

# On the Super Saint Petersburg Paradox

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On the Super Saint Petersburg Paradox

## The St. Petersburg Paradox

Daniel Bernoulli (1738):

$X = 2^k$ , with prob.  $2^{-k}$ ,  $k = 1, 2, \dots$

$x :$	2	4	8	16	32	...
$p(x) :$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	...

- Pay  $c$ .
- Receive  $X$ .
- Note  $EX = \infty$ .

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# The St. Petersburg Paradox

$X = 2^k$ , with prob.  $2^{-k}$ ,  $k = 1, 2, \dots$

$x :$	2	4	8	16	32	...
$p(x) :$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	...

- Are you willing to pay any price  $c$  to receive  $X - c$ ?
- Some say No.
- That's clearly wrong since  $E(X - c) = \infty$ .
- But would you pay  $c = \$1,000,000$ ?
- So the answer should be Yes. But why?
- What do we mean?

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## Previous Resolutions of the Paradox

- Bernoulli Utility  $u(X)$ . Let  $u(X) = \ln X$ .  
Now  $E u(X) = E \ln X$ .  
Maximize utility.
- Samuelson (1977): Objection to arbitrariness of log utility.
- Not enough money in world.
- Feller

$$\sum_{i=1}^n X_i - nC$$

But who is offering repeated bets? And where is  $nC$  coming from?

- Menger, Super Saint Petersburg paradox,  
 $P(X = 2^{2^k-1}) = 2^{-k}$ .

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On the Super Saint Petersburg Paradox

- Maximum growth rate
- Relative growth rate (for Super St. Pete)
- Competitive optimality
- Investment always positive.
- Fast drop-off of investment for high costs  $c$ .
- Resolution of Super St. Pete.
- Eyeball test

## Growth rate optimality

Kelly (1956)

- Horse race:

Horses	$V :$	1,	2,	...	$m$	
Probabilities	$p :$	$p_1,$	$p_2,$	...	$p_m$	
Odds (m for 1)	$o :$	$m,$	$m,$	...	$m$	
Bets	$b :$	$b_1,$	$b_2,$	...	$b_m,$	$b_i \geq 0, \sum_i b_i = 1$

- Wealth factor:  $S = b_i m$ , with probability  $p_i$ .

$$\begin{aligned}
 S_n &= \prod_{j=1}^n S(V_j) = \exp\left(\sum_{j=1}^n \ln S(V_j)\right) \\
 &= \exp\left(n \frac{1}{n} \sum_{j=1}^n \ln S(V_j)\right) \\
 &\rightarrow \exp(nE \ln S(V))
 \end{aligned}$$

## Growth rate optimality

- Kelly(1956):
- We have

$$\begin{aligned}W^* &= \max_{b_i \geq 0, \sum b_i = 1} \sum_{i=1}^m p_i \ln m b_i \\ &= \ln m - H(p), \quad b^* = p\end{aligned}$$

- Thus  $W^* + H(p) = \ln m$
- Side information  $Y: p(x)p(y|x)$   
Increase in growth rate:

$$\Delta W = I(X; Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

Optimal investment:

$$b^*(v|y) = p(v|y)$$

## Growth rate optimality

- Shannon (1966, MIT): Invest  $(b_1, b_2, \dots, b_m)$  in stocks  $(X_1, X_2, \dots, X_m)$  to maximize

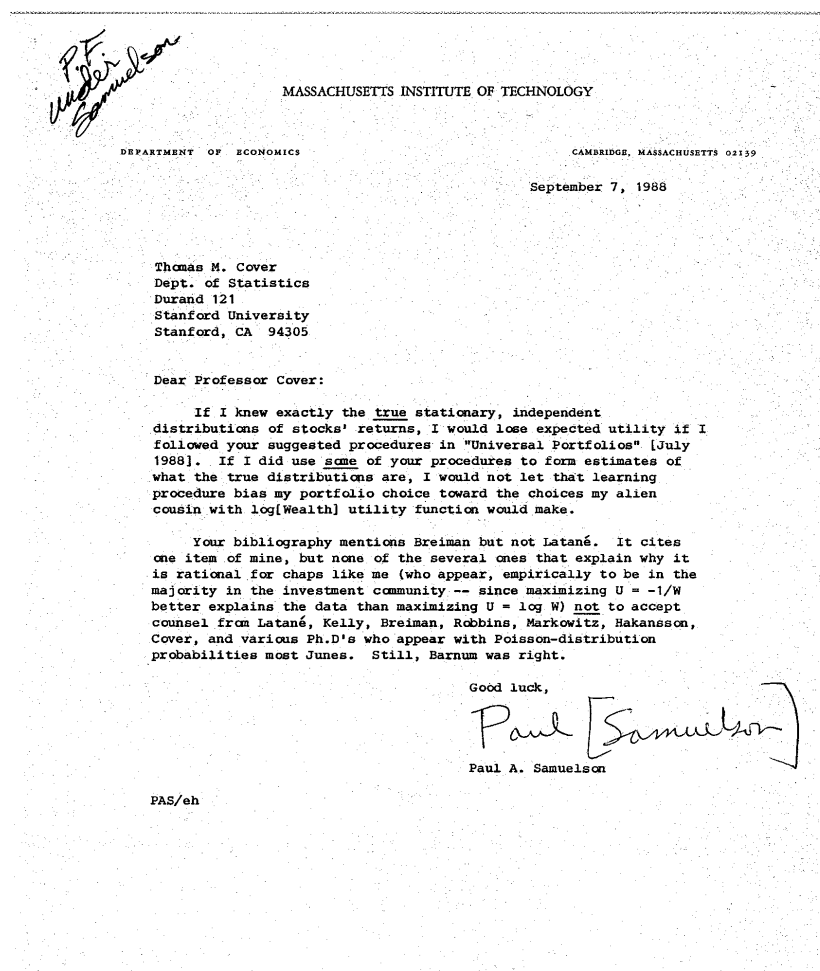
$$W^* = \max_b E \ln b^t X$$

- Breiman (1961)  $b^*$  maximizes growth rate  $W$  and minimizes time for  $S_n \geq A$ .

- Samuelson(1977):  
“Our analysis enables us to dispel a fallacy that has been borrowed into portfolio theory from information theory of the Shannon type”
- Samuelson(1979): “Why we should not make mean log of wealth big though years to act are long”
- Shannon (1986, Brighton) Meetings with Paul Samuelson in 1960's.
- Samuelson vs TC and vs Ziemba (personal correspondence).

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Dear Professor Cover:

If I knew exactly the true stationary, independent distributions of stocks' returns, I would lose expected utility if I followed your suggested procedures in "Universal Portfolios" [July 1988]. If I did use some of your procedures to form estimates of what the true distributions are, I would not let that learning procedure bias my portfolio choice toward the choices my alien cousin with  $\log[\text{Wealth}]$  utility function would make.

Your bibliography mentions Breiman but not Latané. It cites one item of mine, but none of the several ones that explain why it is rational for chaps like me (who appear, empirically to be in the majority in the investment community -- since maximizing  $U = -1/W$  better explains the data than maximizing  $U = \log W$ ) not to accept counsel from Latané, Kelly, Breiman, Robbins, Markowitz, Hakansson, Cover, and various Ph.D's who appear with Poisson-distribution probabilities most Junes. Still, Barnum was right.

Good luck,

Paul [Samuelson]

Paul A. Samuelson

PAS/eh

## Growth rate optimality

**Investment is exchanging one random variable for another.**

Invest proportion  $b$  of current wealth in Saint Petersburg. Keep the rest in cash.

$$1 \rightarrow 1 - b + \frac{bX}{c}$$

Choose  $b$  yielding highest growth rate.

Define

$$W(b, c) = E \log\left(\bar{b} + \frac{bX}{c}\right)$$

Then

$$S_n \doteq 2^{nW(b,c)}$$

## Growth rate optimality

$$S_n \doteq 2^{nW(b,c)}$$

Maximize over  $0 \leq b \leq 1$ . Then

$$S_n^* \doteq 2^{nW(b^*(c),c)}$$

Maximizing investment  $b$ :

$$W(b, c) = E \ln\left(\bar{b} + \frac{bX}{c}\right)$$
$$\frac{\partial W(b, c)}{\partial b} = E \frac{\frac{X}{c} - 1}{\bar{b} + \frac{bX}{c}} = 0$$

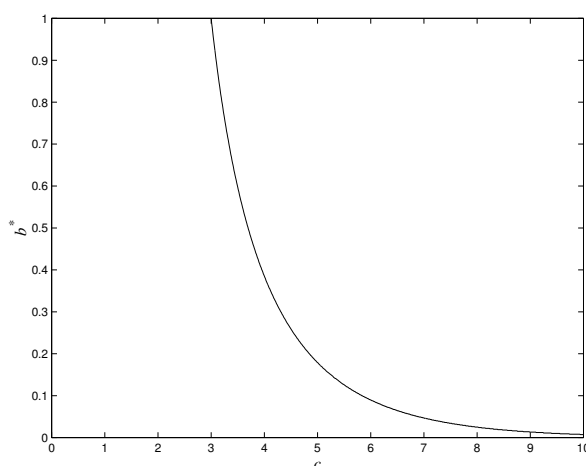
Put proportion  $b^*$  of wealth in St. Pete.

$$\text{Optimal investment } b^* : E \frac{1}{\bar{b} + \frac{bX}{c}} = 1 \quad (*)$$

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## Growth rate optimality



**Figure:** The growth optimal proportion  $b^*$  of wealth invested in the Saint Petersburg gamble as a function of the cost  $c$ .

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# Growth rate optimality

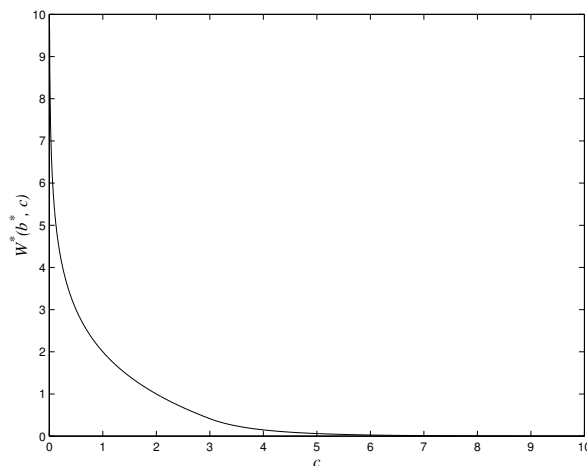


Figure: Growth rate  $W^*(b^*, c)$  as a function of  $c$ .

# Growth rate optimality

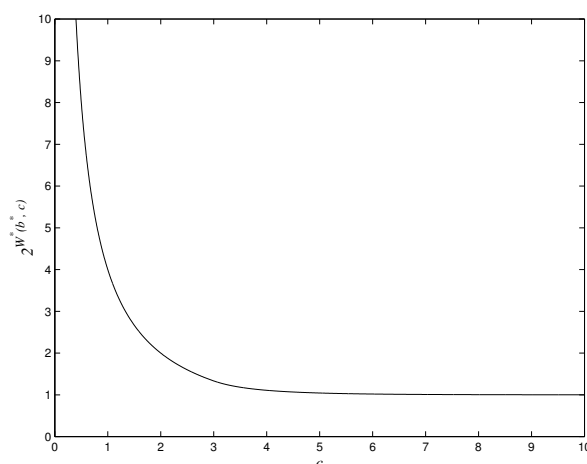


Figure: The wealth factor  $2^{W^*(b^*, c)}$  as a function of  $c$ . Note that  $2^{W^*}$  is not much greater than 1 for costs  $c \geq 4$



# Eye ball test

- Cash:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$



Figure: cash

- St. Pete:

$$\begin{pmatrix} 2 & 4 & 8 & 16 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$

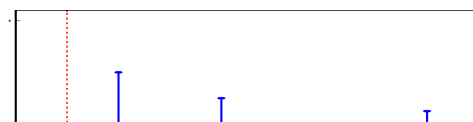


Figure: All St. Pete.

- Mixture:  $S(b) = 1 - b + \frac{bX}{c}$

$$\begin{pmatrix} \bar{b} + \frac{2b}{c} & \bar{b} + \frac{4b}{c} & \dots \\ \frac{1}{2} & \frac{1}{4} & \dots \end{pmatrix}$$

- Cash =  $S(0)$ , and St. Pete =  $S(1)$ .

## Example

$$c = 3$$

$$\text{Then } b^*(c) = 1, W^* = 0.4150, 2^{W^*} = 1.3333$$

$$\text{Cash} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$

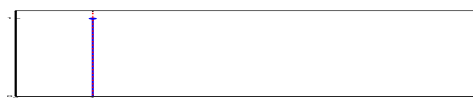


Figure: Cash

$$\text{All St. Pete}/c = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} & \frac{8}{3} & \frac{16}{3} & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$

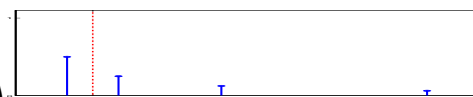


Figure: All St. Pete/3

$$S(b^*) = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} & \frac{8}{3} & \frac{16}{3} & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$



Figure: Optimal mixture:  $c=3$ ,  $b^* = 1$ ,  $2^{W^*} = 1.3333$

# Example

$$c = 4$$

$$\text{Then } b^*(c) \cong 0.384, 2^{W^*} = 1.1086$$

$$\begin{aligned} \text{Cash: } S(0) &= \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix} \\ \text{All St. Pete/4: } S(1) &= \begin{pmatrix} \frac{1}{2} & 1 & 2 & 4 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix} \\ S(b^*) &= \begin{pmatrix} 0.85 & 1 & 1.3 & 1.9 & 3.1 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \dots \end{pmatrix} \end{aligned}$$

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# Example



Figure: Cash



Figure: All St. Pete/4



Figure: Optimal mixture:  $c=4$ ,  $b^* = 0.38$ ,  $2^{W^*} = 1.1086$

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On the Super Saint Petersburg Paradox

## Example

C=4 (continued)

$$S(b^*) = \begin{pmatrix} 0.85 & 1 & 1.3 & 1.9 & 3.1 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \dots \end{pmatrix}$$

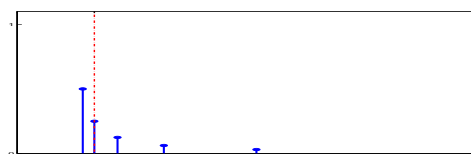


Figure: Optimal mixture:  $c=4$

Growth optimal  $b^*(c)$  is somewhat conservative.

$$P(S(b^*) = 0.85) = \frac{1}{2} \text{ vs } P(S(1) = \frac{1}{2}) = \frac{1}{2}.$$

## Example

$c = 8$

Then  $b^*(c) \cong 0.02484$ ,  $S^* = 0.97516 + 0.02484 \frac{X}{8}$

All St. Pete/8:

$$S(1) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 1 & 2 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$

All cash:

$$S(0) = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$

Growth optimal mixture:

$$S(b^*) = \begin{pmatrix} 0.978 & 0.984 & 1 & 1.032 & 1.096 & 1.224 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} \\ \\ 1.480 & 1.992 & 3.016 & 5.064 & \dots \\ \frac{1}{128} & \frac{1}{256} & \frac{1}{512} & \frac{1}{1024} & \dots \end{pmatrix}$$

$$2^{W^*} = 1.004278.$$

# Example

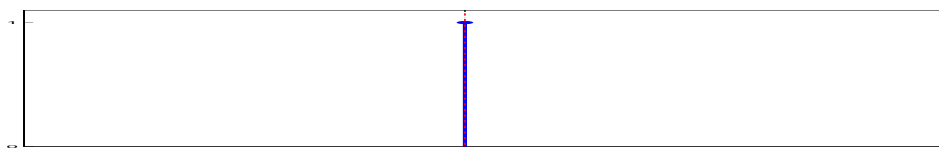


Figure: Cash

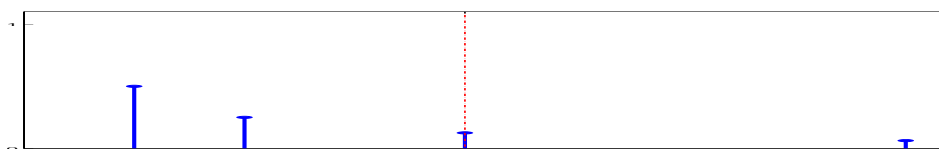


Figure: All St. Pete/8

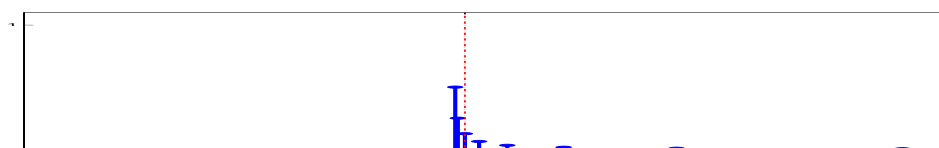


Figure: Optimal mixture:  $c=8$ ,  $b^* = 0.024$ ,  $2^{W^*} = 1.0043$

# Example

$c=8$ . Growth optimal mixture:

$$S(b^*) = \left( \begin{array}{cccccc} 0.978 & 0.984 & 1 & 1.032 & 1.096 & 1.224 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} \\ & & 1.480 & 1.992 & 3.016 & 5.064 & \dots \\ & & \frac{1}{128} & \frac{1}{256} & \frac{1}{512} & \frac{1}{1024} & \dots \end{array} \right)$$

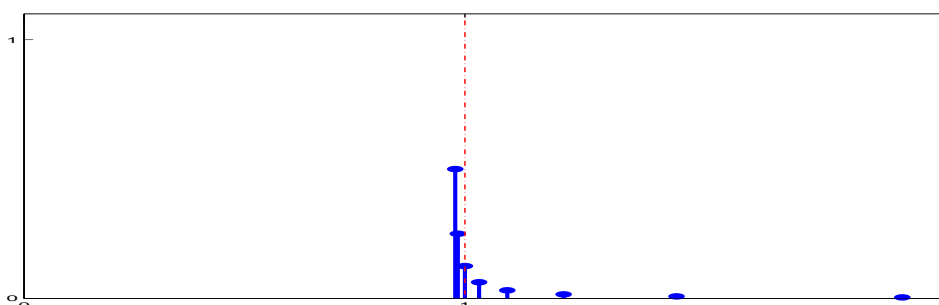


Figure: Optimal mixture:  $c=8$

$$b^* = 0.024, 2^{W^*} = 1.004278.$$

# Critical costs and r-th mean

r-th mean:  $\mu_r = (E|X|^r)^{\frac{1}{r}}$ .

**Arithmetic mean:**  $\mu_1 = EX$ .

**Geometric mean:**  $\mu_0 = \exp(E \ln X)$ .

**Harmonic mean:**  $\mu_{-1} = \frac{1}{E(\frac{1}{X})}$ .

## Laws of Large numbers

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mu_1$$
$$\left( \prod_{i=1}^n X_i \right)^{\frac{1}{n}} \rightarrow \mu_0$$

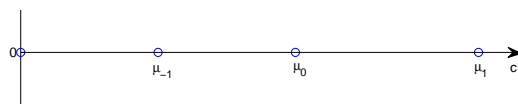


Figure: Various cases for the cost  $c$

# Critical Costs

For what costs  $c$  do you want some or all of Saint Petersburg?

$$\frac{\partial W(b, c)}{\partial b} = E \frac{\frac{X}{c} - 1}{\frac{X}{c} + b \frac{X}{c}}$$

$$\text{All: } b^* = 1 : \frac{\partial W(b, c)}{\partial b} > 0, \quad \text{if } c < \frac{1}{E(\frac{1}{X})}$$

$$\text{Some: } b^* = 0 : \frac{\partial W(b, c)}{\partial b} > 0, \quad \text{if } c < EX$$

# Competitive Optimality

Stocks:  $X \sim F(x), x \in \mathcal{R}^m$ .

Portfolio:  $b = (b_1, b_2, \dots, b_m), b_i \geq 0, \sum b_i = 1$ .

$b^* : \max_b E \ln b^t X$ .

$$S^* = b^{*t} X$$

$$S = b^t X$$

**Theorem (Bell and TC, 1980):**

$$b^* : E \frac{X_i}{b^{*t} X} = 1, i = 1, \dots, m$$

$$E\left(\frac{S}{S^*}\right) \leq 1, \forall b,$$

$$\Pr\{S \geq tS^*\} \leq \frac{1}{t}$$

$$\Pr\{S \geq \mathcal{U}_{[0,2]} S^*\} \leq \frac{1}{2}$$

## Competitive optimality

Thus no other portfolio  $b$  can beat  $\mathcal{U}b^*$  with probability greater than  $\frac{1}{2}$ .

Therefore,  $b^*$  is both growth-rate optimal and competitively optimal.

# Relative Growth Rate and Super Saint Petersburg

## Super Saint Petersburg

$$P(X = 2^{2^k-1}) = 2^{-k} \quad \begin{array}{l} x : \quad 2 \quad 8 \quad 128 \quad 32768 \quad \dots \\ p(x) : \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \dots \end{array}$$

Even when  $S_n(b) \nearrow \infty$  for any proportion  $b \in [0, 1]$ , there is a rationale for not choosing  $b = 1$ .

Compare investor  $(\bar{b}, b)$  with investor  $(\frac{1}{2}, \frac{1}{2})$ .

Then,

$$\frac{S_n(b)}{S_n(\frac{1}{2})} = \exp \left( nE \ln \frac{\bar{b} + \frac{bX}{c}}{\frac{1}{2} + \frac{1}{2} \frac{X}{c}} \right)$$

Define the relative growth rate

$$\Delta(b, \tilde{b}) = E \ln \frac{\bar{b} + \frac{bX}{c}}{\tilde{b} + \frac{\tilde{b}X}{c}}$$

# Relative Growth Rate and Super Saint Petersburg

Note:

- $\Delta(b_1, b_2) = W(b_1) - W(b_2)$  when  $W(b)$  is finite.
- $\Delta$  is always finite.
- $\Delta(b_1, b_2) = \Delta(b_1, b_3) + \Delta(b_3, b_2)$
- $\max_b \Delta(b_1, b_2)$  is achieved by  $b^*(c)$ , the solution to

$$E \frac{1}{\bar{b} + \frac{bX}{c}} = 1. \quad (*)$$

NOTE: Same equation for  $b^*$  as before.

- In repeated plays,

$$\frac{S_n(b^*(c))}{S_n(\frac{1}{2})} \doteq \exp \left( n\Delta(b^*, \frac{1}{2}) \right) \nearrow \infty$$

Also note that  $b^*(c) = 1$ , for all prices  $c \leq \mu_0$ .

# Relative Growth Rate and Super Saint Petersburg

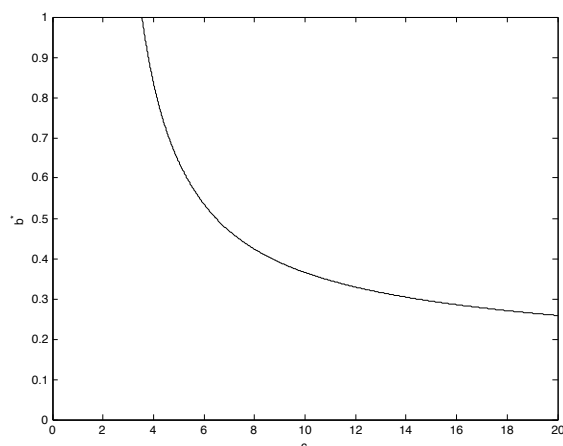


Figure: Optimal portfolio  $b^*$  as a function of  $c$  for Super Saint Petersburg.

## Example: Super St. Pete

cost  $c = 8$ .

Super St. Pete (2, 8, 128, ...)

Super St. Pete/8:

$S(1) =$

$$\begin{pmatrix} \frac{1}{4} & 1 & 16 & 4096 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$

$b^*(8) = 0.4238$

Growth optimal mixture:

$S(b^*) =$

$$\begin{pmatrix} 0.682 & 1 & 7.356 & 1736.44 & \dots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \dots \end{pmatrix}$$

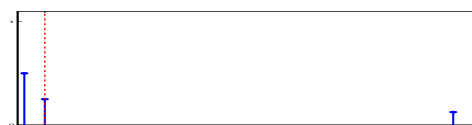


Figure: Super St. Pete/8

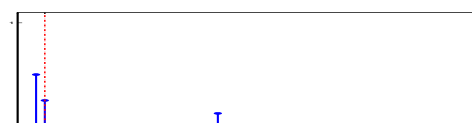


Figure: Optimal mixture:  $c=8$



## Conclusions

Let  $X$  be Saint Petersburg or Super Saint Petersburg.  
Let  $b(c)$  maximize relative growth rate

$$\Delta(b \parallel \frac{1}{2}) = E \ln \frac{\bar{b} + \frac{bX}{c}}{\frac{1}{2} + \frac{1}{2} \frac{X}{c}}$$

Thus optimal proportion invested in  $X$  is  $b^*(c)$ , solution to

$$E \frac{1}{\bar{b} + \frac{bX}{c}} = 1.$$

## Conclusions

For Saint Petersburg and Super Saint Petersburg,





- Want  $X$  at any price  $c$ , i.e.  $b^*(c) > 0, \forall c$ .

$$E \frac{1}{\bar{b} + \frac{bX}{c}} = 1.$$






- $S^* = 1 - b^*(c) + b^*(c) \frac{X}{c}$  is *growth rate* optimal.
- $S^*$  is *relatively* growth rate optimal.
- $S^*$  is *competitively* optimal in one-shot investment.
- $\frac{S_n^*}{S_n} \nearrow \infty$  in repeated plays.
- $S^*$  passes eyeball test.
- Want  $X$  at any price, but less and less as price goes up.

- Kartik Venkat
- Gowtham Kumar
- Lei Zhao


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