Topological Properties of Orthostochastic Matrices*

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ABSTRACT

The set of \( n \times n \) orthostochastic matrices with the topology induced by the Euclidean matrix is shown to be compact and path-connected. For \( n < 3 \), the set of orthostochastic matrices is identical to the set of doubly stochastic matrices. In this paper, it is shown that for \( n \geq 3 \) the orthostochastic matrices are not everywhere dense in the set of doubly stochastic matrices, thus answering a question of L. Mirsky in his survey article on doubly stochastic matrices \[2\].

Denote by \( \mathcal{U}(n) \) the set of \( n \times n \) unitary matrices, and let \( \mathcal{D}(n) \) represent the set of doubly stochastic (d.s.) matrices (those matrices with non-negative real entries and row and column sums equal to 1). Define a matrix function \( f \) from the set of \( n \times n \) complex matrices to the set of \( n \times n \) real matrices

\[
f : \mathbb{C}^{n \times n} \rightarrow \mathbb{R}^{n \times n}, \quad \text{by} \quad f(A) = |a_{ij}|^2.
\]

**PROPOSITION.** \( f \) is continuous with respect to the usual topologies.

Equivalently, considering \( \mathbb{R}^{n \times n} \) with the product topology, \( \pi_{ij} \circ f \) is continuous for all projections \( \pi_{ij} \) of \( \mathbb{R}^{n \times n} \) onto \( \mathbb{R} \). But \( (\pi_{ij} \circ f)(A) = |a_{ij}|^2 = a_{ij} \overline{a}_{ij} \), so \( \pi_{ij} \circ f \) is a composition of continuous functions; hence it is continuous.

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DEFINITION. A matrix $O$ is called orthostochastic (o.s.) if and only if there exists a unitary matrix $U$ satisfying $|u_{ij}|^2 = (o_{ij})$.

The set of $n \times n$ o.s. matrices $O(n)$ is just the continuous image of the set of $n \times n$ unitary matrices; that is, $O(n) = f(U(n))$. Consequently, information about $O(n)$ follows from results for $U(n)$.

**Lemma.** $O(n)$ is a compact, path-connected subspace of $C^{n \times n}$.

**Proof.**

(i) **Compactness.** $O(n) \subset I^{n \times n}$, the unit cube, which is compact. So it suffices to establish that $O(n)$ is closed. Introduce a notation for the inner product of two columns of a complex matrix by

$$g_{ij} : C^{n \times n} \to C \quad \text{with} \quad g_{ij}(A) = \sum_{k=1}^{n} a_{ki} \bar{a}_{kj}.$$  

Then define

$$S_{ii} = \{ A \in C^{n \times n} : g_{ii}(A) = 1 \} = g_{ii}^{-1}(\{1\})$$

$$S_{ij} = \{ A \in C^{n \times n} : g_{ij}(A) = 0 \} = g_{ij}^{-1}(\{0\}) \quad \text{for} \quad i \neq j.$$  

The singleton sets $\{0\}$ and $\{1\}$ are closed in the metric space $C$. Furthermore, each $g_{ij}$, being a composition of continuous functions, is continuous. Hence, each $S_{ij}$ is closed, since it is just the inverse image of a closed set under a continuous mapping. From the characterization of the set of unitary matrices as those complex matrices with orthonormal columns, it follows that

$$O(n) = \bigcap_{i \neq j} S_{ij}.$$  

Since $O(n)$ is a finite intersection of closed sets, $O(n)$ is closed and therefore compact.

(ii) **Path-connectedness.** It is first shown that the set of diagonal unitary matrices is path-connected. The unit circle in the complex plane, $S^1 = \{ z \in C : |z| = 1 \}$, is path-connected. Therefore the product set $\prod_{i=1}^{n} S^1$ is also path-connected. Now the diagonal unitary matrices (the complex matrices with zero entries everywhere except for elements of unit modulus along the main diagonal) form a subspace which is homeomorphic to $\prod_{i=1}^{n} S^1$.

By the spectral theorem, any $U \in O(n)$ is unitarily similar to a diagonal unitary matrix:

$$U = V^*DV,$$
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and accordingly,

\[ \mathcal{U}(n) = \left\{ U \in C^{n \times n} : U = V^* DV \text{ with } D, V \in \mathcal{U}(n) \text{ and } D \text{ diagonal} \right\} \]

\[
= \bigcup_{V \in \mathcal{U}(n)} \left( \bigcup_{D \in \mathcal{U}(n) \text{ diagonal}} \{ V^* DV \} \right)
\]

\[
= \bigcup_{V} T_V, \quad \text{where } T_V = \bigcup_{D \in \mathcal{U}(n) \text{ diagonal}} \{ V^* DV \}.
\]

Now \( T_V \), the continuous image of the path-connected set of diagonal unitary matrices, is path-connected. Choosing \( D = I \) in the similarity transformation gives that \( I \in T_V \) for all \( V \). Consequently, as is well known, \( \mathcal{U}(n) \) is path-connected, since \( \mathcal{U}(n) \) is the union of the path-connected sets \( T_V \), each of which contains the common point \( I \).

Employing standard theorems from topology, the results for \( \mathcal{O}(n) \) are obtained readily, as illustrated below.

**Theorem 1.** \( \mathcal{O}(n) \) is a compact, path-connected subspace of \( R^{n \times n} \).

**Proof.** Since \( \mathcal{U}(n) \) is compact and path-connected, its continuous image \( f(\mathcal{U}(n)) = \mathcal{O}(n) \) enjoys the same properties.

From the fact that the row and column vectors of a unitary matrix have a norm of 1, one sees that every orthostochastic matrix is doubly stochastic. In a paper on doubly stochastic matrices [2], L. Mirsky posed the question of whether the orthostochastic matrices are dense in the set of doubly stochastic matrices. This question can be answered easily once it is established that \( \mathcal{O}(n) \) is a proper subset of \( \mathfrak{D}(n) \) for \( n \geq 3 \).

**Theorem 2.**

\[ \mathcal{O}(n) = \mathfrak{D}(n) \quad \text{for } n = 1, 2. \]

\[ \mathcal{O}(n) \neq \mathfrak{D}(n) \quad \text{for } n \geq 3. \]

**Proof.** For \( n = 1 \), let \( U = (1) \in \mathcal{U}(n) \). Then \( O = f(U) = (1) \in \mathcal{O}(1) \) and
Let $\mathcal{O}(1) = \{ (1) \} \subset \mathcal{O}(1)$. In the case $n=2$, let

$$U_\alpha = \begin{bmatrix} \alpha^{1/2} & (1-\alpha)^{1/2} \\ -(1-\alpha)^{1/2} & \alpha^{1/2} \end{bmatrix} \in \mathcal{O}(2), \quad 0 \leq \alpha \leq 1,$$

then

$$O_\alpha = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix} = f(U_\alpha) \in \mathcal{O}(2).$$

However,

$$\mathcal{O}(2) = \{ (a_{ij}) \in R^{2\times2} : a_{ij} \geq 0, a_{11} + a_{12} = a_{12} + a_{22} = a_{11} + a_{21} = a_{11} + a_{22} = 1 \}$$

$$= \left\{ \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix} : 0 \leq \alpha \leq 1 \right\} \subset \mathcal{O}(2).$$

For larger $n$, it suffices to show that there exists a d.s. matrix which is not o.s. Consider the matrix

$$D(3) = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in \mathcal{O}(3),$$

which is attributed to A. J. Hoffmann [1]. Let

$$U = (u_{ij}) \in f^{-1}(D(3)).$$

Then

$$g_{12}(U) = u_{11}u_{12} + u_{21}u_{22} + u_{31}u_{32} = u_{31}u_{32}$$

since $|u_{11}|^2 = |u_{22}|^2 = 0$. Since neither $u_{31}$ nor $u_{32}$ is zero (as $|u_{31}|^2 = |u_{32}|^2 = 1/2$), $g_{12}(U) \neq 0$. That is, $D(3)$ is not the image under $f$ of any unitary matrix, hence $D(3) \notin \mathcal{O}(3)$. 
By defining
\[ D(n) = \begin{pmatrix} D(3) & 0 \\ 0 & I_{n-3} \end{pmatrix} \in \mathcal{D}(n), \]

analogous reasoning proves that \( D(n) \not\subset \mathcal{O}(n). \)

**Theorem 3.** \( \mathcal{O}(n) \) is not dense in \( \mathcal{D}(n) \) for \( n \geq 3. \)

**Proof.** Suppose \( \mathcal{O}(n) \) were dense in \( \mathcal{D}(n) \), that is, \( \text{Cl} \mathcal{O}(n) = \mathcal{D}(n) \). Theorem 1 states that \( \text{Cl} \mathcal{O}(n) = \mathcal{O}(n) \). It follows that \( \mathcal{O}(n) = \mathcal{D}(n) \), which contradicts Theorem 2. Therefore, \( \text{Cl} \mathcal{O}(n) \neq \mathcal{D}(n) \) for \( n \geq 3 \) and the theorem is proved.

**References**


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