Topological Properties of Orthostochastic Matrices*

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ABSTRACT

The set of $n\times n$ orthostochastic matrices with the topology induced by the Euclidean matric is shown to be compact and path-connected. For $n<3$, the set of orthostochastic matrices is identical to the set of doubly stochastic matrices. In this paper, it is shown that for $n\geqslant 3$ the orthostochastic matrices are not everywhere dense in the set of doubly stochastic matrices, thus answering a question of L. Mirsky in his survey article on doubly stochastic matrices [2].

Denote by $\mathcal{U}(n)$ the set of $n\times n$ unitary matrices, and let $\mathcal{D}(n)$ represent the set of doubly stochastic (d.s.) matrices (those matrices with nonnegative real entries and row and column sums equal to 1). Define a matrix function f from the set of $n\times n$ complex matrices to the set of $n\times n$ real matrices

$$f: C^{n\times n}\to R^{n\times n}, \quad \text{by} \quad f(A)=|a_{ij}|^2.$$

PROPOSITION. *f is continuous with respect to the usual topologies.*

Equivalently, considering $R^{n\times n}$ with the product topology, $\pi_{ij}\circ f$ is continuous for all projections π_{ij} of $R^{n\times n}$ onto R. But $(\pi_{ij}\circ f)(A)=|a_{ij}|^2=a_{ij}\bar{a}_{ij}$, so $\pi_{ij}\circ f$ is a composition of continuous functions; hence it is continuous.

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DEFINITION. A matrix O is called orthostochastic (o.s.) if and only if there exists a unitary matrix U satisfying $(|u_{ij}|^2)=(o_{ij})$.

The set of $n\times n$ o.s. matrices $\mathcal{O}(n)$ is just the continuous image of the set of $n\times n$ unitary matrices; that is, $\mathcal{O}(n)=f(\mathcal{U}(n))$. Consequently, information about $\mathcal{O}(n)$ follows from results for $\mathcal{U}(n)$.

LEMMA. *$\mathcal{U}(n)$ is a compact, path-connected subspace of $C^{n\times n}$.*

Proof.

(i) *Compactness.* $\mathcal{U}(n)\subset I^{n\times n}$, the unit cube, which is compact. So it suffices to establish that $\mathcal{U}(n)$ is closed. Introduce a notation for the inner product of two columns of a complex matrix by

$$g_{ij}: C^{n\times n}\to C \qquad \text{with} \quad g_{ij}(A)=\sum_{k=1}^{n} a_{ki}\bar{a}_{kj}.$$

Then define

$$S_{ii}=\{A\in C^{n\times n}: g_{ii}(A)=1\}=g_{ii}^{-1}(\{1\})$$

$$S_{ij}=\{A\in C^{n\times n}: g_{ij}(A)=0\}=g_{ij}^{-1}(\{0\}) \qquad \text{for} \quad i\neq j.$$

The singleton sets $\{0\}$ and $\{1\}$ are closed in the metric space C. Furthermore, each g_{ij}, being a composition of continuous functions, is continuous. Hence, each S_{ij} is closed, since it is just the inverse image of a closed set under a continuous mapping. From the characterization of the set of unitary matrices as those complex matrices with orthonormal columns, it follows that $\mathcal{U}(n)=\cap_{i,j}S_{ij}$. Since $\mathcal{U}(n)$ is a finite intersection of closed sets, $\mathcal{U}(n)$ is closed and therefore compact.

(ii) *Path-connectedness.* It is first shown that the set of diagonal unitary matrices is path-connected. The unit circle in the complex plane, $S^1=\{z\in C: |z|=1\}$, is path-connected. Therefore the product set $\prod_{i=1}^{n}S^1$ is also path-connected. Now the diagonal unitary matrices (the complex matrices with zero entries everywhere except for elements of unit modulus along the main diagonal) form a subspace which is homeomorphic to $\prod_{i=1}^{n}S^1$.

By the spectral theorem, any $U\in\mathcal{U}(n)$ is unitarily similar to a diagonal unitary matrix:

$$U=V^{*}DV,$$

and accordingly,

$$\mathcal{U}(n)=\{U\in C^{n\times n}: U=V^{*}DV \text{ with } D,V\in\mathcal{U}(n) \text{ and } D \text{ diagonal}\}$$

$$=\bigcup_{V\in\mathcal{U}(n)}\left[\bigcup_{\substack{D\in\mathcal{U}(n)\\ \text{diagonal}}}\{V^{*}DV\}\right]$$

$$=\bigcup_{V} T_V, \qquad \text{where} \quad T_V=\bigcup_{\substack{D\in\mathcal{U}(n)\\ \text{diagonal}}}\{V^{*}DV\}.$$

Now T_V, the continuous image of the path-connected set of diagonal unitary matrices, is path-connected. Choosing $D=I$ in the similarity transformation gives that $I\in T_V$ for all V. Consequently, as is well known, $\mathcal{U}(n)$ is path-connected, since $\mathcal{U}(n)$ is the union of the path-connected sets T_V, each of which contains the common point I. ■

Employing standard theorems from topology, the results for $\mathcal{O}(n)$ are obtained readily, as illustrated below.

THEOREM 1. *$\mathcal{O}(n)$ is a compact, path-connected subspace of $R^{n\times n}$.*

Proof. Since $\mathcal{U}(n)$ is compact and path-connected, its continuous image $f(\mathcal{U}(n))=\mathcal{O}(n)$ enjoys the same properties. ■

From the fact that the row and column vectors of a unitary matrix have a norm of 1, one sees that every orthostochastic matrix is doubly stochastic. In a paper on doubly stochastic matrices [2], L. Mirsky posed the question of whether the orthostochastic matrices are dense in the set of doubly stochastic matrices. This question can be answered easily once it is established that $\mathcal{O}(n)$ is a proper subset of $\mathcal{D}(n)$ for $n\geqslant 3$.

THEOREM 2.

$$\mathcal{O}(n)=\mathcal{D}(n) \qquad \textit{for} \quad n=1,2.$$

$$\mathcal{O}(n)\neq\mathcal{D}(n) \qquad \textit{for} \quad n\geqslant 3.$$

Proof. For $n=1$, let $U=(1)\in\mathcal{U}(n)$. Then $O=f(U)=(1)\in\mathcal{O}(1)$ and

$\mathcal{D}(1)=\{(1)\}\subset\mathcal{O}(1)$. In the case $n=2$, let

$$U_\alpha=\begin{bmatrix}\alpha^{1/2} & (1-\alpha)^{1/2}\\ -(1-\alpha)^{1/2} & \alpha^{1/2}\end{bmatrix}\in\mathcal{U}(2),\qquad 0\leqslant\alpha\leqslant 1,$$

then

$$O_\alpha=\begin{pmatrix}\alpha & 1-\alpha\\ 1-\alpha & \alpha\end{pmatrix}=f(U_\alpha)\in\mathcal{O}(2).$$

However,

$$\mathcal{D}(2)=\{(a_{ij})\in R^{2\times 2}:a_{ij}\geqslant 0,\, a_{11}+a_{12}=a_{12}+a_{22}=a_{11}+a_{21}=a_{21}+a_{22}=1\}$$

$$=\left\{\begin{pmatrix}\alpha & 1-\alpha\\ 1-\alpha & \alpha\end{pmatrix}:0\leqslant\alpha\leqslant 1\right\}\subset\mathcal{O}(2).$$

For larger n, it suffices to show that there exists a d.s. matrix which is not o.s. Consider the matrix

$$D(3)=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\in\mathcal{D}(3),$$

which is attributed to A. J. Hoffmann [1]. Let

$$U=(u_{ij})\in f^{-1}(D(3)).$$

Then

$$g_{12}(U)=u_{11}\bar{u}_{12}+u_{21}\bar{u}_{22}+u_{31}\bar{u}_{32}=u_{31}\bar{u}_{32}$$

since $|u_{11}|^2=|u_{22}|^2=0$. Since neither u_{31} nor $\bar{u}_{32}$ is zero (as $|u_{31}|^2=|u_{32}|^2=\frac{1}{2}$), $g_{12}(U)\neq 0$. That is, $D(3)$ is not the image under f of any unitary matrix, hence $D(3)\not\in\mathcal{O}(3)$.

By defining

$$D(n)=\left(\begin{array}{c|c} D(3) & 0 \\ \hline 0 & I_{n-3} \end{array}\right)\in \mathcal{D}(n),$$

analogous reasoning proves that $D(n)\notin \mathcal{O}(n)$.

THEOREM 3. *$\mathcal{O}(n)$ is not dense in $\mathcal{D}(n)$ for $n \geqslant 3$.*

Proof. Suppose $\mathcal{O}(n)$ were dense in $\mathcal{D}(n)$, that is, $\mathrm{Cl}\,\mathcal{O}(n)=\mathcal{D}(n)$. Theorem 1 states that $\mathrm{Cl}\,\mathcal{O}(n)=\mathcal{O}(n)$. It follows that $\mathcal{O}(n)=\mathcal{D}(n)$, which contradicts Theorem 2. Therefore, $\mathrm{Cl}\,\mathcal{O}(n)\neq\mathcal{D}(n)$ for $n\geqslant 3$ and the theorem is proved.

REFERENCES

1 M. Marcus, Some properties and applications of doubly stochastic matrices, *Am. Math. Mon.*, **67** (1960), 215–221.

2 L. Mirsky, Results and problems in the theory of doubly-stochastic matrices, *Z. Wahrscheinlichkeitstheorie* **1** (1963), 319–334.

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