Channel Knowledge at the Tx: Outline

• Linear Pre-filtering
  – Full Knowledge - Bit Rate Maximization, Error Rate Minimization
  – Partial Knowledge - Bit Rate Maximization, Error Rate Minimization

• Selection at the Transmitter
  – SM and Diversity Coding
  – Antenna Selection

• Exploiting Imperfect Channel Knowledge
• The signal model for a frequency-flat channel is

\[ y = \sqrt{\frac{E_s}{M}}HWx + n = \sqrt{\frac{E_s}{M}}Hs + n \]

with \( \mathcal{E}\{xx^H\} = I_m \) and \( s = Wx \) and \( R_{ss} = WW^H \), \( M = M_T = M_R \)

• The pre-filter matrix (or beamformer) \( W \) must satisfy the power constraint at the Tx.

• Metrics are either the bit rate rate or the error rate.
Linear Pre-filtering - Power Constraints

- Sum Average Power Constraint: For a sum average transmit power constraint of $E_s$,

$$\|\mathbf{W}\|_F^2 = \text{Tr}(\mathbf{R}_{ss}) = M$$

- Mixed Power Constraint: Peak power constraint of $E_{\text{peak}}$ per Tx antenna in addition to a sum average power constraint

$$\|\mathbf{w}_j\|_F^2 = [\mathbf{R}_{ss}]_{j,j} \leq \frac{E_{\text{peak}}M}{E_s}$$

$$\|\mathbf{W}\|_F^2 = M$$

where $\mathbf{w}_j$ is the $j^{th}$ ($j = 1, 2, \cdots, M$) row of $\mathbf{W}$. 
Maximum Rate Pre-filtering (Full Knowledge)

- Capacity of MIMO channel given by

\[ C = \max_{R_{ss}} \log_2 \det \left( I_M + \frac{E_s}{MN_o} H R_{ss} H^H \right) \text{ bps/Hz}. \]

  - Sum average power constraint - \( R_{ss}^{\text{opt}} \) from water-pouring. (Lect. 5)
  - Mixed power constraint - \( R_{ss}^{\text{opt}} \) is convex and can be found by numerical methods.

- On calculating \( R_{ss}^{\text{opt}} \), the optimal pre-filtering matrix is given by

\[ W^{\text{opt}} = Q^{\text{opt}} \Lambda^{1/2}_{\text{opt}} \]

where \( Q^{\text{opt}} \Lambda^{\text{opt}} Q^{H}_{\text{opt}} \) is the eigendecomposition of \( R_{ss}^{\text{opt}} \).
Pre-Filter (Sum Average Power Constraint)

Modal Decomposition.

- Optimal $R_{ss}$ for sum average power constraint $\text{Tr}(R_{ss}) = M$ is given by
  \[
  R_{ss} = VR_{\tilde{s}\tilde{s}}^{opt}V^H
  \]
  where $R_{\tilde{s}\tilde{s}}^{opt}$ is a diagonal matrix found from waterpouring and $H = U \Sigma V^H$

- The optimal pre-filter becomes
  \[
  W^{opt} = Q_{opt}A_{opt}^{1/2} = V \left( R_{\tilde{s}\tilde{s}}^{opt} \right)^{1/2}
  \]

- Multiplying the received signal $\tilde{y}$ by $U^H$, we get
  \[
  \tilde{y} = \sqrt{\frac{E_s}{M}} \Sigma \tilde{s} + \tilde{n}
  \]
  with $\tilde{n} = U^H n$ and $\tilde{s} = \left( R_{\tilde{s}\tilde{s}}^{opt} \right)^{1/2} x$. 

Channel Modes
Error Rate Minimization (Full Knowledge)

- With sum power constraint, modal decomposition is optimal with
  \( W_{opt} = Q_{opt} \Lambda_{opt}^{1/2} = V \Lambda_{opt}^{1/2} \). (For maximum rate we know \( \Lambda_{opt} = R_{ss}^{opt} \))

- The power allocation problem for other criteria can be posed in terms of SNR in each mode and its influence of SER.

- The problem can also be posed in terms of a weighted sum of inverse SNRs or mean squared estimation errors [Sampath et. al. 2001]
Maximum Rate Pre-filtering (Partial Knowledge)

- We assume a channel model given by $H = H_w R_t^{1/2}$ where the Tx correlation matrix $R_t$ is known, but $H$ unknown.

- We can only optimize statistics, i.e. either outage or ergodic capacity.

- Optimizing the ergodic capacity, our objective is given by
  \[
  \bar{C} = \max_{\mathbf{w}} \mathcal{E} \left\{ \log_2 \det \left( I_{MR} + \frac{\rho}{M} H \mathbf{w} \mathbf{w}^H H^H \right) \right\}.
  \]
  and the sum power constraint requires $\| \mathbf{w} \|_F^2 = M$

- Capacity is achieved when
  \[
  \mathbf{w}^{opt} = Q_{R_t} \Lambda_{\mathbf{W}}^{1/2}
  \]
  where $Q_{R_t}$ is the eigenvector matrix of $R_t$ (i.e., $R_t = Q_{R_t} \Lambda_{R_t} Q_{R_t}^H$) and $\Lambda_{\mathbf{W}}$ is a diagonal power allocation matrix that satisfies
  \[
  \text{Tr}(\Lambda_{\mathbf{W}}) = M.
  \]
Maximum Rate Pre-filtering (Partial Knowledge) - (ctd)

- We should transmit along the eigenvectors of $\mathbf{R}_t$ (beamform)
- Optimal power allocation $\Lambda_w$ is a numerical maximization subject to the trace constraint.
  - Approximate solution waterpours on the weighted eigenmodes of $\mathbf{R}_t$ called “stochastic waterpouring” [Gorokhov, ’00]
  - If $\mathbf{R}_t$ approaches rank one, range of SNR exists where optimal strategy is to direct power in dominant eigenmode of $\mathbf{R}_t$. 
Error Rate Minimization (Partial Knowledge)

- We again assume $R_t$ is known but not $H$. The effective channel is

$$H = H_w R_t^{1/2} W$$

- We optimize the average Pair-wise Error Probability (PEP) upper-bounded by (see lecture 9)

$$P(S^{(i)} \rightarrow S^{(j)}) \leq \left( \frac{1}{\det \left( I_M + \frac{\rho}{4M} E_{i,j}^H W^H R_t W E_{i,j} \right)} \right)^M.$$

where $E_{i,j} = S^{(i)} - S^{(j)}$ ($M_T \times T$) is an error word.

- Under a sum average power constraint, choose $W$ to maximize

$$\max_{\text{Tr}(WW^H) = M} \det \left( I_M + \frac{\rho}{4M} E_{i,j}^H W^H R_t W E_{i,j} \right).$$
Error Rate Minimization (Partial Knowledge) - ctd

- Assuming $\mathbf{R}_t$ and $\mathbf{E}_{i,j}$ are full-rank, the optimal $\mathbf{W}$ satisfies

$$\mathbf{W}^{opt} = \mathbf{Q}_{\mathbf{R}_t} \Lambda_{\mathbf{W}}^{1/2} \mathbf{Q}_{\mathbf{E}_{i,j}}^H,$$

with $\mathbf{E}_{i,j} \mathbf{E}_{i,j}^H = \mathbf{Q}_{\mathbf{E}_{i,j}} \Lambda_{\mathbf{E}_{i,j}} \mathbf{Q}_{\mathbf{E}_{i,j}}^H$ and $\Lambda_{\mathbf{W}}$ is a diagonal matrix whose diagonal elements can be computed using waterpouring.

- For OSTBC codes with $\mathbf{E}_{i,j} \mathbf{E}_{i,j}^H = d_{\text{min}}^2 \mathbf{I}_M$

$$\mathbf{W}_{\text{OSTBC}}^{opt} = \mathbf{Q}_{\mathbf{R}_t} \Lambda_{\mathbf{W}}^{1/2},$$

which is simply to signal on the modes of $\mathbf{R}_t$. 
Rate Performance Based on Channel Knowledge

Ergodic capacity comparison based on degree of channel knowledge available to the transmitter

\[ R_t = \begin{bmatrix} 1 & 0.9 & 0.81 & 0.729 \\ 0.9 & 1 & 0.9 & 0.81 \\ 0.81 & 0.9 & 1 & 0.9 \\ 0.729 & 0.81 & 0.9 & 1 \end{bmatrix} \]
Precoding for Alamouti coding based on knowledge of $R_t$ improves performance.

$$R_t = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$
- Combines transmit MRC and Alamouti coding given the channel covariance.
- We model the channel as

\[ h^T = [h_1^T \ h_2^T] \]
The channel is perfectly correlated across antennas within a sub-array and uncorrelated across sub-arrays with

\[
\mathbf{R} = \mathcal{E}\{\mathbf{h}\mathbf{h}^H\} = \begin{bmatrix}
\mathcal{E}\{h_1 h_1^H\} & 0_{N,N} \\
0_{N,N} & \mathcal{E}\{h_2 h_2^H\}
\end{bmatrix}.
\]

The eigenvectors of \( \mathbf{R} \) are given by the channel response at each sub-array

\[
\begin{bmatrix}
h_1 \\
0_{N,1}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0_{N,1} \\
h_2
\end{bmatrix}.
\]

An optimal scheme is beamforming (transmit MRC) at each sub-array with Alamouti coding across sub-arrays

This scheme extracts all the available array gain and diversity gain.
Selection between SM and Diversity Coding

- We assume Spatial Multiplexing and Alamouti coding are available at Tx.
- We keep the rate constant (by changing Tx constellation) and choose the scheme with the better $P_e$.
- Approximating the instantaneous $P_e$, we have

\[
P_{SM}^{\text{inst}} \approx \left(\sqrt{\frac{\rho}{2M_T}} D_{\text{min,SM}}^2\right) \quad P_{\text{OSTBC}}^{\text{inst}} \approx \left(\sqrt{\frac{\rho}{2M_T}} D_{\text{min,OSTBC}}^2\right)
\]

| $D_{\text{min,SM}}^2 = \min_{i,j} \|H(s^{(i)} - s^{(j)})\|_F^2$ | $D_{\text{min,OSTBC}}^2 = \|H\|_F^2 d_{\text{min,OSTBC}}^2$ |

- $d_{\text{min,scheme}}^2$ is the squared minimum distance of the transmit scalar constellation when the scheme is used. $D_{\text{min,scheme}}$ is defined for the separation of the vector constellation points.
Selection between SM and Diversity Coding - (ctd)

- We can bound $D_{min,SM}^2$ by

$$D_{min,SM} \geq \lambda_{min} d_{min,SM}^2$$

where $\lambda_{min}$ is the minimum eigenvalue of $HH^H$.

- For $P_{OSTBC}^{inst} \leq P_{SM}^{inst}$ then

$$D_{min,OSTBC}^2 \geq D_{min,SM}^2 \Rightarrow \kappa^2 \geq \frac{d_{min,SM}^2}{d_{min,OSTBC}^2}$$

$$\kappa^2 = \frac{\|H\|_F^2}{\lambda_{min}}$$

- $\kappa$ is the Demmel Condition number of matrix $H$

- The Alamouti code is chosen if $\kappa$ above a threshold and vice-versa.
Performance of Switched (OSTBC, SM)

Comparison of switched (OSTBC, SM) transmission with fixed OSTBC and fixed SM
- System with $M_T$ transmit antennas and $P$ transmit RF chains
- A total of $\binom{M_T}{P}$ distinct choices which we index using $i$.
- Maximum Information Rate
  - Our objective is to
    \[
    C = \max_{i, \mathbf{R}_{ss}} \log_2 \det \left( \mathbf{I}_M + \frac{\rho}{P} \mathbf{H}_i \mathbf{R}_{ss} \mathbf{H}_i^H \right),
    \]
    with $\text{Tr}(\mathbf{R}_{ss}) = P$ and where $\mathbf{R}_{ss}$ ($P \times P$) covariance matrix.
Antenna Selection Performance

Ergodic Capacity with transmit antenna selection as a function of selected antennas, P and SNR,

\[ M_T = 4 \]
Minimum SER with Alamouti Coding

- We assume an OSTBC is used for transmission over the $M_R \times P$ link.
- The received SNR $\eta$

$$\eta = \frac{\rho}{F} \|H_i\|_F^2$$

- The $P$ columns of $H$ that maximizes $\|H_i\|_F^2$ are the optimal antenna subset.
- It can be shown that

$$\frac{\rho}{P} \|H\|_F^2 \geq \eta^{opt} \geq \frac{\rho}{P M_T} \|H\|_F^2.$$

- The upper- and lower-bounds of the received SNR depend on the squared Frobenius norm of the channel
- This shows that selection provides the same diversity order $M_T M_R$
Receive Antenna Selection

- Need to search over all antenna sub-sets to maximize the performance metric.
- Perform search subject to channel state and power constraints.
- Shows no loss in diversity order for Alamouti code.

Selecting 2 out of 3 receive antennas delivers full diversity order, AC
Exploiting Imperfect Channel Knowledge

- We have a MISO channel with $M_T = 2$ and $h = h_w$
- The Rx has perfect channel knowledge while the Tx has an imperfect channel estimate $\hat{h}$
- We assume the correlation between the true and estimated channels at both Tx antennas is the same
  \[
  \rho = \mathcal{E}\{h_i\hat{h}_i^*\}, \ i = 1, 2.
  \]
- The distribution of $h$ conditioned on $\hat{h}$ is Gaussian with mean and covariance
  \[
  m_{h|\hat{h}} = \rho \hat{h},
  \]
  \[
  R_{hh|\hat{h}} = (1 - |\rho|^2)I_{M_T}.
  \]
- $\rho$ is a measure of the channel error at the Tx with $\rho = 1$ perfect and $\rho = 0$ not useful.
Exploiting Imperfect Channel Knowledge - ctd

- Given a signal model

\[ y = \sqrt{\frac{E_s}{2}} hWx + n, \]

with \( x \) an Alamouti codeword. \( x = \begin{bmatrix} x_0 & -x_1^* \\ x_1 & x_0^* \end{bmatrix}, \)

- We desire to find \( W \) that minimizes the PEP given the conditional statistics \( (m_{h|h}, R_{hh|h}) \)

- For \( \rho = 0 \), \( W_{opt} = I_2 \), i.e. standard Alamouti coding.

- For \( \rho = 1 \) \( W_{opt} = \begin{bmatrix} \sqrt{2} \frac{h^H}{\|h\|_F} & 0_{2,1} \end{bmatrix} \).

- Hence the actual transmitted codeword \( S = WX \) when \( \rho = 1 \) implies transmit-MRC.

\[ S = \begin{bmatrix} \sqrt{2} \frac{h^H}{\|h\|_F} x_0 \\ \sqrt{2} \frac{h^H}{\|h\|_F} x_1 \end{bmatrix}. \]