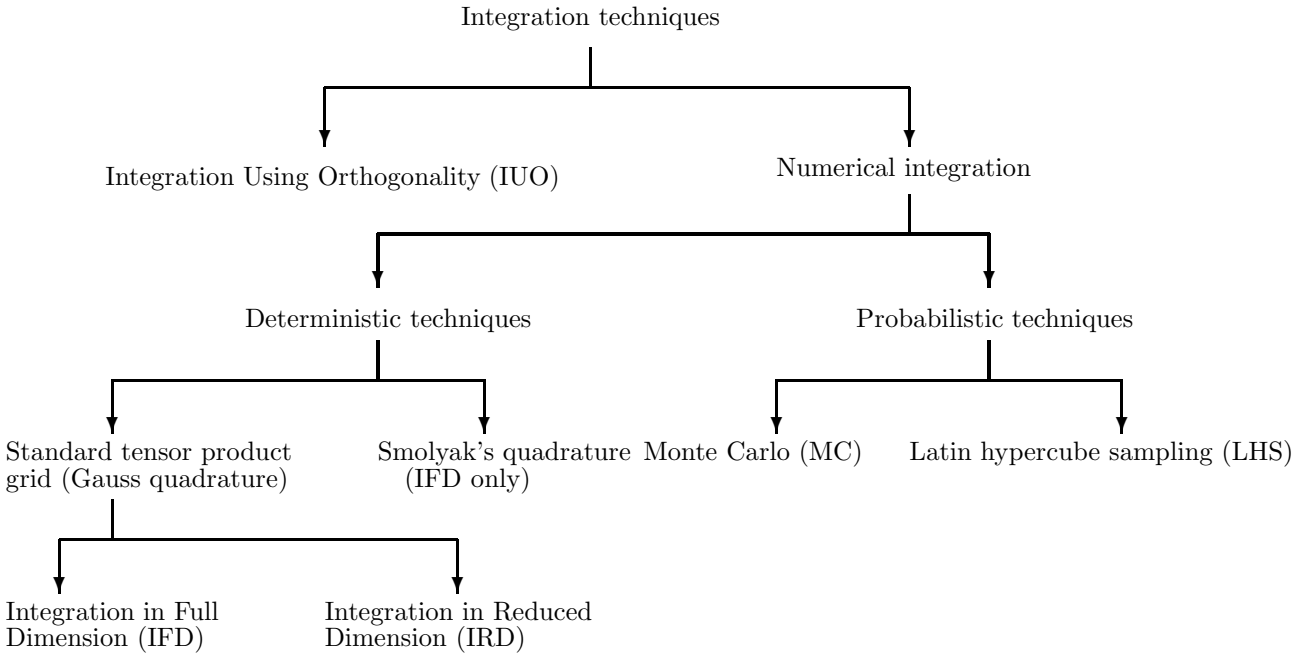


Figure 1. Classification of the integration techniques used in this paper.

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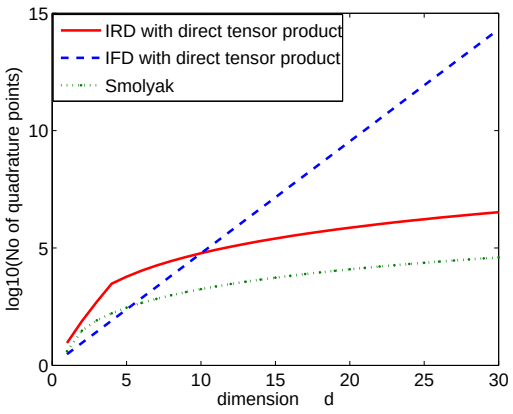


Figure 2. Number of quadrature points required by three different deterministic integration schemes to compute exactly the fourth-order polynomial expansion of a random function.

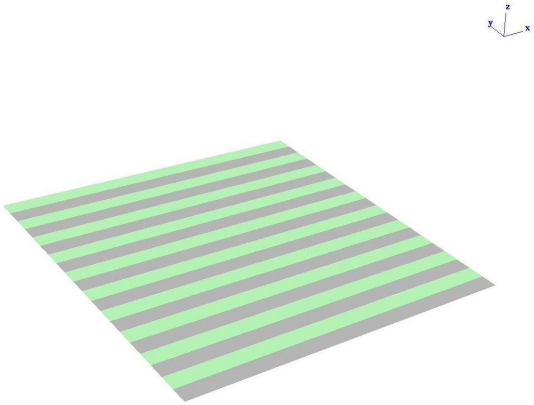

Figure 3. Plate problem.

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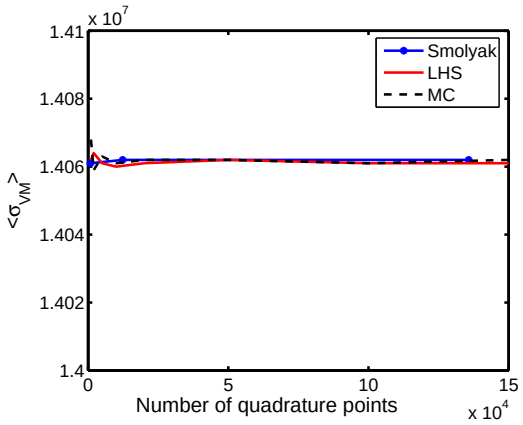


Figure 4. Mean von Mises stress $\bar{\sigma}_{VM}$ (in Pa) at an arbitrarily chosen node: convergence of various schemes.

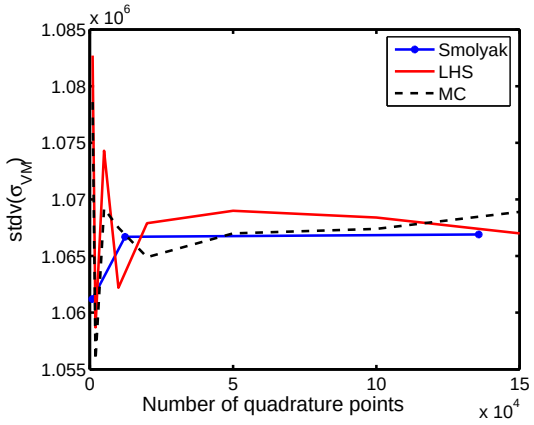


Figure 5. $stdev(\sigma_{VM})$ (in Pa) at the same node as in previous figure: convergence of various schemes.

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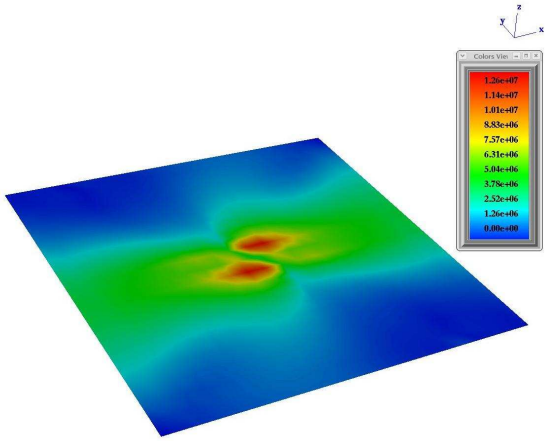


Figure 6. $stdev(\sigma_{VM})$ (in Pa): computation using Smolyak's quadrature with $M = 4$.

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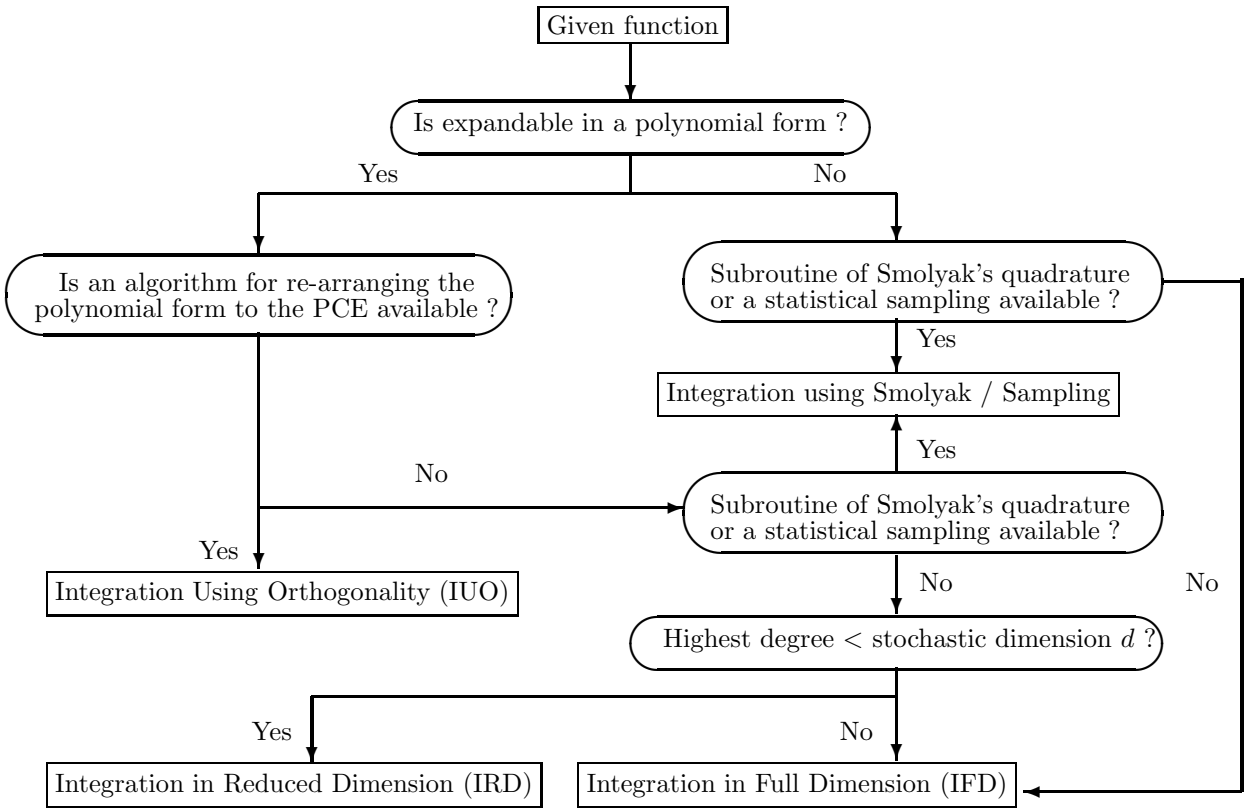


Figure 7. A guideline for choosing an integration technique.

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