General Drag Coefficient for Flow over Spherical Particles

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A generalized physics-based expression for the drag coefficient of spherical particles moving in a fluid is developed. The proposed correlation incorporates essential rarefied effects, low-speed hydrodynamics, and shock-wave physics to accurately model the particle-drag force for a wide range of Mach and Knudsen numbers (and therefore Reynolds number). Owing to the basis of the derivation in physics-based scaling laws, the proposed correlation embeds gas-specific properties and has explicit dependence on the ratio of specific heat capacities. The correlation is applicable for arbitrary particle relative velocity, particle diameter, gas pressure, gas temperature, and surface temperature. Compared with existing drag models, the correlation is shown to more accurately reproduce a wide range of experimental data. Finally, the new correlation is applied to simulate particle trajectories in high-speed dusty flows, relevant to a spacecraft entering the Martian atmosphere. The enhanced surface heat flux due to particle impact is found to be sensitive to the particle drag model.

A N UNDERSTANDING of particle and droplet migration in fluid flows is important in quantitatively describing a number of engineered systems. In particular, for high-speed flight vehicles, dust particles and water droplets in the atmosphere as well as spalling fibers from the vehicle’s surface may alter surface heating rates and erode the heat shield’s surface [1]. This is particularly relevant for future Mars and Earth entry missions, where particulate matter in dust storms [2], droplets, or ice crystals may be present. In addition to particle impacts on high-speed flight vehicles, another application where particles accelerated to supersonic speeds impact on substrates is particle-based coating processes, such as cold spray deposition [3], plasma spray deposition [4], and aerosol deposition [5]. The quality and resulting properties of the coating are strongly dependent on the particle size and particle impact velocity. In monitoring particle trajectories in all high-speed environments, accurate drag force calculations are essential; finite particle inertia yields appreciable velocity differences between particles and the surroundings at shock fronts and near surfaces. However, even if the high-speed gas flows themselves are in the continuum regime, the flow relative to the motion of the particle may be in the rarefied regime. Theoretical models for the drag force on particles have been derived only for a limited range of conditions, whose extension to a wide range of conditions is dependent on strictly empirical relations [6]. Building upon prior theoretical work in understanding drag on a spherical particle from low to high speed, and continuum to rarefied flows, the purpose of this work is to develop physics-based expression for the drag coefficient, applicable under general conditions, i.e., over a wide range of Mach (\(M_\infty\)) number and Knudsen (\(Kn_\infty\)) numbers (and hence Reynolds number \(Re_\infty \propto M_\infty / Kn_\infty\)).

In principle, the drag force \(F_d\) on a spherical particle of radius \(R\) depends on the freestream relative velocity \(U_\infty\) between the particle and the fluid, the density \(\rho_\infty\), viscosity \(\mu_\infty\), temperature \(T_\infty\) of the fluid, and surface temperature \(T_p\) of the particle. The coefficient of drag \(C_d\) is defined as

\[
C_d = \frac{F_d}{\frac{1}{2} \rho_\infty U_\infty^2 A}
\]  

where \(A = \pi R^2\) for a spherical particle. For restricted ranges of \(Re_\infty\), \(M_\infty\) or \(Kn_\infty\), theoretical evaluation of the drag correlation is possible

Nomenclature

- \(C_d\) = coefficient of drag
- \(C_{d_1}\) = compressible coefficient of drag
- \(C_{d_\infty}\) = incompressible coefficient of drag
- \(C_{d_{M_\infty}}\) = coefficient of drag in hypersonic, continuum limit
- \(C_{d_{s\infty}}\) = coefficient of drag from postshock pressure
- \(D\) = diameter of the sphere
- \(F_d\) = drag force
- \(Kn\) = Knudsen number
- \(M\) = Mach number
- \(R\) = radius of the sphere
- \(Re\) = Reynolds number
- \(s\) = Mach number scaled by \(\sqrt{\gamma / 2}\)
- \(T\) = temperature
- \(U\) = relative speed of flow and particle
- \(W_r\) = rarefaction parameter
- \(\gamma\) = ratio of specific heats
- \(\delta\) = boundary-layer thickness
- \(\mu\) = viscosity of fluid
- \(\rho\) = density of fluid

Subscripts

- \(p\) = particle property
- \(s\) = postshock variables
- \(\infty\) = freestream

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and has been performed. For instance, at low \( R_e \) (\( \ll 1 \)) and within
the incompressible flow regime (\( M_{\infty} < 0.3 \)), when the drag is pri-
marily due to viscous effects, Stokes relation (\( C_d = 24 / R_e \)) esti-
mates the drag force accurately \[2\]. An extension of this theory to
to moderate Reynolds number (\( R_e \leq 5 \)) has been proposed by Oseen
\[8\], which has been further extended for \( R_e \leq 200 \) by Schiller and
Naumann \[9\]. Similarly, when \( Kn_{\infty} \gg 1 \) (i.e., flow is in the free-
molecular regime), Epstein \[10\] derived an expression for \( C_d \) for
small \( M_{\infty} \) numbers. Using the methodology developed by Patterson
\[11\] for estimating drag on flat plate and cylinder in a free-molecular
flow, Baines et al. \[12\] derived a closed-form expression for \( C_d \) on
sphere for all \( M_{\infty} \) for free-molecular flow. Further generalizations
of \( C_d \) to a wide range of \( R_e \) and \( M_{\infty} \), however, have relied on ad hoc
interpolations \[6,13\] between different regimes or neural-network-
based empirical formulizations \[14\]. Davuluri et al. \[15\] have recently
proposed a blended model by combining several empirical corre-
lations available in the literature. Part of the reason for empiricism in
these generalizations is that intermediate regimes encompass a range
of complex physical mechanisms, including inertial effects at high
\( R_e \), viscous effects at low \( R_e \), strong gas compression across
shock waves in high-speed (or high \( M_{\infty} \)) flows, and noncontinuum
effects due to fewer intermolecular collisions and particle–gas collis-
ions in the rarefied regime (\( Kn_{\infty} > 1 \)). Including these complex
physical mechanisms and, more importantly, transitions between
these regimes, is essential to develop a physics-based generalization
of the drag correlation.

We develop a consistent physics-based generalized drag correla-
tion by incorporating the contribution of these complex mechanisms
through physical scaling laws to describe relevant regimes. For
continuum (\( Kn_{\infty} < 0.01 \)) and slip (\( 0.01 < Kn_{\infty} < 0.1 \)) flows, the
Navier–Stokes equations are applicable and coordinate transforma-
tions, such as the Howarth–Stewartson transformation and a novel
transformation for estimating effective postshock conditions behind a
curved wall, are employed to include transitions between different
continuum regimes. To correct the continuum result in the tran-
sitional Knudsen number regime (for \( 0.1 < Kn_{\infty} < 1 \)), physical
scalings derived from higher-order Burnett-type continuum equations
\[16,17\] are used. For rarefied (\( Kn > 1 \)) flow, owing to the
difficulty of obtaining the solution of the Boltzmann equation,
the proposed correlation in the transitional regime isbridged to the
closed-form expression \[11\] of the drag in free molecular limit
(\( Kn \gg 1 \)). The parameters introduced in the resulting scaling laws
are determined using available experimental measurements and first-
principles-based simulations obtained via direct simulation Monte
Carlo (DSMC) method \[18,19\]. One feature of the proposed corre-
lation is its explicit dependence on the gas type, specifically on the
ratio of specific heat capacities, which is absent in existing corre-
lations. We apply the proposed correlation to simulate a high-speed
dusty flow over a sphere to show that the surface heating rate is
sensitive to the drag correlation used for estimating trajectories of
dust particles. Although the current formulation only considers
spherical objects, generalizations are possible \[20\].

II. Theory and Approach

The physical mechanisms at play and their relative contributions
to the overall drag may vary depending on flow regime. For instance,
viscous forces dominate drag force for low \( R_e \), whereas inertial
forces dominate as \( R_e \) increases. Flow regimes are divided based
on compressibility and rarefaction effects. The Mach number \( M_{\infty} \)
demarcates regimes based on compressibility effects, from the in-
compressible regime (\( M_{\infty} < 0.3 \)), to compressible (\( 0.3 < M_{\infty} \leq 1.0 \)),
supersonic (\( M_{\infty} > 1.0 \)), and hypersonic regimes (\( M_{\infty} > 5.0 \)). Simi-
larly, rarefaction effects are characterized by the Knudsen number
\( Kn_{\infty} \), which is defined as the ratio of mean free path to the charac-
teristic length scale \( 2R \). \( Kn_{\infty} \) can also be expressed in terms of \( M_{\infty} \) and
\( R_e \) as

\[
Kn_{\infty} = \frac{M_{\infty}}{R_e} \frac{\sqrt{\gamma \pi}}{2}
\]  

(2)

As shown in Fig. 1, low \( Kn \) (\( Kn < 0.01 \)) corresponds to the continuum
regime, followed by the slip regime (\( 0.01 < Kn < 0.1 \)), the transition
regime (\( 1 < Kn < 10 \)), and the free-molecular regime (\( Kn > 10 \)). In
the following subsections, the aforementioned flow regimes are con-
sidered and physics-based scaling laws for a generalized drag corre-
lation are derived.

A. Continuum Regime Formulation

We first consider the continuum regime (\( Kn < 0.01 \)) for any \( M_{\infty} \),
provided that \( R_e \) remains less than the critical transition condition,
\( R_e \ll 10^2 \) (i.e., before the onset of boundary-layer turbulence
effects). In terms of organization, incompressible flow is considered
first, followed by compressible flow, which is further divided into
subsonic, supersonic, and hypersonic regimes.

1. Incompressible Flow

For sufficiently high \( Re \) (\( \gg 1 \)), as viscous forces are confined
within a thin region called the boundary layer near the wall, Abraham
\[21\] considered the drag force on an effective sphere (of radius
\( R + \delta \)), which includes the boundary-layer (\( \delta \)) region as shown in
Fig. 2a. The drag force on the effective sphere is only a function of
freestream momentum flux, and the effects of viscosity on this
extended sphere can be ignored. Therefore, on the effective sphere,
\( C_d \) is equal to a constant (\( C_0 \)), which is independent of \( R_e \). The
drag force on the actual sphere (\( F_d \)) can then be expressed as

\[
F_d = C_0 \left( 1 + \frac{\delta}{R} \right)^2 \frac{\rho_0 U_\infty^2}{2}
\]  

(3)

where \( F_d \) is drag force and \( \delta \) is boundary-layer thickness. The dimen-
sional analysis arguments for obtaining the expression for drag force
in Eq. (3) are also supported by mathematical arguments \[22\] and
related discussion can be found in Ref. \[23\]. The coefficient of drag
(\( C_0 \)) from \( F_d \) on the original sphere can be expressed as

\[
C_d = C_0 \left( 1 + \frac{\delta_0}{(R_{e})^{2/5}} \right)^2
\]  

(4)

where \( \delta_0 = \delta_0 / R_{e} \), the superscript “ic” denotes the incompres-
sible regime, and \( R_{e} = \rho_0 U_\infty D / \mu \) for diameter \( D \) of the
sphere, and \( C_0 \delta_0 = 24 \), which reduces to Stokes’s result for low
\( R_e \). Abraham \[21\] found \( \delta_0 = 9.06 \) based on fitting the correlation
to limited experimental data.

2. Subsonic Compressible Flow (\( M_{\infty} \leq 1.0 \))

Compressibility results in density changes, which for \( M_{\infty} < 1.0 \)
can be accounted for by applying a correction to the incompressible
formulation. We map the boundary layer of a weakly compressible
flow to an equivalent incompressible flow using the Howarth \[24\] and
Stewartson \[25\] transformation. The schematic showing the transfor-
mation (\( U_\infty \rightarrow U \), \( \rho_0 \rightarrow \tilde{\rho} \), \( T_\infty \rightarrow \tilde{T} \)) is presented in Fig. 2b,
and the algebraic details are given in the Appendix. Including compress-
ibility effects, the expression for the drag coefficient is

\[
C_d = C_0 \Theta(M) \left( 1 + \frac{\delta_0}{(R_{e})^{2/5}} \right)^2
\]  

(5)

where

\[
\Theta(M) = \left( \frac{\tilde{T}}{T_\infty} \right)^{(\gamma-1)/\gamma} = \left[ 1 + (\gamma - 1) \frac{M^2}{2} \right]^{(\gamma-1)/\gamma}
\]  

(6)

This expression reduces to the incompressible case when \( M_{\infty} \ll 1 \) and
\( R_e \) is small.
where $\tilde{R}e(\tilde{R}e, M) = Re\Theta(M)^{(r+1)−2(r−1)\omega/2}$

(7)

where $\tilde{R}e$ and $\Theta$ are, respectively, equivalent to $\tilde{R}e(\ldots)$ and $\Theta(\ldots)$ evaluated at freestream conditions ($R_{e,\infty}$, $M_{\infty}$). $r$ is the ratio of specific heat capacities, and $\omega$ is the exponent in the power-law dependence of viscosity ($\mu \propto T^\omega$) on the temperature. Before we proceed to the supersonic flow regime, we point out that accurate predictions in the subsonic regime require modification of the parameter $\delta_0$ to 9.4 from 9.06. The modified $\delta_0$ does not modify incompressible drag appreciably (shown in Fig. 3) compared with the original value of $\delta_0$.

3. Supersonic Flow ($1 \leq M_{\infty} \leq 5$)

When a flow becomes supersonic, a shock wave causes rapid compression of the gas. To extend the drag model to supersonic conditions, the same formulation developed in Eq. (5) is employed, with an additional contribution to the drag ($C_{dp}$) due to pressure changes across the shock. The effect of compression due to the shock wave is approximated by using a set of effective flow variables. First, the Rankine–Hugoniot conditions across a normal shock are used to obtain postshock conditions (given in the Appendix A.2) ($U_{\infty} \to U_s$, $\rho_{\infty} \to \rho_s$, $T_{\infty} \to T_s$, $p_{\infty} \to p_s$). To approximate effects of the non-normal (hyperbolic) structure of the shock, we introduce mapping functions $\alpha(M_{\infty})$ and $\phi(M_{\infty})$ to transform postshock variables ($U_s \to aU_s$, $\rho_s \to (1/\alpha)\rho_s$, $T_s \to \phi T_s$, $T_s \to \phi T_s$, $p_s \to p_s^\phi$, $M_s \to M_s^\phi$) as shown in Fig. 2c. The inverse scaling ($\alpha$ and $1/\alpha$ for density and velocity, respectively, ensures mass conservation. The function $\phi$ is obtained by the constraint that the transformation preserves postshock Mach number $M_s$, yielding $\phi = \alpha^2$. Using the transformation of the flow variables, the pressure change term $C_{dp}$ can be approximated as

$$C_{dp} = C_1 \left( \frac{p_s^\phi - p_{\infty}}{\rho_{\infty}U_{\infty}^2} \right)$$

(8)

where $p_s^\phi$ is the effective pressure and $C_1$ is the scaling for the contribution of the pressure change across a curved shock. The pressure term $C_{dp}$ does not contain an effective area term, which includes boundary-layer thickness, because the pressure does not significantly vary within the boundary layer and therefore is applied directly on the original sphere of radius $R$. Expressing the pressure change in terms of the difference in momentum flux between freestream (at $\rho_{\infty}$ and $U_{\infty}$) and the effective momentum flux (at $\rho_s/\alpha$ and $aU_s$) after the shock wave, we obtain

$$C_{dp} = C_1 \left( 1 - \frac{aU_s}{U_{\infty}} \right)$$

(9)

Therefore, the expression for overall drag coefficient becomes

$$C_d = C_1 \left( 1 - \frac{aU_s}{U_{\infty}} \right) + C_0 \Theta_s \left( 1 + \frac{\delta_0}{(R_{e_s})^{1/2}} \right)^2$$

(10)

where the second term in Eq. (10) is the same as in Eq. (5) but is now evaluated at the effective Mach number $M_s$ and $R_{e_s}$. $M_s$ is invariant under the proposed transformation, and $R_{e_s}$ is given by

$$R_{e_s} = \frac{1}{a^2} \left( \frac{T_{\infty}}{T_s} \right)^{\omega/2} \Theta_s^r \left( 1 - 1 - 2(r-1)\omega/2 \right)$$

(11)

where $\Theta_s$ is equivalent to $\Theta$ evaluated at effective conditions after transformation. The temperature ratio raised to the power $\omega$ appears...
in Eq. (11) due to the variation of viscosities across the shock. Although the estimation of the transport coefficients at high temperature remains an active area of research interest [29, 30], the power law with $\omega = 0.74$ is the most widely used relation for viscosity at high temperatures for air (>600 K). Furthermore, although the Stewarnton–Howarth transformation, used to derive the function $\Theta$, uses $\omega = 1$ for algebraic simplification, the correlation is not sensitive to the magnitude of $\omega$ in the $\Theta$ function. The algebraic details for $Re$, are provided in the Appendix (refer to Sec. A.2). There are two unknown parameters $C_1$ and $\alpha$, which are required in Eq. (10) to determine $C_d$. Because of the difficulty of analytically obtaining a solution for $\alpha$, we rely on heuristic arguments to determine its functional form. For $M_\infty \gg 1$, the effective flow to remain continuum [$Kn_\infty < 0.1$, see Eq. (2)], $Re$, should not reduce to zero, implying that $\alpha \propto 1/M_\infty^2 (T_\infty/T_s \propto 1/M_\infty^2$ for $M_\infty \gg 1)$. A simple formulation for $\alpha$ satisfying this condition (in addition to $\alpha$ being unity for $M_\infty = 1$) is

$$\alpha = \frac{1}{\alpha_0 M_\infty + 1 - \alpha_0} \quad (12)$$

where $\alpha_0$ is an unknown parameter. Recall that $C_1$ is a scaling factor for the pressure term, and it is obtained from the solution for the hypersonic limit in the next subsection.

4. Hypersonic Limit ($M_\infty \geq 5$)

In hypersonic flows, the postshock gas is in thermal and chemical nonequilibrium, and the simplified assumption of calorically perfect gas used in supersonic flows becomes inaccurate. In fact, nonequilibrium reaction chemistry modeling, calculation of transport coefficients, and development of appropriate boundary conditions for hypersonic flows remain open areas of research. The objective of the current subsection is to find a scaling of the drag coefficient in the hypersonic regime. An approximation of this limit has been obtained by Hornung et al. [31] under the assumption of negligible viscous effects. Hornung et al. show that the magnitude of the drag coefficient approaches a nearly constant value at high $M_\infty$, which is also supported by earlier theoretical investigations by Lighthill [32] and experimental data. In the current work, we use the hypersonic limit ($C^{M_\infty}_d \approx 0.9$; the reader is referred to Fig. 17 in Ref. [31]) for further details) to estimate the expression for $C_1$ in the proposed formulation [Eq. (10)] for the drag correlation. Substitution of variables ($U_\omega/ U_\infty$, $M_\infty^2$, $\Theta$, and $\alpha$) in the limit of $M_\infty \gg 1$ in Eq. (10) and equating the corresponding expression for $C_d$ to $C^{M_\infty}_d$ yields the following relation for $C_1$:

$$C_1 = C^{M_\infty}_d - C^0_d (1 + [(\gamma - 1)/2\gamma (\gamma - 1)]^{(\gamma - 1)/\gamma})^{-1} \approx 1/\alpha_0 M_\infty (1 - 1/\gamma + 1) \quad (13)$$

For more algebraic details or the exact high-Mach-number limit of the variables, we refer the reader to Sec. A.3 of the Appendix. At this stage our continuum formulation for the expression of the drag coefficient is complete [Eq. (10)], with only one unknown parameter ($\alpha_0$). Before we estimate $\alpha_0$, the extension of the model to rarefied regime is presented next.

B. Theoretical Development: Rarefaction Regime

In rarefied flows ($Kn_\infty > 0.01$; see Fig. 1), due to fewer gas molecule collisions near the surface, the bulk velocity and temperature of the gas do not equilibrate with the surface velocity and surface temperature, respectively. This results in a finite velocity slip and temperature jump at the wall. To account for these noncontinuum effects, velocity slip (and temperature jump) at the wall is employed along with the Navier–Stokes equations. At higher $Kn_\infty (>0.5)$, the stress tensor (and heat flux vector) does not depend linearly on the velocity gradient (and temperature gradient); therefore, the Navier–Stokes equations become inaccurate in the high $Kn_\infty$ regime. If $M_\infty$ is also high, as molecules travel larger distances without collisions at high $Kn_\infty$, the shock layer and boundary layer merge with each other. Although the Boltzmann equation is applicable for arbitrary degrees of rarefaction ($\forall Kn_\infty$), the equation is computationally expensive to solve. However, approximate scaling laws relevant to the drag correlation can still be informed from the Boltzmann equation, which is adequate for the current work and is described in the next subsections.

1. Rarefaction Correction ($Kn_\infty$) at Low $M_\infty$

For low-speed flows, the Boltzmann equation has been theoretically solved in an approximate manner by Phillips [33] for estimation of the drag force on a sphere. Phillips expressed $C_d = f_{Kn}$ $C_d^0$, where $f_{Kn}$ is a multiplicative rarefaction correction to the continuum drag coefficient $C_d^0$. For low-speed flows, numerically identical to the expression by Phillips [33], an alternative simple closed-form expression for $f_{Kn}$ was proposed by Davies [34] (and for nonspherical particles in Ref. [35]) as follows:

$$f_{Kn} = \frac{1}{1 + Kn_\infty [A_1 + A_2 \exp(-A_1/Kn_\infty)]} \quad (14)$$

where $A_1 = 2.514$, $A_2 = 0.8$, and $A_3 = 0.55$. These coefficients are based on data from Millikan [36] and are valid for all Knudsen numbers at low $M_\infty$ numbers [37]. Basically, $f_{Kn}$ reduces to the high-$Kn$ number drag expression for high $Kn$ ($\gg 1$) and unity at low $Kn_\infty < 0.01$. An intuitive argument for the justification of Eq. (14) using a reduced sphere due to slip effects is presented in the Appendix (Sec. A.4). Next, we develop the correction function for high $Kn_\infty$ and high $M_\infty$.

2. Rarefaction Correction for the Slip (0.01 ≤ $Kn_\infty < 0.1$) and Transition (0.1 ≤ $Kn_\infty < 10$) Regime at High $M_\infty$

For high-Mach-number flows ($M_\infty > 0.3$), an analytical solution of the Boltzmann equation is challenging, and in many cases, impossible. Several sets of nonlinear constitutive relations and higher-order moment equations have been proposed [17,38,39] as computationally efficient alternatives to the Boltzmann equation for modeling slip and transition regimes. For high-speed flow over a sphere, Singh and Schwartzentruber [40,41] mathematically showed that the ratio of the nonlinear to linear (used in the Navier–Stokes equations) constitutive relations dominantly depends on a nondimensional number called $W_f$, which is given by

$$W_f = W_f (1 + \frac{T_s}{T_p})^\omega$$

with $W_f = M_\infty^{2\omega} Re = Kn_\infty M_\infty^{2\omega - 1} / \sqrt{\gamma} M_\infty > 1 \quad (15)$

and $T_p$ is surface temperature. Note that the parameter $W_f$ was originally proposed by Wang et al. [42] and $W_f$ is its extension, developed by considering supersonic flows [40] in addition to hypersonic regime. In fact, an approximate contribution of the higher order (up to infinite order in terms of $Kn_\infty$) constitutive relations has been shown as a scaling factor to the heat transfer coefficient. At high $Kn_\infty$, the gas does not fully thermally accommodate with the wall, and the drag force depends explicitly on the surface temperature as also evident in Eq. (15). We employ the same correction to $f_{Kn}$, which modifies the high-Knudsen-number correction using an additional term based on $W_f$:

$$f_{Kn, W_f} = \frac{1}{1 + Kn_\infty [A_1 + A_2 \exp(-A_1/Kn_\infty)]} \frac{1}{1 + \alpha_{hoc} W_f} \quad (16)$$

where $\alpha_{hoc}$ is an unknown parameter, and the subscript hoc refers to higher-order correction. Equation (16) corrects the drag coefficient for rarefaction effects at moderate $Kn_\infty$ and at high $M_\infty$.

3. Rarefaction Correction for Free-Molecular Regime ($Kn_\infty > 10$) at High $M_\infty$

As a final step, we bridge the correlation to the analytically obtained $C_d^0$ for the free-molecular regime. An analytical expression
for \( C_d \) valid for free-molecular flow developed by Baines et al. [12] using the methodology proposed by Patterson [11] is given by

\[
C_d(f_m) = \frac{2(1-\epsilon)}{3\epsilon} \sqrt{\frac{T_p}{T_\infty}} + \frac{2(1-\epsilon)}{3\epsilon} \sqrt{\frac{T_p}{T_\infty}}
\]

where \( s = M_\infty \sqrt{\gamma/2} \), and \( \epsilon \) is the accommodation coefficient. The particle surface is assumed to be totally diffusely reflective for which \( \epsilon = 0 \). Singh and Schwartzentruber [41] proposed a bridging function between a purely empirical drag correlation for high \( M_\infty (>) 1 \) in the transition and free-molecular regimes using the inverse Cheng's parameter, \( K_\infty [45] \). Mathematically, \( 1/K_\infty \mu_T/\mu_T/\mu_T \), where \( \mu_T \) is the average of \( T_c \) and \( T_p \) and \( \mu_T \) is evaluated at \( T^* \). Substituting power-law relations for viscosity, it can be shown that the inverse of \( K_\infty \) is proportional to \( W^2 \), both of which depend on the wall temperature that becomes important as the degree of rarefaction increases. In this work, we employ \( Br(\alpha W^2) \) as the correlation parameter to bridge the proposed general physics-based drag correlation to the free-molecular expression,

\[
Br = W^2 \frac{M_\infty^{2s-1} + 1}{M_\infty^{2s-1}}
\]

\[ Br \] has the desirable properties that for \( M_\infty \gg 1 \), \( Br \to W^2 \) and for \( M_\infty \ll 1 \), \( Br \to Kn_\infty \). Using a rational polynomial function as a plausible bridging function, the expression for full drag correlation is defined as

\[
C_d = C_d(\eta f_{Kn, Wr} \frac{1}{1 + Br^\eta}) + C_d(f_m) \frac{Br^\eta}{1 + Br^\eta}
\]

where \( \eta \) is an unknown parameter. Equation (19) is the main result of the present work. There are three unknown parameters \( a_0, a_{hoc} \), and \( \eta \), which are obtained in the next subsection. For the sake of brevity, the complete set of equations, for the entire correlation, are also provided separately in Sec. C of the Appendix.

### C. Estimation of \( a_0, a_{hoc} \), and \( \eta \)

Least-square fitting was used to determine the unknown parameters based on available experimental data and relevant simulations. A summary of the data used for fitting unknown parameters is compiled in this section. Bailey and Hiatt [26] determined drag on spherical bodies for a wide range of Mach and Reynolds numbers (0.1, \( M_\infty < 6 \) and \( 10^0 \leq Re_{in} < 10^6 \)) in a ballistic range. Bailey [46] also determined the drag on spheres for higher Reynolds numbers using a similar setup. Sreekanth [47] determined the drag on spherical bodies in the transitional Knudsen number regime (\( M_\infty = 2 \) and \( 0.1 < Kn_\infty < 0.8 \)) using a wind tunnel setup. Zarin and Nicholls [27] determined the drag on spherical bodies in a subsonic wind tunnel using magnetic suspension for a range of Reynolds numbers (0.1, \( M_\infty = 0.57 \) and \( 4 \times 10^3 < Re_{in} < 5 \times 10^5 \)). Kissell and Overell used computational fluid dynamics (CFD) and DSMC, respectively, to determine the drag on a spherical body [48–50]. Aroyo measured spherical drag in Berkeley’s low-density wind tunnel for Mach numbers of roughly 2, 4, and 6 for \( 10^3 < Re_{in} < 10^5 \) [51]. Goin and Lawrence [28] also used a ballistic range to determine drag for subsonic spheres (0.1, \( M_\infty < 1 \)) for a range of Reynolds numbers (2, \( 10^2 < Re_{in} < 10^3 \)). Charters and Thomas [52] also used a ballistic range setup and determined drag on spheres for a larger range of Mach numbers at relatively high Reynolds numbers (0.3, \( M_\infty < 4 \) and 9.3, \( 10^5 < Re_{in} < 1.3 \times 10^6 \)). Hodges [53] determined drag on supersonic and hypersonic spheres (\( 2 < M_\infty < 10 \)) for very high Reynolds numbers (\( Re_{in} > 3 \times 10^9 \)). May and Witt [54] also found the drag on spherical bodies, but for lower Reynolds numbers and Mach numbers (0.8, \( M_\infty < 4.7 \) and \( 1.1 \times 10^3 < Re_{in} < 8.4 \times 10^5 \)). The obtained parameters from least-square fitting are listed in Table 1.

### III. Results and Discussion

In this section, we compare the proposed drag model with state-of-the-art models. We then present the drag correlation results for different species type (monatomic, diatomic, and triatomic gases). Lastly, we apply the proposed model to investigate the sensitivity to surface heating rates for dusty flow over a sphere, relevant for high-speed flows in Martian atmosphere.

#### A. Comparison to State-of-the-Art Models

We compare the proposed model to standard state-of-the-art models in the literature. Because the proposed model is based on physics-based scaling relations, the proposed model is compared with the most widely used semi-empirical models developed by Henderson [6] and Loth’s [13] models.

The equations for both models are reported in Sec. II of the Appendix.

In Fig. 4a, the proposed model is compared with the Henderson model [6] and experimental data as a function of Mach number for a range of Reynolds number corresponding to the continuum regime. For subsonic and hypersonic regimes, both the Henderson and the proposed model fit the data adequately. However, near the transonic and supersonic regimes, the proposed model predicts the experimental data more accurately than the Henderson model. This is not surprising because the Henderson model interpolates the drag in these regimes \( (1 \geq M_\infty \geq 1.75) \), whereas the proposed model has systematically incorporated shock-wave physics.

Figure 4b compares the proposed model with the experimental data and Loth model for a wide range of \( Re_{in} \) as a function of \( M_\infty \) in the continuum regime. Although the Loth model predicts the subsonic regime as accurately as the proposed model, the disagreement at high \( M_\infty \) numbers is significant, specifically near transonic and supersonic regimes. The Loth model is based on fitting the data such that \( C_d \) remains invariant if \( M_\infty \) is varied but \( Re_{in} \) is kept fixed at 45. The proposed model does significantly better compared with the Loth model at all \( M_\infty \). In terms of the entire data considered, the relative \( L_2 \) norms of the errors of the proposed correlation, Henderson model, and Loth model, when compared with the experimental data are 7.6%, 15.8%, and 12.1%, respectively.

Next, we compare the proposed model with Henderson model and Loth model for a range of \( Re_{in} \) in Fig. 5, which includes data from continuum to free-molecular regime for three different freestream \( M_\infty \).

All three models compare reasonably well with the data with discrepancies in certain regimes. Firstly, the proposed model predicts accurate drag coefficients in the free-molecular (low \( Re \)) limit for subsonic data supported by the DSMC data at these conditions. However, in the limit of very high \( Kn \), the Knudsen number correction from Eq. (14) scales correctly compared with the free-molecular limit, but differs by a constant factor. Loth’s model approaches this limit as opposed to the free-molecular limit, which is the reason for the discrepancy for the subsonic data. Although the proposed model agrees well with the experimental data, it deviates slightly in the transitional regime. The proposed model under- and overpredicts over a short range of

#### Table 1 Parameters required in the proposed correlation

| \( \delta_0 \) | Boundary-layer thickness scaling | 9.4 |
| \( a_0 \) | Shock-curvature parameter | 0.356 |
| \( A_1, A_2, A_3 \) | Low-speed rarefaction correction | 2.514, 0.8, 0.55 |
| \( \eta \) | Bridging function modulator | 1.8 |
| \( \eta_{hoc} \) | High-speed rarefaction correction | 1.27 |
Because monatomic gases have higher $\gamma$ compared to triatomic gases, they have fewer energy modes for thermal energy redistribution generated due to the conversion of bulk energy across a shock wave. Therefore, the resulting higher postshock pressure and velocity for monatomic gases [see Eqs. (23) and (22)] results in a higher drag coefficient. The experimental data from Åroesty [51] correspond to a monatomic gas (argon) and therefore provide a test-bed to investigate the sensitivity of the drag to $\gamma$. Figure 6a shows that the proposed correlation for a monatomic gas is in reasonably good agreement with experimental data.

C. Sensitivity of Drag Correlation to Surface Heat Flux for a High-Speed Dusty Gas Flow over a Sphere

In this subsection, the proposed correlation is applied to model trajectories of dust particles in a high-speed flow over a sphere. Understanding high-speed particle-laden flows is relevant for Martian entry, where dust storms are frequent [2], and it is also relevant for hypersonic flight within Earth’s atmosphere, where particulates, droplets, or ice crystals may be present. Recently, Ching et al. [1] simulated high-speed nitrogen flow seeded with dust particles over a sphere, showing that the surface heat flux is amplified by the presence of the dust particles due to interphase momentum and energy transfer and inelastic particle-wall collisions. Interestingly, the extent of the amplification of surface heat flux was found to be sensitive to the employed drag correlations. In addition to the drag correlation, the accuracy of the surface heat flux depends on several factors, such as the employed heat flux correlation [56], and other physical mechanisms. In this work, we perform sensitivity analysis focusing on the drag correlation, and therefore apply the proposed drag correlation to the same flow configuration considered in Ref. [1] and compare the surface heat flux obtained with other drag models.

Compressible Navier–Stokes equations are employed for the transport of mass, momentum, and energy of the carrier gas in conjunction with the ideal gas law. The specific heats at constant volume and pressure are assumed to be constants. For evaluating the dynamic viscosity, Sutherland’s law is used. Chemical and thermal nonequilibrium effects are not considered. Particles, assumed to be smooth, spherical, solid, nonrotating, and of fixed size, have uniform temperature. While particles exchange momentum and energy with the carrier gas, particle–particle interactions are ignored. Further, under the assumption of dilute limit considered in this work, the volume fraction of particles is neglected. Finally, the coefficients of restitution for particle–wall collisions are computed from Ref. [57]. In terms of numerical methods, third-order backward difference and the third-order Adams–Bashforth method are employed to integrate the gas and particles in time, respectively. A third-order-accurate discontinuous Galerkin (DG) scheme for discretizing the Navier–Stokes equations has been shown to have robust heat transfer...
Fig. 6 Comparison of the proposed model using different gas types. Experimental data from Aroesty [51] and CFD simulations from Overell [49] are also included, both of which considered monatomic gases.

Fig. 7 Pure-gas and dusty-gas surface heat flux profiles for a flow over a sphere of diameter 0.012 m and surface temperature of 300 K.
predictive capability [58]. To simulate dust particles suspended in the carrier gas, a Lagrangian particle method under the DG framework [55] has been employed. At each time step, particles are injected at random locations along the inflow boundary. Further simulation details can be found in Ref. [1]. The freestream conditions, identical to those considered in Ref. [55], are characterized by \( M_\infty = 6.1, P_\infty = 1000 \text{ Pa}, \) and \( T_\infty = 68 \text{ K}. \) The dust particles are made of SiO\(_2\) with a density of 2264 kg/m\(^3\), mean diameter of 0.19 \( \mu \text{m} \), and 3\% mass loading ratio (the mass loading ratio is the ratio of the mass flux of the particles to the mass flux of the gas).

Figure 7a shows the temperature contour and the distribution of dust particles for a high-speed nitrogen gas flow over a sphere. In Fig. 7b, the surface heat flux distribution obtained with the proposed drag correlation is compared with that obtained using the Henderson and Loth models for trajectory estimation. Simulations with pure gas (i.e., no dust particles) show lower surface heat flux relative to the dusty gas. Using the proposed correlation, the heat flux predictions deviate by nearly \( \pm 14\% \) from the Henderson and Loth models. The main contributor to the heat flux augmentation is interphase thermal energy transfer. Specifically, the particles (which are all small and therefore have low thermal inertia) heat up rapidly upon crossing the shock. The particles that then reach the colder near-wall region subsequently deposit the acquired thermal energy to the gas. In this manner, low drag enables more particles to reach the near-wall region and/or strike the sphere surface and subsequently have longer residence times in those regions. Therefore, the particles phase deposits more energy in the case of the Loth’s (lowest drag) model compared with the proposed (intermediate drag) and Henderson’s (highest drag) models.

IV. Conclusions

A general drag coefficient model for spherical particles moving in a fluid has been developed, applicable for arbitrary particle relative velocity, particle diameter, gas pressure, gas temperature, and surface temperature. In addition to freestream Mach number \( M_\infty \) and Reynolds number \( Re_\infty \), the proposed drag model encapsulates explicit dependence on the particle’s surface temperature, the ratio of specific heat capacities (\( \gamma \)), a noncontinuum parameter \( (W_f \propto M_\infty^2/Re_\infty) \), and the power-law exponent (\( \omega \)) of viscosity \( (\mu \propto T^{\omega}) \). The correlation is formulated with incorporating simple physics-based scaling laws to model low-speed hydrodynamics, high-speed shock-wave physics, and noncontinuum effects due to rarefied gas dynamics. Free parameters introduced in the scaling laws are obtained by using experimental data for drag coefficients from the literature. The proposed correlation is demonstrated to be in better agreement with experimental data compared with widely used drag models.

The dependence of the drag model on \( \gamma \) and therefore on the gas composition is investigated. The distinct trend of higher drag coefficients for monatomic gas compared with diatomic gas in experimental measurements is captured quantitatively by the proposed drag model.

Finally, the proposed drag model is applied to evaluate trajectories of particles in a simulation of high-speed dusty flow over a sphere. Enhancement of the surface heat transfer coefficient due to particle–flow and particle–surface interactions is found to be significantly different when using the newly proposed correlation compared with state-of-the-art empirical drag models.

### Appendix A: Mathematical Details Related to the Transformations Used in Derivation of the Drag Model

In this section, we provide additional algebraic details required in the derivation of the proposed drag model.

#### A.1. Mapping Weakly Compressible Boundary-Layer to Incompressible-Boundary-Layer Variables

Howarth [24] and Stewartson [25] introduced a transformation to map a compressible laminar boundary layer to an equivalent incompressible boundary layer. The required transformed variables, denoted by \( \alpha \), in terms of free-stream variables can be expressed as

\[
\frac{\tilde{T}}{T_\infty} = 1 + \frac{(y - 1)M_\infty^2}{2} ; \quad \frac{\tilde{\rho}}{\rho_\infty} = \left( \frac{\tilde{T}}{T_\infty} \right)^{1/(y-1)} \tag{A1}
\]

#### A.2. Transformation of Variables Across a Shock Wave in the Supersonic Regime

Postshock conditions across a normal shock can be obtained via Rankine–Hugoniot [59,60] conditions, which are

\[
\frac{T_s}{T_\infty} = \frac{[(y - 1)M_\infty^2 + 2][yM_\infty^2 - (y - 1)]}{(y + 1)^2M_\infty^2} \tag{A2}
\]

\[
\frac{P_s}{P_\infty} = \frac{2y}{\gamma + 1} \frac{M_\infty^2 - \frac{y - 1}{\gamma + 1}}{\frac{y - 1}{\gamma + 1}} \tag{A3}
\]

\[
\frac{U_s}{U_\infty} = \frac{2 + (y - 1)M_\infty^2}{(y + 1)M_\infty^2} \tag{A4}
\]

Using the jump conditions, postshock \( M_s \) can be written as

\[
M_s = \sqrt{\frac{(y - 1)M_\infty^2 + 2}{2yM_\infty^2 - (y - 1)}} \tag{A5}
\]

To obtain Reynolds number \( Re_\infty \), the postshock Reynolds number \( Re_s \) is first obtained using the Rankine–Hugoniot conditions, and then transformed using the proposed transformation for curvature effects to obtain \( Re_s^{ef} \).

\[
Re_s^{ef} = Re_\infty \left( \frac{T_\infty}{\tilde{T}_s} \right)^{\omega} \tag{A6}
\]

The second step is to employ the Stewartson–Howarth transformation for weak compressibility effects corresponding to the postshock Mach number \( M_s \) using Eq. (7):

\[
Re_s = Re_\infty\Theta(M)^{(y+1/2y)-(y-1)\omega} = Re_\infty \left( \frac{T_\infty}{\tilde{T}_s} \right)^{\omega} \tag{A7}
\]

#### A.3. Drag Coefficient in the Hypersonic Limit

The additional variables \( (U_s/U_\infty, M_s^2, \Theta, \text{ and } \alpha) \) in the limit of \( M_\infty \gg 1 \) required in Eq. (10) to estimate \( C_d \) are

\[
\frac{U_s}{U_\infty} \bigg|_{M_\infty \gg 1} = \frac{\gamma - 1}{\gamma + 1} \tag{A8}
\]

\[
M_s^2 \bigg|_{M_\infty \gg 1} = \frac{\gamma - 1}{2\gamma} \tag{A9}
\]

\[
\Theta \bigg|_{M_\infty \gg 1} = \left[ 1 + \frac{(y - 1)M_\infty^2}{2} \right]^{(y-1)/2y} = \left[ 1 + \frac{(y - 1)^2}{4\gamma} \right]^{y/(y-1)} \tag{A10}
\]

\[
\alpha \bigg|_{M_\infty \gg 1} = \frac{1}{\alpha_0M_\infty^{2\gamma}} \tag{A11}
\]

Employing variables from Eqs. (A8–A11) in the expression for \( C_d \) in Eq. (10) yields

\[
C_d(M_s, Re_s) \bigg|_{M_\infty \gg 1} \Rightarrow C_1 \left( 1 - \alpha_0 \frac{\gamma - 1}{\gamma + 1} \right) + A_s \left[ 1 + \frac{(y - 1)^2}{4\gamma} \right]^{y/(y-1)} = C_d^{M_s} \nonumber
\]

\[
\Rightarrow C_1 = \frac{C_d^{M_s} - C_0 [1 + (y - 1)^2/4\gamma]^{y/(y-1)}}{1 - (1/\alpha_0M_\infty^{2\gamma})(y - 1/\gamma + 1)} \tag{A12}
\]
A.4. Slip Effects and Their Relevance to Drag

In view of our approach (shown in Fig. 2d), we present an intuitive argument supporting Eq. (14) in the slip flow regime. The effect of velocity slip at the wall can be understood by considering a sphere of smaller radius that has no slip at the wall. The drag coefficient $C_d$, for a sphere with slip boundary condition can be obtained using the drag on the sphere of smaller radius with no slip:

$$C_d = C_d \left(1 - \frac{\delta_{Kn}}{R}\right)^2$$  \hspace{1cm} (A13)

where the $\delta_{Kn}$ is the Knudsen layer thickness. A simple scaling for $\delta_{Kn}$ based on a stationary wall in terms of slip velocity $u_s$ and velocity gradient at the wall $(\partial u/\partial y)|_w$ can be expressed as $\delta_{Kn} = (u_s)/(\partial u/\partial y)$. Here, $u_s$ is the slip velocity and the wall is assumed stationary. The approximation to $u_s$ using Maxwell’s slip model in terms of slip coefficient $C_{Kn}$ and the mean free-path $\lambda$ is given by

$$U_s = C_{Kn} \lambda \frac{\partial u}{\partial y}|_w$$  \hspace{1cm} (A14)

Equations (A13) and (A14) can be combined to yield an approximate expression for $\delta_{Kn}$ as

$$\frac{\delta_{Kn}}{R} = C_{Kn} \frac{Kn}{R}$$  \hspace{1cm} (A15)

Substituting the above expression for $\delta_{Kn}$ from Eq. (A15) in Eq. (A13), $C_d$ reduces to

$$C_d = C_d (1 - C_{Kn} \frac{Kn}{R})^2$$  \hspace{1cm} (A16)

This Knudsen correction in the limit of small Knudsen number reduces to

$$C_d \approx C_d (1 - 2C_{Kn} \frac{Kn}{R})$$  \hspace{1cm} (A17)

The expression for $C_d$ in Eq. (A17) is identical to the result obtained theoretically by Basset [61].

Appendix B: Expressions for the Henderson and Loth Models

**Henderson model**:

$$C_d(M_{\infty}, Re_{\infty}, T_w/T_\infty) = 24 \left[Re_{\infty} + \left(4.33 + \frac{3.65 - 1.53 T_w/T_\infty}{1 + 0.353 T_w/T_\infty} \exp \left(-0.247 \frac{Re_{\infty}}{s} \right) \right] \right]^{-1}$$

$$+ \exp \left(\frac{0.5 M_{\infty}}{\sqrt{Re_{\infty}}} \left[4.5 + 0.38 (0.03 Re_{\infty} + 0.48 \sqrt{Re_{\infty}}) + 1.03 Re_{\infty} + 0.48 \sqrt{Re_{\infty}} \right] + 0.1 M_{\infty}^2 + 0.2 M_{\infty}^3 \right) + \left[1 - \exp \left(-\frac{M_{\infty}}{Re_{\infty}} \right) \right] 0.6 \sqrt{\frac{T_p}{T_\infty}} \frac{1}{s}$$

$$= 0.9 + (0.34/M_{\infty}^2) + 1.86(M_{\infty}/Re_{\infty})^{1/2}[2 + (2/s) + (1.058/s)(T_w/T_\infty)^{1/2} - (1/s^2)]$$

$$M_{\infty} > 1.75$$

$$= C_d(1, Re_{\infty}) + 4 \left(\frac{M_{\infty} - 1}{C_d(1.75, Re_{\infty}) - C_d(1, Re_{\infty})} \right)$$  \hspace{1cm} (B1)
where

\[ W' = W_f \left(1 + \frac{T_p}{T_i}\right)^{\alpha} \text{and} M_\infty > 1 \quad (C3) \]

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